# Some topological properties associated with measurable cardinals

by

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**§ 1. Introduction** (1). H. J. Keisler and A. Tarski have introduced in [5] the symbols $\mathcal{C}_1$, $\mathcal{C}_1^*$ and $\mathcal{C}_2$ to denote the following classes of cardinal numbers.

$\mathcal{C}_1$: the class of all cardinals $\alpha$ for which every $\alpha$-complete ultrafilter on (the discrete space of cardinality) $\alpha$ is principal;

$\mathcal{C}_1^*$: the class of all cardinals $\alpha$ for which either $\alpha$ is a singular cardinal or some $\alpha$-complete filter on (the discrete space of cardinality) $\alpha$ cannot extend to an $\alpha$-complete ultrafilter;

$\mathcal{C}_2$: the class of all cardinals $\alpha$ for which some $\alpha$-complete filter on some set cannot extend to an $\alpha$-complete ultrafilter (2).

The class inclusion $\mathcal{C}_1 \subset \mathcal{C}_1^*$ is obvious, and it is clear also that $\alpha \in \mathcal{C}_2$ whenever $\alpha$ is regular and $\alpha \in \mathcal{C}_1^*$; it is an open question whether either of these class inclusions is proper. Keisler and Tarski in [5] have studied these (and other) classes, using a certain binary relation $\mathcal{R}$ on the class of all cardinals. From our point of view, which focuses on the Stone–Čech compactification of discrete spaces of certain large cardinalities, it is convenient to modify the relation $\mathcal{R}$ of [5] and introduce the binary relation $\mathcal{S}$ on the class of all cardinals as follows.

Definition. Let $\alpha$ and $\beta$ be cardinal numbers. Then $\alpha \mathcal{S} \beta$ provided there is, on (the discrete space of cardinality) $\beta$, an $\alpha$-complete filter that cannot extend to an $\alpha$-complete ultrafilter.

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(1) Notation and terminology not explained here will be given in § 2.

(2) That the class $\mathcal{C}_2$ coincides with the one defined in [5] follows from the proof of Theorem 5.6 in [5]. The classes $\mathcal{C}_1$, $\mathcal{C}_2$ are identical with the classes of all cardinals satisfying properties $\boldsymbol{P}_3$, $\boldsymbol{P}_4$ respectively, of [1].

Notice that $\alpha \in \mathfrak{C}_2$ if and only if $\alpha \mathfrak{S} \beta$ for some cardinal $\beta$, and for regular $\alpha$ that $\alpha \in \mathfrak{C}_1^*$ if and only if $\alpha \mathfrak{S} \alpha$.

The object of the present paper is to characterize the set-theoretic relation $\mathfrak{S}$ in topological terms. Using the (topological) concept of type, we consider the way in which the set of all non-principal $\alpha$-complete ultrafilters on (the discrete space of cardinality) $\beta$ is embedded in the set $\Psi_\alpha(\beta)$ of all $\alpha$-uniform ultrafilters on the space, where both $\Omega_\alpha(\beta)$ and $\Psi_\alpha(\beta)$ are regarded as subspaces of the Stone-Čech compactification of a discrete space of cardinality $\beta$. Specifically, we prove the following two results for cardinals $\alpha$ not in $\mathfrak{C}_1$; the first of these describes when $\alpha \mathfrak{S} \beta$ for $\beta \geqslant \alpha$, and the second tells when $\alpha \mathfrak{S} \alpha$. (In the body of the text these two results, together with certain other equivalences, appear as Theorems 3.6 and 3.9 respectively.)

THEOREM A. *If $\alpha \notin \mathfrak{C}_1$ and $\alpha \leqslant \beta$, then $\alpha \mathfrak{S} \beta$ fails if and only if $\Omega_\alpha(\beta)$ is $C^*$-embedded in $\Psi_\alpha(\beta)$ and each open subset of $\Psi_\alpha(\beta)$ containing $\Omega_\alpha(\beta)$ contains densely an open set whose type does not exceed $\alpha$.*

THEOREM B. *If $\alpha \notin \mathfrak{C}_1$ and $\alpha^+ = 2^\alpha$, then $\alpha \mathfrak{S} \alpha$ fails if and only if $\Omega_\alpha(\alpha)$ is $C^*$-embedded in $\Psi_\alpha(\alpha)$.*

The last section of this paper contains some additional results, holding for all cardinals $\alpha$ not belonging to the class $\mathfrak{C}_1$. They relate to the local behavior of uniform ultrafilters on (the discrete space of cardinality) $\alpha$, and to the cardinalities of certain subsets of ultrafilters on (the discrete space of cardinality) $\alpha$.

In a direction different from ours, but nevertheless topological, there are product-space theorems about measurable cardinals given by Keisler–Tarski [5] (Theorem 4.27, for example) and by Monk–Scott [6]. We observe here, as in [11], that certain of the results of [1], [3], [5] and [6] were anticipated by Parovičenko in [9] and [10]. We shall not attempt at this time to relate our "Stone-Čech compactification" theorems to the "Tychonoff product-space" theorems given there and in [7].

**§ 2. Definitions and lemmas.** Ordinal numbers are denoted by $\xi$ and $\zeta$. Each ordinal concides with the set of all smaller ordinals, so that the condition $\xi < \zeta$ is equivalent to the condition $\xi \in \zeta$. Nevertheless, we will make the notational distinction between the first ordinal 0 and the empty set $\emptyset$. A cardinal number is an initial ordinal. Cardinals are denoted by $\alpha$, $\beta$, $\gamma$. The first infinite cardinal is $\omega$. The least cardinal greater than $\alpha$ is denoted by $\alpha^+$. $\alpha$ is a limit cardinal if it is not of the form $\beta^+$. A cardinal $\alpha$ is regular if it cannot be expressed as the sum of fewer than $\alpha$ cardinals each smaller than $\alpha$. $\beta^\alpha$ denotes the cardinal number of the set of all mappings from $\alpha$ to $\beta$. We set $\beta^{\underline{\alpha}} = \sum\{\beta^\gamma\colon \gamma < \alpha\}$. The cardinality of a set $A$ will be denoted by $|A|$. For each cardinal number $\alpha$, we shall denote by $D(\alpha)$ the (topologically unique) discrete topological space of cardinality $\alpha$. As sets, $\alpha$ and $D(\alpha)$ are indistinguishable.



If $\alpha$ and $\beta$ are cardinals, then a filter $p$ on the set $D(\beta)$ will be called $\alpha$-uniform if $|F| \geqslant \alpha$ whenever $F \in p$. A filter $p$ on the set $D(\beta)$ will be called $\alpha$-complete provided that $\bigcap q \in p$ whenever $q \subset p$ and $|q| < \alpha$. Every filter $p$ on $D(\beta)$ for which $\bigcap p \neq \emptyset$ is, of course, $\alpha$-complete for all $\alpha$. A filter on $D(\beta)$ not properly contained in any filter on $D(\beta)$ will be called an ultrafilter. We shall be interested in the non-principal ultrafilters on $D(\beta)$, i.e., in those ultrafilters $p$ for which $\bigcap p = \emptyset$. The set of non-principal $\alpha$-complete ultrafilters on $D(\beta)$, and the set of $\alpha$-uniform ultrafilters on $D(\beta)$, will be denoted respectively by the symbols $\Omega_\alpha(\beta)$, $\Psi_\alpha(\beta)$. The inclusion $\Omega_\alpha(\beta) \subset \Psi_\alpha(\beta)$ is obvious.

If $\omega \leqslant \alpha \leqslant \beta$, then $\Psi_\alpha(\beta) \neq \emptyset$. Indeed, the family of complements of all subsets of $D(\beta)$ of cardinality less than $\alpha$ forms a filter, any extension of which to an ultrafilter on $D(\beta)$ produces an element of $\Psi_\alpha(\beta)$. Further, for $\alpha = \beta = \omega$, $\Omega_\omega(\omega) = \Psi_\omega(\omega) \neq \emptyset$. In contrast, it is not known whether any uncountable cardinal $\alpha$ exists for which $\Omega_\alpha(\alpha) \neq \emptyset$. Restated: It is not known whether there is any uncountable cardinal number not in the class $\mathfrak{C}_1$. An uncountable cardinal not belonging to $\mathfrak{C}_1$ will be called Ulam-measurable or, simply, measurable. It is precisely for these cardinals $\alpha$ that there exists a measure $\mu$ defined on $D(\alpha)$ with the following properties: (a) each subset of $D(\alpha)$ is $\mu$-measurable; (b) if $S \subset D(\alpha)$, then $\mu(S) = 0$ or $\mu(S) = 1$; (c) if $x \in D(\alpha)$, then $\mu(\{x\}) = 0$; (d) $\mu(D(\alpha)) = 1$; (e) if $\mu(S_i) = 0$ for each $i \in I$ with $|I| < \alpha$, then $\mu(\bigcup_{i\in I} S_i) = 0$. (Our terminology in this respect differs from the traditional one, given in [2] for example, according to which $\alpha$ is called measurable if there is a countably additive measure satisfying conditions (a), (b), (c) and (d). According to the classical Ulam–Tarski theorem (cf. [14]) the first cardinal number measurable in the sense of [2], is in fact Ulam-measurable). If $\alpha$ is a measurable cardinal then $\alpha$ is strongly inaccessible in the sense that $\alpha$ is a regular cardinal and that $2^\beta < \alpha$ whenever $\beta < \alpha$ (cf. [13]); in particular, $\alpha = \alpha^{\underline{\alpha}} = 2^{\underline{\alpha}}$.

The notations $\mathrm{cl}_X Y$, $\mathrm{int}_X Y$ denote the (topological) closure, interior of the subspace $Y$ of the topological space $X$ in $X$, respectively.

A subset $Y$ of a topological space $X$ is said to be $C^*$-embedded in $X$ if each bounded continuous real-valued function on $Y$ extends continuously to $X$. With every completely regular Hausdorff topological space we associate its Stone–Čech compactification, denoted $\beta(X)$ and characterized by the following properties: (a) $\beta(X)$ is a compact Hausdorff space; (b) $X$ is (homeomorphic with) a dense subspace of $\beta(X)$; (c) $X$ is $C^*$-embedded in $\beta(X)$. (The space $\beta(X)$ is constructed and discussed in [2].)

For each discrete space $D$, the Stone–Čech compactification $\beta(D)$

of $D$ will be regarded as the set of ultrafilters on $D$, topologized so that for each subset $A$ of $D$ and each ultrafilter $p$ on $D$ we have

$$p \in \mathrm{cl}_{\beta(D)} A \quad \text{if and only if} \quad A \in p .$$

(In case $p$ is in fact an element of $D$ itself, our notation has identified $p$ with the ultrafilter on $D$ consisting of all subsets of $D$ to which $p$ belongs.) From the fact that $X$ is $C^*$-embedded in $\beta(X)$, it follows that if $A$ is any open-and-closed subset of $X$ then $\mathrm{cl}_{\beta(X)} A$ is also open-and-closed in $\beta(X)$. In particular $A$ and $D \backslash A$ have disjoint closure in $\beta(D)$ whenever $D$ is discrete and $A \subset D$, so that each set of the form $\mathrm{cl}_{\beta(D)} A$, with $A \subset D$, is open-and-closed in $\beta(D)$.

DEFINITION. If $D$ is a discrete space and $A \subset D$, we set $A^* = (\mathrm{cl}_{\beta(D)} A) \backslash D$. In particular, $D^* = \beta(D) \backslash D$.

2.1. LEMMA. *For each discrete space $D$ the family $\{A^*\colon A \subset D\}$ is a base for the topology on $D^*$.*

Proof. If $U$ is a neighborhood in $D^*$ of $p$, then because $D^*$ is completely regular there is a continuous function $f$ from $D^*$ to $[0, 1]$ for which $f(p) = 0$ and $f \equiv 1$ on $D^* \backslash U$. Since $\beta D$ is a normal space and $D^*$ is closed in it, $f$ extends by Tietze's theorem to a continuous function $g$ mapping $\beta D$ to $[0, 1]$. Defining

$$A = g^{-1}[0, 1/2) \cap D ,$$

we have

$$p \in A^* \subset U .$$

We need some additional facts about the topology of the Stone-Čech compactification of a discrete space.

2.2. LEMMA. *$\Psi_\alpha(\beta)$ is a compact subset of $D(\beta)^*$.*

Proof. To see that $\Psi_\alpha(\beta)$ is closed in $D(\beta)^*$, let $p \in D(\beta)^* \backslash \Psi_\alpha(\beta)$ and find a subset $A$ of $D(\beta)$ for which $|A| < \alpha$ and $A \in p$. Then $A^*$ is a neighbourhood in $D^*$ of $p$, and $A^* \cap \Psi_\alpha(\beta) = \emptyset$.

2.3. LEMMA. *Let $W$ be a subset of $\Psi(\beta)$ which is open-and-closed (in the topology which $\Psi(\beta)$ inherits from $D(\beta)^*$). Then there is a subset $A$ of $D$ for which*

$$W = A^* \cap \Psi_\alpha(\beta) .$$

Proof. For every $p \in W$, let $A_p$ be a subset of $D(\beta)$ such that $p \in A_p^* \cap \Psi_\alpha(\beta) \subset W$. Since $W$ is compact, there is a finite number of elements $p_1, p_2, \ldots, p_n$ in $W$ such that

$$W = \bigcup_{k=1}^{n} A_{p_k}^* \cap \Psi_\alpha(\beta) .$$



Defining $A = \bigcup_{k=1}^{n} A_{p_k}$ we have

$$W = A^* \cap \Psi_\alpha(\beta) ,$$

as desired.

We have already observed that the condition $\alpha \notin \mathfrak{C}_1$, i.e., the condition that $\alpha$ be measurable, is equivalent to the condition $\Omega_\alpha(\alpha) \neq \emptyset$. We pursue this further.

2.4. LEMMA. *Let $\alpha$ be a measurable cardinal number. Then for each cardinal number $\beta$ the set $\Omega_\alpha(\beta)$ is dense in $\Psi_\alpha(\beta)$.*

Proof. We shall suppose that $\alpha \leqslant \beta$, since otherwise both sets in question are empty. For each nonvoid open subset $U$ of $\Psi_\alpha(\beta)$ there is a subset $A$ of $D(\beta)$ for which $|A| = \alpha$ and

$$A^* \cap \Psi_\alpha(\beta) \subset U .$$

Since $\alpha$ is measurable there is an $\alpha$-complete ultrafilter $p$ on $A$ which is not principal. Let $q = \{F \subset D\colon F \cap A \in p\}$. Then $q \in U \cap \Omega_\alpha(\beta)$.

2.5. LEMMA. *Let $\{K_i\}_{i \in I}$ be a collection of closed subsets of $\Omega_\alpha(\beta)$, with $|I| < \alpha$. Then $\bigcup_{i \in I} K_i$ is closed in $\Omega_\alpha(\beta)$.*

Proof. If $p \in \Omega_\alpha(\beta) \backslash \bigcup_{i \in I} K_i$ then there is, for each $i$ in $I$, a subset $A_i$ of $D(\beta)$ with $p \in A_i^*$ and with $A_i^* \cap K_i = \emptyset$. Since each of the sets $A_i$ belongs to $p$ and $|A| < \alpha$, we have $\bigcap_{i \in I} A_i \in p$. Then $(\bigcap_{i \in I} A_i)^* \cap \Omega_\alpha(\beta)$ is a neighborhood of $p$ which misses $\bigcup_{i \in I} K_i$.

A topological space is said to be $\alpha$-compact, where $\alpha$ is a cardinal number, if each of its open covers admits a subcover by fewer than $\alpha$ elements. The relation $\mathfrak{S}$ defined above may easily be characterized in terms of this concept.

2.6. LEMMA. *Let $\alpha$ be measurable, and let $\alpha \leqslant \beta$. Then the relation $\alpha \mathfrak{S} \beta$ is false if and only if $\Omega_\alpha(\beta)$ is $\alpha$-compact.*

Proof. Since $\{A^* \cap \Omega_\alpha(\beta)\colon A \subset D(\beta)\}$ is a base for the relative topology on $\Omega_\alpha(\beta)$, the space $\Omega_\alpha(\beta)$ is $\alpha$-compact if and only if each cover of $\Omega_\alpha(\beta)$ by (open) sets of the form $A^* \cap \Omega_\alpha(\beta)$ admits a subcover by fewer than $\alpha$ sets; this is equivalent to the condition that each collection of (closed) subsets of the form $A^* \cap \Omega_\alpha(\beta)$ has nonvoid intersection provided that each subfamily with fewer than $\alpha$ members has nonvoid intersection; this in turn is equivalent to the condition that each collection of subsets of $D(\beta)$ extends to an $\alpha$-complete ultrafilter provided that each subcollection with fewer than $\alpha$ members does so—i.e., to the condition that $\alpha \mathfrak{S} \beta$ fails.

We remark that the hypothesis that $\alpha$ be measurable cannot be dismissed. Indeed, if $\alpha$ is any regular, nonmeasurable cardinal whatever, then it is easy to see that the relation $\alpha S \alpha$ is valid but that $\Omega_\alpha(\alpha)$, since it is empty, is compact.

According to Lemma 2.5, the intersection of fewer than $\alpha$ open subsets of $\Omega_\alpha(\beta)$ will again be open in $\Omega_\alpha(\beta)$. Any analogous statement about $\Psi_\alpha(\beta)$ is emphatically false, but the following is a suitable substitute.

2.7. LEMMA. *Let $\alpha$ be a measurable cardinal, and let $\{W_i\}_{i\in I}$ be a collection of open-and-closed subsets of $\Psi_\alpha(\alpha)$ for which $(\bigcap_{i\in I} W_i)\cap\Omega_\alpha(\alpha)\neq\emptyset$ and $|I|<\alpha$. Then $\operatorname{int}_{\Psi_\alpha(\alpha)}(\bigcap_{i\in I} W_i)\neq\emptyset$.*

Proof. Lemma 2.3 assures us that for each $i$ in $I$ there is a subset $A_i$ of $D(\alpha)$ such that $W_i = A_i^*\cap\Psi_\alpha(\alpha)$. If $p$ is chosen so that $p\in(\bigcap_{i\in I} W_i)\cap\Omega_\alpha(\alpha)$, then for each $i$ in $I$ we have $A_i\in p$, hence $\bigcap_{i\in I} A_i\in p$. Defining $A=\bigcap_{i\in I} A_i$, we have

$$A^*\cap\Psi_\alpha(\alpha)\subset\operatorname{int}_{\Psi_\alpha(\alpha)}(\bigcap_{i\in I} W_i)\,,$$

as desired.

**§ 3. Characterization of the relation $S$ in topological terms.** According to Lemma 2.6, the relation $S$ can be characterized in topological terms by finding conditions equivalent to the condition that $\Omega_\alpha(\beta)$ be $\alpha$-compact. It is to this project that the present section is devoted. We begin with two lemmas, the first a tool for the proof of the second.

3.1. LEMMA. *If $\Omega_\alpha(\beta)$ is $\alpha$-compact and $U$ is (relatively) open-and-closed in $\Omega_\alpha(\beta)$, then there is a subset $A$ of $D(\beta)$ for which $U=A^*\cap\Omega_\alpha(\beta)$.*

Proof. Because $U$ is open in $\Omega_\alpha(\beta)$ there is for each point $p$ in $U$ a subset $A_p$ of $D(\beta)$ for which $p\in A_p^*\cap\Omega_\alpha(\beta)\subset U$; since $U$ is closed in $\Omega_\alpha(\beta)$, and hence $\alpha$-compact, a subset $X$ of $U$ exists for which $|X|<\alpha$ and

$$U=(\bigcup_{p\in X} A_p^*)\cap\Omega_\alpha(\beta)\,.$$

Writing $A=\bigcup_{p\in X} A_p$, we have $U=A^*\cap\Omega_\alpha(\beta)$: for surely $U\subset A^*\cap\Omega_\alpha(\beta)$, and if $q\in A^*\cap\Omega_\alpha(\beta)$, in which case $A\in q$, then because $q$ is $\alpha$-complete a point $p$ exists in $X$ for which $A_p\in q$, so that

$$q\in A_p^*\cap\Omega_\alpha(\beta)\subset U\,.$$

3.2. LEMMA. (a) *If $\Omega_\alpha(\beta)$ is $\alpha$-compact, then $\Omega_\alpha(\beta)$ is $C^*$-embedded in $\Psi_\alpha(\beta)$;*

(b) *If $\Omega_\alpha(\beta)$ is $C^*$-embedded in $\Psi_\alpha(\beta)$ and $U$ is (relatively) open-and-closed in $\Omega_\alpha(\beta)$, then there is a subset $A$ of $D(\beta)$ for which $U=A^*\cap\Omega_\alpha(\beta)$.*



Proof. We shall suppose that $\omega\leqslant\alpha\leqslant\beta$, the assertions being obvious otherwise.

To prove (a), indeed, we may take $\alpha>\omega$ (since $\Omega_\omega(\beta)=\Psi_\omega(\beta)$). In this case, then, given $f$ in $C^*(\Omega_\alpha(\beta))$ and a real number $r$, the set $f^{-1}(r)$, because it is a $G_\delta$ in $\Omega_\alpha(\beta)$, is open-and-closed in $\Omega_\alpha(\beta)$ by Lemma 2.5. Choosing by Lemma 3.1 a subset $A_r$ of $D(\beta)$ for which $f^{-1}(r)=A_r^*\cap\Omega_\alpha(\beta)$, we wish to extend $f$ to a bounded function $g$ defined throughout $D(\beta)\cap\Omega_\alpha(\beta)$ by writing $g\equiv r$ on $A_r$. Unfortunately the subsets $A_r$ of $D(\beta)$ may fail to be pairwise disjoint, so that $g$ cannot be so defined. To remedy this difficulty, we define

$$B_r=A_r\setminus\bigcup_{s<r} A_s$$

and we define $g$ on $D(\beta)$ by the rule

$$g(x)=\begin{cases} r & \text{if}\quad x\in B_r\,,\\ 0 & \text{if}\quad x\in D(\beta)\setminus\bigcup_r B_r\,.\end{cases}$$

To check that the bounded function $g$ has a continuous extension to $\beta(D(\beta))$ which agrees with $f$ on $\Omega_\alpha(\beta)$, it suffices to check that

$$f^{-1}(r)=B_r^*\cap\Omega_\alpha(\beta)$$

for each real $r$. Fixing $r$ we choose a sequence $s_n$ of real numbers, each less than $r$, with limit $r$, so that

$$B_r=A_r\setminus\bigcup_{n\in\omega}(\bigcup_{s<s_n} A_s)\,.$$

Now if $p\in f^{-1}(r)$ then $p\notin(\bigcup_{s<s_n} A_s)^*$, since otherwise we would have

$$f(p)\leqslant s_n<r=f(p)\,.$$

Thus $\bigcup_{s<s_n} A_s\notin p$, so that ($p$ being $\alpha$-complete, with $\alpha>\omega$) $\bigcup_{s<r} A_s\notin p$; thus

$$p\in B_r^*\cap\Omega_\alpha(\beta)\,,$$

as desired. This completes the proof of (a).

For (b), we note that by hypothesis there is an open-and-closed subset $U'$ of $\Psi_\alpha(\beta)$ for which $U=U'\cap\Omega_\alpha(\beta)$. Applying Lemma 2.3 to $U'$, we see that there is a subset $A$ of $D(\beta)$ for which

$$U=U'\cap\Omega_\alpha(\beta)=(A^*\cap\Psi_\alpha(\beta))\cap\Omega_\alpha(\beta)=A^*\cap\Omega_\alpha(\beta)\,.$$

The following definition is taken from [8].

3.3. DEFINITION. Let $U$ be an open subset of $\Psi_\alpha(\beta)$. The type of $U$, denoted $\tau(U)$, is the smallest cardinal number which indexes a collection of open-and-closed subsets of $\Psi_\alpha(\beta)$ whose union is $U$.

This concept furnishes an easy, though not particularly useful, characterization of the relation $\mathcal{S}$.

3.4. THEOREM. *Let $\omega \leqslant \alpha \leqslant \beta$. The following conditions are equivalent:*

(a) *$\Omega_\alpha(\beta)$ is $\alpha$-compact;*

(b) *if $U$ is open in $\Psi_\alpha(\beta)$ and $\Omega_\alpha(\beta) \subset U \subset \Psi_\alpha(\beta)$, then an open subset $V$ of $\Psi_\alpha(\beta)$ exists for which $\Omega_\alpha(\beta) \subset V \subset U$ and $\tau V < \alpha$.*

Proof. (a) $\Rightarrow$ (b). For each point $p$ in $\Omega_\alpha(\beta)$ there is a subset $A_p$ of $D(\beta)$ for which

$$p \in A_p^* \cap \Psi_\alpha(\beta) \subset U .$$

For the desired set $V$ we choose

$$V = \bigcup_{p \in X} \big(A_p^* \cap \Psi_\alpha(\beta)\big) ,$$

where $X$ is a subset of the $\alpha$-compact space $\Omega_\alpha(\beta)$ chosen so that $|X| < \alpha$ and $\Omega_\alpha(\beta) \subset \bigcup_{p \in X} \big(A_p^* \cap \Psi_\alpha(\beta)\big)$.

(b) $\Rightarrow$ (a). If $\mathcal{W}$ is a cover of $\Omega_\alpha(\beta)$ by (relatively) open subsets, we choose a collection $\mathcal{U}$ of open subsets of $\Psi_\alpha(\beta)$ for which $\mathcal{W} = \{U \cap \Omega_\alpha(\beta)\colon U \in \mathcal{U}\}$ and we define $U = \bigcup \mathcal{U}$. According to (b), there are open-and-closed subsets $V_i$ of $\Psi_\alpha(\beta)$ for which

$$\Omega_\alpha(\beta) \subset \bigcup_{i \in I} V_i \subset U$$

and $|I| < \alpha$. Since each of the sets $V_i$ is compact, each $V_i$ is covered by finitely many elements of $\mathcal{U}$. Thus a subset $\mathcal{U}'$ of $\mathcal{U}$ exists with $|\mathcal{U}'| < \alpha$ and with

$$\Omega_\alpha(\beta) \subset \bigcup \mathcal{U}' .$$

It follows that $\Omega_\alpha(\beta) \subset \bigcup \mathcal{W}'$, where

$$\mathcal{W}' = \{U' \cap \Omega_\alpha(\beta)\colon U' \in \mathcal{U}'\} \subset \mathcal{W} .$$

The following lemma, which furnishes us with a multiplicity of equivalences in Theorem 3.6, shows that under certain circumstances the condition $\tau V \leqslant \alpha$ may, in effect, be replaced by the condition $\tau V < \alpha$.

3.5. LEMMA. *Let $\alpha$ be measurable, and let $\alpha \leqslant \beta$. Suppose that $\Omega_\alpha(\beta)$ is $C^*$-embedded in $\Psi_\alpha(\beta)$ and that $V$ in a dense open subset of $\Psi_\alpha(\beta)$ for which $\tau V \leqslant \alpha$. Then $\Omega_\alpha(\beta) \subset V$, and there is an open subset $V'$ of $\Psi_\alpha(\beta)$ for which*

$$\Omega_\alpha(\beta) \subset V' \subset V \quad \text{and} \quad \tau V' < \alpha .$$

Proof. If $V = \bigcup_{\xi < \alpha} V_\xi$, with each $V_\xi$ a non-void open-and-closed subset of $\Psi_\alpha(\beta)$, we set

$$W = V \cap \Omega_\alpha(\beta) \quad \text{and} \quad W_\xi = V_\xi \cap \Omega_\alpha(\beta) ,$$



so that, from Lemma 2.4, $W$ is dense in $\Omega_\alpha(\beta)$; and $W = \bigcup_{\xi < \alpha} W_\xi$. We set

$$W'_\xi = W_\xi \setminus \bigcup_{\zeta < \xi} W_\zeta ;$$

each of the sets $W'_\xi$ is, by Lemma 2.5, open-and-closed in $\Omega_\alpha(\beta)$. In order that we may complete the proof of the present lemma, we define

$$S = \{\xi\colon W'_\xi \neq \emptyset\}$$

and we consider two cases.

Case I: $|S| < \alpha$. Then $W$, which is $\bigcup_{\xi \in S} W'_\xi$, is a union of fewer than $\alpha$ open-and-closed subsets of $\Omega_\alpha(\beta)$; hence $W$ is closed in $\Omega_\alpha(\beta)$. Since $W$ is dense in $\Omega_\alpha(\beta)$ we have $W = \Omega_\alpha(\beta)$, i.e., $V \supset \Omega_\alpha(\beta)$. We set $V' = \bigcup_{\xi \in S} V_\xi$, so that

$$\Omega_\alpha(\beta) = \bigcup_{\xi \in S} W'_\xi \subseteq \bigcup_{\xi \in S} V_\xi = V' \subseteq V$$

and $\tau V' \leqslant |S| < \alpha$, as desired.

Case II. $|S| = \alpha$. We show that this is impossible. According to Lemma 3.2 (b) there is for each $\xi$ in $S$ a subset $A_\xi$ of $D(\beta)$ for which $W'_\xi = A_\xi^* \cap \Omega_\alpha(\beta)$. Since $W'_\xi \cap W'_\zeta = \emptyset$ whenever $\xi$ and $\zeta$ are distinct elements of $S$ we have always $|A_\xi \cap A_\zeta| < \alpha$ so that, setting

$$B_\xi = A_\xi \setminus \bigcup_{\zeta < \xi} A_\zeta ,$$

we have $|A_\xi \setminus B_\xi| < \alpha$, whence $B_\xi^* \cap \Psi_\alpha(\beta) = A_\xi^* \cap \Psi_\alpha(\beta)$. In particular, then,

$$B_\xi^* \cap \Omega_\alpha(\beta) = A_\xi^* \cap \Omega_\alpha(\beta) = W'_\xi$$

for each $\xi$ in $S$.

Now for each $\xi$ in $S$ we choose a point $p_\xi$ in $B_\xi$. We set $P = \{p_\xi\colon \xi \in S\}$. We have $|P| = \alpha$, so that $P^*$ meets the dense subset $W$ of $\Omega_\alpha(\beta)$. The desired contradiction is given by the computation

$$\begin{aligned} \emptyset \neq P^* \cap W = P^*\Big(\bigcup_{\xi \in S} W'_\xi\Big) &= \bigcup_{\xi \in S} (P^* \cap W'_\xi) \\ &= \bigcup_{\xi \in S} \big(P^* \cap B_\xi^* \cap \Omega_\alpha(\beta)\big) = \bigcup_{\xi \in S} \emptyset = \emptyset , \end{aligned}$$

where $P^* \cap B_\xi^* = \emptyset$ because $P^* \cap B_\xi^* = (P \cap B_\xi)^*$ and $|P \cap B_\xi| = 1$.

Juxtaposing the results given by the preceding lemmas, we obtain the following theorem.

3.6. THEOREM. *Let $\alpha$ be measurable, and let $\alpha \leqslant \beta$. Then the following conditions are equivalent:*

(a) *$\Omega_\alpha(\beta)$ is $\alpha$-compact—i.e., the relation $\alpha \mathcal{S} \beta$ fails;*

(b) *$\Omega_\alpha(\beta)$ is $C^*$-embedded in $\Psi_\alpha(\beta)$; and if $U$ is open in $\Psi_\alpha(\beta)$ and $\Omega_\alpha(\beta)\subset U\subset\Psi_\alpha(\beta)$, then an open subset $V$ of $\Psi_\alpha(\beta)$ exists for which $\Omega_\alpha(\beta)\subset V\subset U$ and $\tau V<\alpha$;*

(c) *$\Omega_\alpha(\beta)$ is $C^*$-embedded in $\Psi_\alpha(\beta)$; and if $U$ is open in $\Psi_\alpha(\beta)$ and $\Omega_\alpha(\beta)\subset U\subset\Psi_\alpha(\beta)$, then an open subset $V$ of $\Psi_\alpha(\beta)$ exists for which $\Omega_\alpha(\beta)\subset V\subset U$ and $\tau V\leqslant\alpha$;*

(d) *$\Omega_\alpha(\beta)$ is $C^*$-embedded in $\Psi_\alpha(\beta)$; and if $U$ is open in $\Psi_\alpha(\beta)$ and $\Omega_\alpha(\beta)\subset U\subset\Psi_\alpha(\beta)$, then a dense open subset $V$ of $\Psi_\alpha(\beta)$ exists for which $V\subset U$ and $\tau V<\alpha$;*

(e) *$\Omega_\alpha(\beta)$ is $C^*$-embedded in $\Psi_\alpha(\beta)$; and if $U$ is open in $\Psi_\alpha(\beta)$ and $\Omega_\alpha(\beta)\subset U\subset\Psi_\alpha(\beta)$, then a dense open subset $V$ of $\Psi_\alpha(\beta)$ exists for which $V\subset U$ and $\tau V\leqslant\alpha$.*

Proof. The implication (a) $\Rightarrow$ (b) is given by 3.2 (a) and 3.4, and (b) $\Rightarrow$ (a) also follows from 3.4. Since $\Omega_\alpha(\beta)$, and hence each of its supersets, is dense in $\Psi_\alpha(\beta)$, the implications (b) $\Rightarrow$ (c) $\Rightarrow$ (e) and (b) $\Rightarrow$ (d) $\Rightarrow$ (e) are formal truisms. That (e) $\Rightarrow$ (b) follows directly from 3.5.

3.7. Corollary. *Let $\alpha$ be measurable, let $\alpha\leqslant\beta$, and suppose that $\Omega_\alpha(\beta)$ is $C^*$-embedded in $\Psi_\alpha(\beta)$. If $\Omega_\alpha(\beta)$ is $\alpha^+$-compact, then $\Omega_\alpha(\beta)$ is $\alpha$-compact.*

Proof. It suffices to show that for each cover $\mathfrak{U}$ of $\Omega_\alpha(\beta)$ by (relatively) open-and-closed subsets there is a subcover $\mathfrak{U}'$ for which $|\mathfrak{U}'|<\alpha$. We may suppose that $|\mathfrak{U}|\leqslant\alpha$. From Lemma 3.2 (b) there is a family $\mathfrak{V}$ of open-and-closed subsets of $\Psi_\alpha(\beta)$ for which $|\mathfrak{V}|\leqslant\alpha$ and $\mathfrak{U}=\{W\cap\Omega_\alpha(\beta)\colon W\in\mathfrak{V}\}$. We set $V=\bigcup\mathfrak{V}$, so that $\tau V\leqslant|\mathfrak{V}|\leqslant\alpha$, and we find by 3.5 an open subset $V'$ of $\Psi_\alpha(\beta)$ for which

$$\Omega_\alpha(\beta)\subset V'\subset V\quad\text{and}\quad\tau V'<\alpha\,.$$

Since $V'$ is a union of fewer than $\alpha$ compact sets, there is a subfamily $\mathfrak{V}'$ of $\mathfrak{V}$ for which

$$V'\subset\bigcup\mathfrak{V}'\subset V\quad\text{and}\quad|\mathfrak{V}'|<\alpha\,.$$

We use $\mathfrak{U}'=\{W\cap\Omega_\alpha(\beta)\colon W\in\mathfrak{V}'\}$.

3.8. Remarks. It is worthwhile to remark on a couple of equivalences called to mind by Theorem 3.6.

The first of these is the equivalence for any measurable cardinal $\alpha$ of the $\alpha$-compactness of the space $\Omega_\alpha(\beta)$ with the $\alpha$-compactness of the space $D(\beta)\cup\Omega_\alpha(\beta)$. (This may be established by remarking first that $\Omega_\alpha(\beta)$ is a closed subset of $D(\beta)\cup\Omega_\alpha(\beta)$, so that the former is $\alpha$-compact whenever the latter is. If $\{W_i\}_{i\in I}$ is a cover of $D(\beta)\cup\Omega_\alpha(\beta)$ by open-and-closed sets and if $J$ is a subset of $I$ for which $|J|<\alpha$ and $\Omega_\alpha(\beta)\subset\bigcup_{i\in J}W_i$, then $\bigcup_{i\in J}W_i$ must contain every point of $D(\beta)$, with fewer than $\alpha$ exceptions. For if $A\subset D(\beta)$ and $|A|=\alpha$ and $A\cap(\bigcup_{i\in J}W_i)=\emptyset$, then $A^*\cap\Omega_\alpha(\beta)$ is a nonvoid open-and-closed subset of $\Omega_\alpha(\beta)$ disjoint from $\bigcup_{i\in J}W_i$. Thus $D(\beta)\cup\Omega_i(\beta)$ is $\alpha$-compact whenever $\Omega_\alpha(\beta)$ is.

The second equivalence is the following observation: In order that $\Omega_\alpha(\beta)$ be $C^*$-embedded in $\Psi_\alpha(\beta)$, it is necessary and sufficient that each two-valued continuous function on $\Omega_\alpha(\beta)$ be continuously extendable to a two-valued continuous function on $\Psi_\alpha(\beta)$. The necessity being clear, we can check this equivalence by showing that if each open-and-closed subset of $\Omega_\alpha(\beta)$ is the intersection with $\Omega_\alpha(\beta)$ of an open-and-closed subset of $\Psi_\alpha(\beta)$, then each bounded continuous real-valued function on $\Omega_\alpha(\beta)$ extends continuously to $\Psi_\alpha(\beta)$. This is precisely the argument given in the proof of Lemma 3.2(a), and we shall not repeat it.

We have been unable to determine whether under the hypotheses of Theorem 3.6 the condition that $\Omega_\alpha(\beta)$ is $C^*$-embedded in $\Psi_\alpha(\beta)$ is itself sufficient to guarantee that $\Omega(\beta)$ is $\alpha$-compact. In the special case $\alpha=\beta$, and assuming that $\alpha^+=2^\alpha$, the desired implication is provided by the following result, proved by one of us in Corollary 6.3 of [8]: Let $\alpha$ be a regular cardinal with $\alpha^+=2^\alpha$, and let $V$ be an open subset of $\Psi_\alpha(\alpha)$. Then $V$ is $C^*$-embedded in $\Psi_\alpha(\alpha)$ iff $V$ contains densely an open subset $V'$ for which $\tau V'<\alpha$. Specifically, then, we have the following result.

3.9. Theorem. *Let $\alpha$ be a measurable cardinal number, and suppose that $\alpha^+=2^\alpha$. Then the following assertions are equivalent:*

(1) *$\Omega_\alpha(\alpha)$ is $\alpha$-compact, i.e. $\alpha\notin C_1^*$;*

(2) *$D(\alpha)\cup\Omega_\alpha(\alpha)$ is $\alpha$-compact;*

(3) *$\Omega_\alpha(\alpha)$ is $C^*$-embedded in $\Psi_\alpha(\alpha)$* — i.e., $\beta(\Omega_\alpha(\alpha))=\Psi_\alpha(\alpha)$;

(4) *each two-valued continuous function on $\Omega_\alpha(\alpha)$ extends continuously to a two-valued continuous function or $\Psi_\alpha(\alpha)$.*

Proof. The equivalences (1)$\Leftrightarrow$(2) and (3)$\Leftrightarrow$(4) have been discussed in the preceding remarks, while the equivalence (1)$\Leftrightarrow$(3) is given by Theorem 3.6 and the result cited from [8].

The Hewitt realcompactification of a topological space $X$, defined and discussed in detail in [2], is that unique realcompact space $\upsilon(X)$ containing $X$ densely with the property that each real-valued continuous function defined on $X$ extends continuously to $\upsilon(X)$. If $\varkappa$ denotes the first uncountable measurable cardinal then $D(\varkappa)$ is not realcompact and $\Omega_\varkappa(\varkappa)$ coincides with the space $\upsilon(D(\varkappa))\setminus D(\varkappa)$. For the special case $\alpha=\varkappa$, then, Theorem 3.9 takes the form given in 3.10 below.

We intend no disrespect to the reader in remarking at this point, in connection with both 3.9 and 3.10, that if a number of conditions are equivalent then either all are true or all are false. We are not able at this time to make a more substantive comment on this matter; in the general context of 3.9, and even in the particular case handled in 3.10, we do not know if it is „all true", or „all false", that the four conditions are. (In the case of 3.10, this is simply the question whether or not the first measurable cardinal $\varkappa$ belongs to the class $\mathfrak{C}_1^*$ defined in the Introduction.)

3.10. COROLLARY. *Suppose that $\varkappa^+ = 2^\varkappa$. Then the following assertions are equivalent:*

(1) *$\upsilon(D(\varkappa))\setminus D(\varkappa)$ is $\varkappa$-compact;*

(2) *$\upsilon(D(\varkappa))$ is $\varkappa$-compact;*

(3) *$\upsilon(D(\varkappa))\setminus D(\varkappa)$ is $C^*$-embedded in $\Psi_\varkappa(\varkappa)$ — i.e., $\beta(\upsilon(D(\varkappa))\setminus D(\varkappa)) = \Psi_\varkappa(\varkappa)$;*

(4) *each two-valued continuous function on $\upsilon(D(\varkappa))\setminus D(\varkappa)$ extends continuously to a two-valued continuous function on $\Psi_\varkappa(\varkappa)$.*

**§ 4. Some additional results.** The question of when $\Omega_a(\beta)$ is $C^*$-embedded in $\Psi_a(\beta)$ is settled by Theorem 3.3 only relative to the failure of the condition $aS\beta$, even in the special case $a = \beta$. In Theorem 4.2 below, we shall give a complete and satisfying answer to the following simpler question: For what points $p$ in $\Psi_a(a)$ is $\Psi_a(a)\setminus\{p\}$ $C^*$-embedded in $\Psi_a(a)$?

4.1. LEMMA. *Let $a$ be a measurable cardinal number for which $a^+ = 2^a$, and let $\{W_\xi\}_{\xi<a}$ be a collection of open subsets of $\Psi_a(a)$ for which $(\bigcap_{\xi<a} W_\xi) \cap \Omega_a(a) \neq \emptyset$. Then*

$$\operatorname{int}_{\Psi_a(a)}(\bigcap_{\xi<a} W_\xi) \neq \emptyset .$$

Proof. We consider the open set $U = \bigcup_{\xi<a} (\Psi_a(a)\setminus W_\xi)$, a union of $a$ open-and-closed subsets of $\Psi_a(a)$. If $\tau(U) < a$, then the nonvoid set $\bigcap_{\xi<a} W_\xi$ has nonvoid interior in $\Psi_a(a)$ by Lemma 2.7. If $\tau(U) = a$ then from Theorem 3.2 of [8] it follows that there is an open subset $V$ of $\Psi_a(a)$, dense in $U$, such that $\tau(V) < a$. There is, then, again by Lemma 2.7, a nonvoid open subset of $\Psi_a(a)$ missing $V$.

4.2. THEOREM. *Let $a$ be a measurable cardinal, and let $p \in \Psi_a(a)$.*

(1) *If $p \notin \Omega_a(a)$, then $\Psi_a(a)\setminus\{p\}$ is $C^*$-embedded in $\Psi_a(a)$;*

(2) *If $a^+ = 2^a$ and $p \in \Omega_a(a)$, then $\Psi_a(a)\setminus\{p\}$ is not $C^*$-embedded in $\Psi_a(a)$.*

Proof. (1) Since $p \notin \Omega_a(a)$, there is for some $\gamma < a$ a collection $\{A_\xi\colon \xi < \gamma\}$ of elements of $p$ for which $\bigcap_{\xi<\gamma} A_\xi \notin p$. Without loss of generality we may assume that $\bigcap_{\xi<\gamma} A_\xi = \emptyset$. Let $V = \bigcup_{\xi<\gamma} ((D(a)\setminus A_\xi)^* \cap \Psi_a(a))$. Then $V$ is a dense open subset of $\Psi_a(a)$ for which $\tau(V) \leqslant \gamma$, so from Theorem 3.1 of [8] it follows that $V$ is $C^*$-embedded in $\Psi_a(a)$. Hence, the intermediate space $\Psi_a(a)\setminus\{p\}$ is $C^*$-embedded in $\Psi_a(a)$.



(2) Assume that $p \in \Omega_a(a)$ and that $\Psi_a(a)\setminus\{p\}$ is $C^*$-embedded in $\Psi_a(a)$. From Corollary 6.3 of [8], there is a family $\{V_\xi\}_{\xi<a}$ of open-and-closed subsets of $\Psi_a(a)$ whose union is a dense subspace of $\Psi_a(a)\setminus\{p\}$. Let $W_\xi = \Psi_a(a)\setminus V_\xi$ for all $\xi < a$; then the family $\{W_\xi\}_{\xi<a}$ satisfies the conditions of Lemma 4.1 above, but it is clear that

$$\operatorname{int}_{\Psi_a(a)}(\bigcap_{\xi<a} W_\xi) = \emptyset .$$

4.3 COROLLARY. *Let $a \notin \mathfrak{C}_1^*$, and assume that $a^+ = 2^a$. If $p \in \Omega_a(a)$, then $\Omega_a(a)\setminus\{p\}$ is not $C^*$-embedded in $\Omega_a(a)$.*

Proof. Assume that $\Omega_a(a)\setminus\{p\}$ is $C^*$-embedded in $\Omega_a(a)$ for some $p \in \Omega_a(a)$. We shall derive a contradiction, by proving that $\Psi_a(a)\setminus\{p\}$ is $C^*$-embedded in $\Psi_a(a)$. Indeed, let $f$ be a bounded, real-valued continuous function on $\Psi_a(a)\setminus\{p\}$. Let $g$ be its restriction to $\Omega_a(a)\setminus\{p\}$. By our assumption, there is a continuous extension $G$ of $g$ to $\Omega_a(a)$. From Theorem 3.5 it follows that there is a continuous extension $F$ of $G$ to $\Psi_a(a)$. From Lemma 2.4, it follows that $F$ is, in fact, an extension of $f$.

We conclude with the computation of the cardinalities of certain of the subsets of $\beta(D(a))$ considered above. It is a well-known result of Hausdorff [4] and Pospíšil [10] that for every infinite cardinal $a$, $|\beta(D(a))| = 2^{2^a}$. The equality $|\Psi_a(a)| = 2^{2^a}$, valid for every infinite cardinal $a$, appears as exercise 12I in [2].

4.4. THEOREM. *Let $a$ be a measurable cardinal. Then*

(a) $$|\beta(D(a))\setminus\Psi_a(a)| = a ;$$

(b) $$|\Psi_a(a)\setminus\Omega_a(a)| = 2^{2^a} \quad \text{for} \quad a > \omega ;$$

(c) $$|\Omega_a(a)| \geqslant 2^a .$$

Proof. (a) We notice that $a \leqslant |\beta(D(a))\setminus\Psi_a(a)| \leqslant \Sigma\{a^\gamma\cdot 2^{2^\gamma}\colon \gamma < a\} = a$.

(b) We express $D(a)$ in the form

$$D(a) = \bigcup_{n\in\omega} D_n ,$$

where the sets $D_n$ are pairwise disjoint sets for which $|D_n| = a$. We set

$$V_n = D_n^* \cap \Psi_a(a) ,$$

so that $|V_n| = 2^{2^a}$. We well-order $V_n$ according to the cardinal number $2^{2^a}$:

$$V_n = \{p_{n,\xi}\colon \xi < 2^{2^a}\} ;$$

having done so we select, for each $\xi$, an accumulation point $q_\xi$ of the sequence $\{p_{n,\xi}\colon n \in \omega\}$. Then $q_\xi \in \Psi_a(a)$ because the latter set is compact, and $q_\xi \neq q_{\xi'}$ whenever $\xi \neq \xi'$ (because the set

$$\{p_{n,\xi}\colon n \in \omega\} \cup \{p_{n,\xi'}\colon n \in \omega\}$$

is $C^*$-embedded in $\Psi_\alpha(\alpha)$). So it remains only to show that always $q_\xi \notin \Omega_\alpha(\alpha)$. But for each $n$ and each $\xi$ we have $D_n \notin q_\xi$, so that $D \setminus D_n \in q_\xi$; yet

$$\bigcup_{n\in\omega} (D\setminus D_n) = \emptyset \notin q_\xi .$$

This shows that $q_\xi \notin \Omega_{\omega_1}(\alpha)$, so surely $q_\xi \notin \Omega_\alpha(\alpha)$.

(c) Note that $\alpha = 2^{\mathfrak{c}}$. It follows from a result of Sierpiński [13] and Tarski (Théorème 7 in [12]) that there is a family $\{A_i\}_{i\in I}$ of subsets of $D(\alpha)$, satisfying the following conditions: $|A_i| = \alpha$ for all $i \in I$; $|A_i \cap A_j| < \alpha$ for all $i, j \in I$, $i \neq j$; $|I| = 2^\alpha$. Let $V_i = A_i^* \cap \Omega_\alpha(\alpha)$ for all $i \in I$. It is clear that $V_i \neq \emptyset$ for all $i \in I$, and that $V_i \cap V_j = \emptyset$ whenever $i \in I$ and $j \in I$ and $i \neq j$. Thus $|\Omega_\alpha(\alpha)| \geqslant 2^\alpha$.

We have been unable to compute precisely the cardinality of $\Omega_\alpha(\alpha)$, even under the condition that $\alpha \notin \mathrm{C}_1^*$. For an arbitrary measurable cardinal $\alpha$, however, we can compute the density character of each of the three spaces considered in Theorem 4.4. (The density character of a space $X$, denoted $\boldsymbol{d}(X)$, is by definition the smallest cardinal number which is the cardinal number of a dense subset of $X$.)

4.5 THEOREM. *Let $\alpha$ be a measurable cardinal. Then*

(a) $$\boldsymbol{d}\big(\beta(D(\alpha))\setminus\Psi_\alpha(\alpha)\big) = \alpha;$$

(b) $$\boldsymbol{d}\big(\Psi_\alpha(\alpha)\setminus\Omega_\alpha(\alpha)\big) = 2^\alpha \quad \text{for} \quad \alpha > \omega;$$

(c) $$\boldsymbol{d}\big(\Omega_\alpha(\alpha)\big) = 2^\alpha.$$

Proof. The inequality $\leqslant$ in (a) is clear, since in fact $|\beta(D(\alpha))\setminus\Psi_\alpha(\alpha)| = \alpha$. The inequality $\geqslant$ follows from the fact that no set of cardinality $\beta$ can be dense in a Hausdorff space whose cardinality exceeds $2^{2^\beta}$, while $2^{2^\beta} < \alpha$ whenever $\beta < \alpha$. The inequalities $\leqslant$ of (b) and (c) can be established by choosing, for each subset $A$ of $D(\alpha)$ such that $|A| = \alpha$, a point $p_A$ in $A^* \cap \Omega_\alpha(\alpha)$ and a point $q_A$ in $A^* \cap \big(\Psi_\alpha(\alpha)\setminus\Omega_\alpha(\alpha)\big)$. The sets $\{p_A\colon |A| = \alpha\}$, $\{q_A\colon |A| = \alpha\}$, each of cardinality $2^\alpha$, are dense in the spaces $\Omega_\alpha(\alpha)$, $\Psi_\alpha(\alpha)\setminus\Omega_\alpha(\alpha)$ respectively.

The family $\{A_i\}_{i\in I}$ described in the proof of Theorem 4.4 (c) has the property that the sets $A_i^* \cap \Psi_\alpha(\alpha)$ are pairwise disjoint nonvoid open subsets of $\Psi_\alpha(\alpha)$. Since each of these sets meets both $\Omega_\alpha(\alpha)$ and $\Psi_\alpha(\alpha)\setminus\Omega_\alpha(\alpha)$, the inequalities $\geqslant$ of (b) and (c) both follow.

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