# Atomic mappings on irreducible Hausdorff continua

by

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**Introduction.** In 1935, Knaster showed [3] that there is a monotone open mapping of a compact, irreducible, metric continuum onto an arc with each point inverse nondegenerate and, also, a mapping from a compact, irreducible, metric continuum onto an arc with each point inverse an arc. In 1949, Moise showed [5] that there is no open mapping of a compact, irreducible, metric continuum onto an arc with each point inverse an arc, and, in 1953, Dyer showed [2] that there is no open mapping of such a continuum onto an arc with each point inverse a decomposable continuum. In 1966, Thomas [6] and the author [4] showed independently that there is no atomic mapping from a compact, irreducible, metric continuum onto an arc such that each point inverse is a nondegenerate, hereditarily decomposable, chainable continuum. In this note we show that if $K$ is a compact metric continuum, there is an atomic mapping from a compact, separable, first-countable, irreducible, Hausdorff continuum onto an arc such that each point inverse is homeomorphic to $K$. By an atomic mapping we mean a mapping $f$ such that if $C$ is a continuum which is a subset of the domain of $f$, then $f(C)$ is degenerate or $f^{-1}[f(C)]=C$. In the metric case little seems to be known about the corresponding problem, even for monotone mappings. In particular, we do not know if there is a mapping from a compact, irreducible metric continuum onto an arc with each point inverse a simple closed curve. In the following section we describe a general constructive procedure and, in a later section, specialize it to obtain the desired examples.

**A Construction.** Let $M$ denote a compact metric continuum and let $T$ denote a mapping from $M$ onto the closed interval $[-1,1]$. Let $E^1$ denote the set of real numbers and let $S=E^1\times T^{-1}(0)$. We shall next introduce a topology on $S$, determined by $T$, under which $S$ will be a first countable Hausdorff space. If $x$ is in $E^1$ and $U$ is an open subset of $M-[T^{-1}(-1)\cup T^{-1}(1)]$ which intersects $T^{-1}(0)$, let $R(x,U)$ denote the subset of $S$ to which $(t,P)$ belongs if and only if either $t=x$ and $P$ is in $U\cap T^{-1}(0)$, or $P$ is in $T^{-1}(0)$ and $T^{-1}(t-x)\subset U$. The collection of all such subsets of $S$ form a basis for the topology in $S$ and members of this basis will be called regions. As an example, if $M$ is an arc and $T$ is a mapping of $M$ onto $[-1,1]$ such that $T^{-1}(0)$ is an interior arc of $M$ and $T^{-1}(x)$ is degenerate if $x \neq 0$, then the subspace $[0,1]\times T^{-1}(0)$ of $S$ is homeomorphic to $[0,1]\times[0,1]$ with the topology induced by the lexicographic order. Theorems 1, 2, and 3 below yield the well known fact that this space is connected, first countable, compact and Hausdorff. Returning now to the general case we introduce some notation and state two lemmas.

Notation. If $a$ and $b$ are numbers and $a<b$, then $S(a)=\{a\}\times\times T^{-1}(0)$, $S(a,b)=(a,b)\times T^{-1}(0)$, and $S[a,b]=[a,b]\times T^{-1}(0)$.

In the remainder of this paper such sets will be considered as subsets of the space $S$ described above.

LEMMA 1. *If $(t,P)$ is a point of $S$ and $\varepsilon>0$, there is a region $R(t,U)$ such that $(t,P)$ is in $R(t,U)$ and $R(t,U)\subset S(t-\varepsilon,t+\varepsilon)$.*

LEMMA 2. *If $(t,P)$ is a point of a region $R(x,U)$ and $x\neq t$, then there is an $\varepsilon>0$ such that $S(t-\varepsilon,t+\varepsilon)\subset R(x,U)$.*

Proof. Suppose $t\neq x$ and $(t,P)$ is in $R(x,U)$. Then $T^{-1}(t-x)\subset U$, and the set of all numbers in $[-1,1]$ whose preimages under $T$ are contained in $U$ is an open subset of $[-1,1]$ and $t-x$ is neither 1 nor $-1$, so there is an $\varepsilon>0$ such that $T^{-1}(t-x-\varepsilon,t-x+\varepsilon)\subset U$. It follows from the definition of $R(x,U)$ that $S(t-\varepsilon,t+\varepsilon)\subset R(x,U)$.

THEOREM 1. *$S$ is a first countable Hausdorff space.*

Proof. We first show that if the point $(a,P)$ is in each of the regions $R(x,U)$ and $R(y,V)$, then there is a region $R(z,W)$ containing $(a,P)$ and lying in $R(x,U)\cap R(y,V)$. If $x=y\neq a$, applying Lemma 2, there is an $\varepsilon>0$ such that $S(a-\varepsilon,a+\varepsilon)\subset R(x,U)\cap R(x,V)$. By Lemma 1, there is a region $R(a,W)$ containing $(a,P)$ and lying in $S(a-\varepsilon,a+\varepsilon)$. If $x=y=a$, $(a,P)$ is in $R(a,U\cap V)$ and $R(a,U\cap V)\subset R(a,U)\cap\cap R(a,V)$. The other cases follow by similar arguments. Now let $(a,P)$ and $(b,Q)$ denote two points of $S$ and suppose $a\neq b$. Let $\varepsilon=(1/2)|b-a|$. There are, by Lemma 1, regions $R_1$ and $R_2$ containing $(a,P)$ and $(b,Q)$ respectively such that $R_1\subset S(a-\varepsilon,a+\varepsilon)$ and $R_2\subset S(b-\varepsilon,b+\varepsilon)$ and thus $R_1\cap R_2=0$. If $a=b$, then $P\neq Q$ so there are mutually disjoint open subsets $U$ and $V$ of $M-[T^{-1}(-1)\cup T^{-1}(1)]$ containing $P$ and $Q$, respectively. Then $R(a,U)\cap R(a,V)=0$. To show that $S$ is first countable, let $(a,P)$ denote a point of $S$ and let $\{U_i\}$ denote a countable base in $M-[T^{-1}(-1)\cup T^{-1}(1)]$ at the point $P$. Suppose $(a,P)$ is in $R(x,V)$. If $a\neq x$, then there is an $\varepsilon>0$ such that $S(a-\varepsilon,a+\varepsilon)\subset R(x,V)$, and an $n>0$ such that $U_n\cap T^{-1}([-1,-\varepsilon]\cup[\varepsilon,1])=0$. It follows that $R(a,U_n)\subset R(x,V)$. If $a=x$, then $P$ is in $V$ so there is an $n>0$ such that $U_n\subset V$. Then $R(a,U_n)\subset R(a,V)$. We have then that $\{R(a,U_i)\}$ is a countable base at $(a,P)$ in $S$.

That $S[0,1]$ need not be regular, and thus not compact, may be seen with the aid of the following example. Let $M$ denote the subset of $E^2$ to which the point $P$ belongs if and only if either (1) $P$ is in the horizontal interval $[-1,1]\times\{1/2\}$, (2) $P$ is in the vertical interval $\{0\}\times[0,1]$, or (3) $P$ is, for some positive integer $i$, in either the interval $\{1/i\}\times[0,1]$ or the interval $\{-1/i\}\times[0,1]$. For each point $P$ in $M$ let $T(P)$ denote the abscissa of $P$. The following theorem gives a condition under which $S[0,1]$ is compact.

THEOREM 2. *If for each $\varepsilon>0$, there is a $\delta>0$ such that if $|t|<\delta$ and $t\neq 0$, then $\operatorname{diam}[T^{-1}(t)]<\varepsilon$, then $S[0,1]$ is a compact subset of $S$.*

Proof. Let $G$ denote a collection of regions covering $S[0,1]$, and suppose $x$ is in $[0,1]$. We first show that there is $\varepsilon_x>0$ and a finite subset of $G$ covering $S(x-\varepsilon_x,x+\varepsilon_x)$. If there is a point $P$ in $T^{-1}(0)$, a number $t\neq x$, and a region $R(t,U)$ in $G$ containing $(x,P)$, then, by Lemma 2, there is an $\varepsilon_x>0$ such that $S(x-\varepsilon_x,x+\varepsilon_x)\subset R(t,U)$. Otherwise, for each point $P$ in $T^{-1}(0)$, there is an open subset $U_P$ of $M$ such that $R(x,U_P)$ is a region in $G$ containing $(x,P)$. Since $T^{-1}(0)$ is compact, there is a finite subset $H$ of $T^{-1}(0)$ such that $T^{-1}(0)\subset D=\bigcup U_P$ ($P$ in $H$). There is a $\delta>0$ such that $T^{-1}[-\delta,\delta]\subset D$, and $T^{-1}[-\delta,\delta]$ is compact, so there is an $\varepsilon'>0$ such that each open subset of $M$ of diameter less than $\varepsilon'$ which intersects $T^{-1}[-\delta,\delta]$ is a subset of $U_P$ for some point $P$ in $H$. By hypothesis, there is a $\delta'>0$ such that if $|t|<\delta'$ and $t\neq 0$, then $\operatorname{diam}[T^{-1}(t)]<\varepsilon'$. Let $\varepsilon_x=\min\{\delta,\delta'\}$ and let $t$ denote a number such that $0<|t-x|<\varepsilon_x$. Then $|t-x|<\delta'$ and $t\neq x$ so $\operatorname{diam}[T^{-1}(t-x)]<\varepsilon'$. There is an open subset $V$ of $M$ of diameter less than $\varepsilon'$ containing $T^{-1}(t-x)$. $V\cap T^{-1}[-\delta,\delta]\neq 0$ and so there is a point $P$ in $H$ such that $V\subset U_P$. So $T^{-1}(t-x)\subset U_P$ and $S(t)\subset R(x,U_P)$. It follows that $S(x-\varepsilon_x,x+\varepsilon_x)\subset D$. So we have that if $x$ is in $[0,1]$, there is an $\varepsilon_x>0$ and a finite subset of $G$ covering $S(x-\varepsilon_x,x+\varepsilon_x)$. The fact that $[0,1]$ is compact now gives that some finite subset of $G$ covers $S[0,1]$.

LEMMA 3. *Under the hypothesis of Theorem 2, if $H\subset[-1,1]-\{0\}$, $P$ is a point of $T^{-1}(0)$ which is a limit point of $T^{-1}(H)$, and $t$ is a number, then if $R$ is a region containing $(t,P)$, there is a number $h$ in $H$ such that $S(t+h)\subset R$.*

Proof. Let $R(x,U)$ denote a region containing $(t,P)$. If $t\neq x$, by Lemma 2, there is an $\varepsilon>0$ such that $S(t-\varepsilon,t+\varepsilon)\subset R(x,U)$ and there is a number $h$ in $H$ such that $0<|h|<\varepsilon$ so $S(t+h)\subset R(x,U)$. If $t=x$, then $P$ is in $U$ and there is an $\varepsilon>0$ such that every point of $M$ within a distance $\varepsilon$ of $P$ is in $V$. There is a $\delta>0$ such that if $0<|s|<\delta$, then $\operatorname{diam}[T^{-1}(s)]<\varepsilon/2$, and there is a number $h$ in $H$ such that (1) $0<|h|<\delta$, and (2) $T^{-1}(h)$ contains a point within a distance $\varepsilon/2$ of $P$. It follows that $T^{-1}(h)\subset U$ and that $S(t+h)\subset R(t, U)$.

THEOREM 3. *If, in addition to the hypothesis of Theorem* 2, $T^{-1}(0)$ *is connected, then* $S(0,1)$ *is a connected subset of* $S$.

Proof. Suppose $S(0,1)$ is the sum of two mutually separated sets $A$ and $B$. There is a sequence $\{x_i\}$ of distinct numbers converging to a number $x$ such that $S(x)$ is a subset of one of $A$ or $B$ and, for each $i>0$, $S(x_i)$ is a subset of the other. But $T^{-1}(0)$ contains a limit point $P$ of $\bigcup_{i=1}^{\infty} T^{-1}(x_i-x)$ and, by Lemma 3, $(x, P)$ is a limit point of $\bigcup_{i=1}^{\infty} S(x_i)$.

THEOREM 4. *If, in addition to the hypothesis of Theorem* 3, $T^{-1}(0)$ *is a subset of the closure (in* $M$*) of* $M-T^{-1}(0)$, *then the closure (in* $S$*) of* $S(0,1)$ *is a separable continuum, irreducible from* $S(0)$ *to* $S(1)$.

Proof. We first note, with the aid of Lemma 3, that if $R(x, U)$ is a region containing a point of $S(0,1)$, there is a number $t$ in $(0,1)-\{x\}$ such that $S(t)\subset R(x, U)$. And, by Lemma 2, there is an open interval $(a, b)$ in $(0,1)$ such that $S(a,b)\subset R(x, U)$. It follows that if $P$ is in $T^{-1}(0)$ and $C$ is the set of all points $(x, P)$ of $S(0,1)$ such that $x$ is a rational number, then $C$ is a countable dense subset of the closure of $S(0,1)$. It also follows that each region containing a point of $S(0,1)$ but no point of $S(0)\cup S(1)$ separates $S(0,1)$ so that the closure of $S(0,1)$ is an irreducible continuum from $S(0)$ to $S(1)$.

**Mappings on** $S[0,1]$. In this section we let $\pi$ denote the natural projection of $S[0,1]$ onto $[0,1]$. That is, if $(x, P)$ is in $S[0,1]$, $\pi(x, P)=x$. It follows from Lemma 1 that $\pi$ is continuous since $T$ is. If $T^{-1}(0)$ is connected, then $\pi$ is monotone. The following theorem gives a condition under which $\pi$ is atomic.

THEOREM 5. *If in addition to the hypothesis of Theorem* 3, *each point of* $T^{-1}(0)$ *is a limit point of both* $M-T^{-1}[-1,0]$ *and* $M-T^{-1}[0,1]$, *then* $\pi$ *is atomic, and the closure (in* $S$*) of* $S(0,1)$ *is* $S[0,1]$.

Proof. Applying Lemma 3, we note that if $(x, P)$ is in $S[0,1]$, then $(x, P)$ is a limit point of each component of $S[0,1]-S(x)$. Thus if $0\leqslant a<b\leqslant 1$, then $S[a,b]$ is the closure of $S(a,b)$. Furthermore, each region containing a point of $S[a,b]$ but no point of $S(a)\cup S(b)$ separates $S[a,b]$ and so $S[a,b]$ is irreducible from $S(a)$ to $S(b)$. Now if $C$ is a subcontinuum of $S[0,1]$, then there is an $x$ such that $C\subset S(x)$ and $\pi(C)$ is degenerate or there is an interval $[a,b]$ such that $C=S[a,b]$ so that $\pi^{-1}[\pi(C)]=C$.

**Application of the construction.** Our goal in this section is to apply the construction described above to show:

THEOREM 6. *If* $K$ *is a compact metric continuum, there is a compact, separable, first countable continuum* $H$ *in a Hausdorff space* $S$ *and an atomic mapping* $\pi$ *of* $H$ *onto* $[0,1]$ *such that* (1) *if* $x$ *is in* $[0,1]$, $\pi^{-1}(x)$ *is homeomorphic to* $K$, *and* (2) $H$ *is an irreducible continuum from* $\pi^{-1}(0)$ *to* $\pi^{-1}(1)$.



Proof. It is shown in [1] that if $K$ is a compact metric continuum, and $X$ is a locally compact, non-compact metric space, there is a compact metric space $M$ containing a dense subset $X'$ homeomorphic to $X$ and such that $M-X'$ is homeomorphic to $K$. Furthermore, the compactification $M$ obtained in [1] has the additional property that if $C$ is a subcontinuum of $M$ which intersects both $X'$ and $M-X'$, then $C$ contains $M-X'$. Now let $K$ denote a compact metric continuum, let $X=[-1,1]-\{0\}$, and let $M$ denote such a compactification of $X$. Let $T$ denote a mapping of $M$ onto $[-1,1]$ such that $T^{-1}(0)=M-X'$ and such that if $x$ is in $[-1,1]-\{0\}$ then $T^{-1}(x)$ is degenerate. $T$ then satisfies the conditions of Theorems 4 and 5 above so that $S[0,1]$ is the closure of $S(0,1)$ and is a continuum of the type desired and the natural projection of $S[0,1]$ onto $[0,1]$ is atomic.

### References


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*Reçu par la Rédaction le 5. 2. 1969*