Atomic mappings on irreducible Hausdorff continua

by

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Introduction. In 1935, Knaster showed [3] that there is a monotone open mapping of a compact, irreducible, metric continuum onto an arc with each point inverse nondegenerate and, also, a mapping from a compact, irreducible, metric continuum onto an arc with each point inverse an arc. In 1949, Moise showed [5] that there is no open mapping of a compact, irreducible, metric continuum onto an arc with each point inverse an arc, and, in 1953, Dyer showed [2] that there is no open mapping of such a continuum onto an arc with each point inverse a decomposable continuum. In 1966, Thomas [6] and the author [4] showed independently that there is no atomic mapping from a compact, irreducible, metric continuum onto an arc such that each point inverse is a nondegenerate, hereditarily decomposable, chainable continuum. In this note we show that if $K$ is a compact metric continuum, there is an atomic mapping from a compact, separable, first-countable, irreducible, Hausdorff continuum onto an arc such that each point inverse is homeomorphic to $K$.

By an atomic mapping we mean a mapping $f$ such that if $C$ is a continuum which is a subset of the domain of $f$, then $f(C)$ is degenerate or $f^{-1}[f(C)] = C$.

In the metric case little seems to be known about the corresponding problem, even for monotone mappings. In particular, we do not know if there is a mapping from a compact, irreducible metric continuum onto an arc with each point inverse a simple closed curve. In the following section we describe a general constructive procedure and, in a later section, specialize it to obtain the desired examples.

A Construction. Let $M$ denote a compact metric continuum and let $T$ denote a mapping from $M$ onto the closed interval $[-1, 1]$. Let $E^1$ denote the set of real numbers and let $S = E^1 \times T^{-1}(0)$. We shall next introduce a topology on $S$, determined by $T$, under which $S$ will be a first countable Hausdorff space. If $x$ is in $E^1$ and $U$ is an open subset of $M - \{T^{-1}(-1) \cup T^{-1}(1)\}$ which intersects $T^{-1}(0)$, let $R(x, U)$ denote the subset of $S$ to which $(t, F)$ belongs if and only if either $t = x$ and $F$ is in $U \cap T^{-1}(0)$, or $P$ is in $T^{-1}(0)$ and $T^{-1}(t - x) \subset U$. The collection of
all such subsets of $S$ form a basis for the topology in $S$ and members of this basis will be called regions. As an example, if $M$ is an arc and $T$ is a mapping of $M$ onto $[-1,1]$ such that $T^{-1}(0)$ is an interior arc of $M$ and $T^{-1}(x)$ is degenerate or $x \neq 0$, then the subspace $[0,1] \times T^{-1}(0)$ of $S$ is homeomorphic to $[0,1] \times [0,1]$ with the topology induced by the lexicographic order. Theorems 1, 2, and 3 below yield the well known fact that this space is connected, first countable, compact, and Hausdorff. Returning now to the general case we introduce some notation and state two lemmas.

**Notation.** If $a$ and $b$ are numbers and $a < b$, then $S(a) = \{a \times x \mid x \in T^{-1}(0)\}$, $S(a, b) = \{a, b \times x \mid x \in T^{-1}(0)\}$, and $S[a, b] = [a, b] \times T^{-1}(0)$.

In the remainder of this paper such sets will be considered as subsets of the space $S$ described above.

**Lemma 1.** If $(t, P)$ is a point of $S$ and $s > 0$, there is a region $R(t, U)$ such that $(t, P) \in R(t, U)$ and $R(t, U) \subset S(-t - s, t + s)$.

**Lemma 2.** If $(t, P)$ is a point of a region $R(x, U)$ and $x \neq t$, then there is an $s > 0$ such that $S(t - s, t + s) \subset R(x, U)$.

**Proof.** Suppose $x = t$ and $(t, P)$ is a point of $R(x, U)$. Then $T^{-1}(t - s) \subset U$, and the set of all numbers in $[-1,1]$ whose preimages under $T$ are contained in $U$ is an open subset of $[-1,1]$ and $t - s$ is neither $-1$ nor $1$, so there is an $s > 0$ such that $T^{-1}(t - s) \subset U$. It follows from the definition of $R(x, U)$ that $S(t - s, t + s) \subset R(x, U)$.

**Theorem 1.** $S$ is a first countable Hausdorff space.

**Proof.** We first show that if the point $(t, P)$ is in each of the regions $R(x, U)$ and $R(y, V)$, then there is a region $R(z, W)$ containing $(a, P)$ and lying in $R(x, U) \cap R(y, V)$. If $x = y$, then $z = x$, by Lemma 2. If $x \neq y$, then there is an $s > 0$ such that $S(x - s, x + s) \subset R(z, W)$ and $R(z, W) \subset R(x, U) \cap R(y, V)$. By Lemma 1, there is a region $R(a, W)$ containing $(a, P)$ and lying in $S(x - s, x + s)$ and $S(y - s, y + s)$ and $R(a, W) \subset R(z, W)$ and $R(z, W) \subset R(x, U) \cap R(y, V)$. The other cases follow by similar arguments. Let $(a, P)$ and $(b, Q)$ denote two points of $S$ and suppose $a \neq b$. Let $\epsilon = \min\{|a - b|, 1/2\}$. There are, by Lemma 1, regions $R_1$ and $R_2$ containing $(a, P)$ and $(b, Q)$ respectively such that $R_1 \subset S(-a - \epsilon, a + \epsilon)$ and $R_2 \subset S(-b - \epsilon, b + \epsilon)$ and then $R_1 \cap R_2 = 0$. If $a = b$, then $P \neq Q$ so there are mutually disjoint open subsets $U$ and $V$ of $M$ such that $T^{-1}(t - s) \subset U$ and $T^{-1}(t + s) \subset V$, respectively. Then $R(a, U) \cap R(b, V) = 0$. To show that $S$ is first countable, let $(a, P)$ denote a point of $S$ and let $(U_{n})$ denote a countable base in $M - T^{-1}(t - s) \cup T^{-1}(t + s)$ containing $P$ and $Q$, respectively. Then $R(a, U) \cap R(b, V) = 0$. To show that $S$ is first countable, let $(a, P)$ denote a point of $S$ and let $(U_{n})$ denote a countable base in $M - T^{-1}(t - s) \cup T^{-1}(t + s)$ containing $P$ and $Q$, respectively. Then $R(a, U) \cap R(b, V) = 0$. To show that $S$ is first countable, let $(a, P)$ denote a point of $S$ and let $(U_{n})$ denote a countable base in $M - T^{-1}(t - s) \cup T^{-1}(t + s)$ containing $P$ and $Q$, respectively. Then $R(a, U) \cap R(b, V) = 0$. To show that $S$ is first countable, let $(a, P)$ denote a point of $S$ and let $(U_{n})$ denote a countable base in $M - T^{-1}(t - s) \cup T^{-1}(t + s)$ containing $P$ and $Q$, respectively. Then $R(a, U) \cap R(b, V) = 0$.
and (2) $T^{-1}(b)$ contains a point within a distance $\epsilon/2$ of $P$. It follows that $T^{-1}(b) \subset U$ and that $S(t+h) \subset K(t, U)$.

**Theorem 3.** If, in addition to the hypothesis of Theorem 2, $T^{-1}(0)$ is connected, then $S(0,1)$ is a connected subset of $S$.

Proof. Suppose $S(0,1)$ is the sum of two mutually separated sets $A$ and $B$. There is a sequence $(a_i)$ of distinct numbers converging to a number $a$ such that $S(a)$ is not a subset of $A$ or $B$ and, for each $i > 0$, $S(a_i)$ is a subset of the other. But $T^{-1}(0)$ contains a limit point of $P$ of $\bigcup_{i=1}^{\infty} T^{-1}(a_i-x)$ and, by Lemma 3, $(a_i, P)$ is a limit point of $\bigcup_{i=1}^{\infty} S(a_i)$.

**Theorem 4.** If, in addition to the hypothesis of Theorem 3, $T^{-1}(0)$ is a subset of the closure $(M)$ of $M - T^{-1}(0)$, then the closure $(M)$ of $S(0,1)$ is a separable continuum, irreducible from $S(0)$ to $S(1)$.

Proof. We first note, with the aid of Lemma 3, that if $R(x, U)$ is a region containing a point of $S(0,1)$, there is a number $r$ in $(0,1) - \{x\}$ such that $R(r) \subset B(x, U)$. And, by Lemma 2, there is an open interval $(a, b)$ in $(0,1)$ such that $S(a,b) \subset B(x, U)$. It follows that if $P$ is in $T^{-1}(0)$ and $C$ is the set of all points $(x, P)$ of $S(0,1)$ such that $C$ is a rational number, then $C$ is a countable dense subset of the closure of $S(0,1)$. It also follows that each region containing a point of $S(0,1)$ but no point of $S(0) \cup S(1)$ separates $S(0,1)$ so that the closure of $S(0,1)$ is an irreducible continuum from $S(0)$ to $S(1)$.

**Mappings on $S(0,1)$ in $S(0)$**. In this section we let $\pi$ denote the natural projection of $S(0,1)$ onto $[0,1]$. That is, $f(x, P)$ is in $S(0,1), \pi(x, P) = x$. It follows from Lemma 1 that $\pi$ is continuous since $T$ is. If $T^{-1}(0)$ is connected, then $\pi$ is monotone. The following theorem gives a condition under which $\pi$ is atomic.

**Theorem 5.** If, in addition to the hypothesis of Theorem 3, each point of $T^{-1}(0)$ is a subset of both $M - T^{-1}([-1,0])$ and $M - T^{-1}(0,1]$, then $\pi$ is atomic, and the closure $(M)$ of $S(0,1)$ is $S(0,1)$.

Proof. Applying Lemma 3, we note that if $f(T)$ is in $S(0,1)$, then $(x, P)$ is a limit point of each component of $S(0,1) - S(x)$. Thus if $0 < \alpha < b < 1$, then $S(a,b)$ is the closure of $S(a,b)$. Furthermore, each region containing a point of $S(a,b)$ but no point of $S(a,b)$ separates $S(a,b)$ and $S(a,b)$ is irreducible from $S(a,b)$. Now if $C$ is a continuum of $S(0,1)$, then there is an $x$ such that $C \subset S(x)$ and $C \cap \{\alpha\} = \emptyset$ or there is an interval $[a,b]$ such that $C = S(a,b)$ so that $\pi(C) = \emptyset$.

**Application of the construction.** Our goal in this section is to apply the construction described above to show:

**Theorem 6.** If $K$ is a compact metric continuum, there is a compact, separable, first countable continuum $M$ in $K$, and an atomic mapping $\pi$ of $M$ onto $[0,1]$ such that (1) if $x$ is in $[0,1]$, $\pi^{-1}(x)$ is homeomorphic to $K$, and (2) $M$ is an irreducible continuum from $\pi^{-1}(0)$ to $\pi^{-1}(1)$.

Proof. It is shown in [1] that if $K$ is a compact metric continuum, and $X$ is a locally compact, non-compact metric space, there is a compact metric space $M$ containing a dense subset $X'$ homeomorphic to $X$ and such that $M - X'$ is homeomorphic to $K$. Furthermore, the compactification $M$ obtained in [1] has the additional property that if $C$ is a subcontinuum of $M$ which intersects both $X'$ and $M - X'$, then $C$ contains $M - X'$. Now let $K$ denote a compact metric continuum, let $X = [-1,1] - \{0\}$, and let $M$ denote such a compactification of $X$. Let $T$ denote a mapping of $M$ onto $[-1,1]$ such that $T^{-1}(0) = M - X'$ and such that if $x$ is in $[-1,1] - \{0\}$ then $T^{-1}(x)$ is degenerate. $T$ satisfies the conditions of Theorem 4 and 5 above so that $S(0,1)$ is the closure of $S(0,1)$ and is a continuum of the type desired and the natural projection of $S(0,1)$ onto $[0,1]$ is atomic.

**References**


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