On the singularity of Mazurkiewicz in absolute neighborhood retracts

by

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1. Introduction. According to Borsuk [6], a compact metric space which cannot be expressed as a finite or countable union of compact absolute retracts of arbitrarily small diameter has the singularity of Mazurkiewicz. In [6], Borsuk raises the following questions:

1. Suppose \(X\) and \(Y\) are compact metric absolute neighborhood retracts. If \(X\) has the singularity of Mazurkiewicz, then does \(X\times Y\) also have the singularity of Mazurkiewicz?

2. If a polyhedron is represented as a cartesian product, is every factor free from the singularity of Mazurkiewicz?

The purpose of this paper is to give negative solutions to both of these questions. Our solution consists of the following: We give an example of an upper semicontinuous decomposition \(G\) of the 3-sphere \(S^3\) into a null sequence of arcs and points such that if \(X\) is the associated decomposition space, \(X\) has the singularity of Mazurkiewicz. By results of [7] or [9], \(X\times S^{3}\) is homeomorphic to \(S^{3}\times S^{3}\). By a theorem of Borsuk’s [6], \(X\) is a compact absolute neighborhood retract. Hence \(X\) is a compact metric absolute neighborhood retract with the singularity of Mazurkiewicz. \(X\times S^{3}\) is a polyhedron, and no polyhedron has the singularity of Mazurkiewicz [6]. Further, \(X\) is a factor of the polyhedron \(S^{3}\times S^{3}\). Indeed, the triangulable manifold \(S^{3}\times S^{3}\) can be factored into a product of compact absolute neighborhood retracts, one of which has the singularity of Mazurkiewicz.

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In this paper, by “retract” we shall always understand a retract of compact metric space. We use the abbreviations “AR” and “ANR” for “absolute retract” and “absolute neighborhood retract”, respectively.

If \(M\) is a manifold with boundary, then \(\text{Bd } M\) and \(\text{Int } M\) denote the boundary and interior, respectively, of \(M\).
2. Antoine's necklaces. In the construction of the decomposition, we use sets similar to the standard Antoine's necklaces in $S^3$, and which we shall also call "Antoine's necklaces". In Section 3, for each positive integer $r$, we shall construct such a set. In this section, we describe the construction, notation for the construction, and certain auxiliary sets.

Suppose $r$ is a positive integer (fixed in this section). Suppose $\Sigma_r$ is a polyhedral solid torus in $S^3$, and suppose that $(T_{i1}, T_{i2}, ..., T_{im_i})$ is a chain of linked polyhedral unknotted solid tori in $\text{Int} \Sigma_r$ circling $\Sigma_r$ exactly once; see Figure 1. We suppose that if $i = 1, 2, ..., m$, $T_{ri}$ has diameter less than one. If $i = 1, 2, ..., m$, let $(T_{ri}, T_{r_1i}, ..., T_{r_{im_i}})$ be a chain of linked polyhedral unknotted solid tori in $\text{Int} T_{ri}$ circling $T_{ri}$ exactly twice; see Figure 2. We suppose that if $j = 1, 2, ..., m$, $T_{r_1j}$ has diameter less than $\frac{1}{2}$, and then let $(T_{r_1j}, T_{r_2j}, ..., T_{r_{im_i}j})$ be a chain of linked polyhedral unknotted solid tori in $\text{Int} T_{r_1j}$ of diameter less than $\frac{1}{2}$, circling $T_{r_1j}$ exactly once. Let this process be continued, with subsequent chains circling exactly once, and let $M_r, M_{r_1}, M_{r_2}, ...$ denote $\bigcup_{i=1}^{m} T_{ri}, \bigcup_{i=1}^{m} T_{r_1i}, \bigcup_{i=1}^{m} T_{r_2i}, ..., \bigcup_{i=1}^{m} T_{r_{im_i}i}, ...$, respectively.

Let $N_r, N_{r_1}, N_{r_2}, ...$ denote $\bigcap_{i=1}^{m} M_i; N_r$ is an Antoine's necklace of type A circling $\Sigma_r$. Note that $N_r \subset \text{Int} \Sigma_r$.

In the construction of $N_r, T_{ri}, T_{r_1j}, \ldots$, and $T_{r_{im_i}j}$ are the solid tori of the first stage of the construction of $N_r$, the solid tori $T_{r_1j}$, where $1 \leq i \leq m$, and $1 \leq j \leq m$, are the solid tori of the second stage of the construction of $N_r$, and so on.

Fig. 1

Fig. 2

If $n$ is a positive integer, then $n$ is a stage $n$ index in the construction of $N_r$, if and only if there exist integers $i_1, i_2, ..., i_n$, and $i_n$ such that $1 \leq i_1 \leq m$, $1 \leq i_2 \leq m_{i_1}$, ..., and $1 \leq i_n \leq m_{i_{n-1}}$. The statement that $\alpha$ is an index (in the construction of $N_r$) means that for some positive integer $n$, $\alpha$ is a stage $n$ index.

It is easy to see that if $x$ is any point of $\text{Int} \Sigma_r$, we may construct $N_r$ so that $x \in N_r$.

Now we shall describe certain arcs associated with $N_r$. Suppose $i = 1, 2, ..., m$. Consider the first stage torus $T_{ri}$ and the second stage tori $T_{r_1j}, T_{r_2j}, ..., T_{r_{im_i}j}$ lying in $T_{ri}$. It is well known that if $j = 1, 2, ..., m$, there is an arc $a_{ri}$ lying in $\text{Int} T_{ri}$ and containing $N_r \cap T_{r_1j}$.

We shall construct arcs $b_{ri}, b_{r_1j}, ..., b_{r_{im_i}j}$ so that $\bigcup_{i=1}^{m} a_{ri}$ is an arc $A_{ri}$ with certain properties. We regard $T_{ri}$ as a copy...
of $D^2 \times S^1$ where $D^2$ is a disc and $S^1$ is a circle. The copies of $D^2 \times \{0\}$, where $i \leq j$, will be called cross-sections of $T_i$. We give $S^1$ an orientation, "clockwise", and use the induced orientation on the family of all cross sections of $T_i$. We assume that $a_{m_0}, a_{m_1}, \ldots, a_{m_{m_0}}$ are constructed so that if $1 \leq j \leq m_{m_1}$, we may label the endpoints of $a_{m_1}$ by $x_i$ and $y_i$ in such a way that if we start at $x_i$ and go clockwise through the cross sections of $T_i$, these points occur in the order $x_i, y_i, x_i, \ldots, y_i, x_i$, and $y_i, x_i, \ldots, y_i$, respectively. Then $1 \leq j \leq m_{m_0}$, $b_{m_0}$ is to be an arc in Int $T_{m_0}$ from $y_i$ to $x_i$, intersecting precisely those cross-sections of $T_i$ that are encountered in going from $y_i$ to $x_i$ in the clockwise direction.

We suppose the construction done so that all of the cross sections intersecting $b_{m_0}$ is a 3-cell. Further, it is to be true that $(\bigcup_{j=1}^{m_0} b_{j})$ is an arc $A_{m_0}$. Then $N_i \cap T_{m_0} \subseteq A_{m_0}$.

3. Construction of the decomposition. Let $\Sigma_0$ be a polyhedral solid torus in $S^3$. Let $(x_1, x_2, \ldots)$ be a countable dense subset of Int $\Sigma_0$.

Let $J_i$ be a polygonal simple closed curve in Int $\Sigma_0$, circling $\Sigma_0$ exactly once, and containing $x_i$. Let $\Sigma_i$ be a polyhedral tubular neighborhood of $J_i$ lying in Int $\Sigma_0$.

Let $N_i$ be an Antoine's necklace of type $A$ circling $\Sigma_0$ (and hence lying in Int $\Sigma_0$ and circular $\Sigma_0$) such that $x_i \in N_i$ and $J_i$ of the first stage solid tori used in describing $N_i$ has diameter at most 1.

If $i = 1, 2, \ldots, m_{m_0}$, there are no $A_{m_0}$ lying in Int $T_{m_0}$, containing $N_i \cap T_{m_0}$ and constructed as described in Section 2. The area $A_{m_0}, A_{m_1}, \ldots, A_{m_{m_0}}$ are mutually disjoint, and each has diameter less than 1. Let $A_i$ denote $\bigcup_{j=i}^{m} A_{j}$. Let $\Sigma_i$ be a polygonal simple closed curve in Int $\Sigma_0$, circling $\Sigma_0$ exactly once, and containing $x_i$. Let $\Sigma_i$ be a polyhedral tubular neighborhood of $J_i$ lying in Int $\Sigma_0$ and disjoint from $A_i$. Let $N_i$ be an Antoine's necklace of type $A$ circling $\Sigma_0$ such that (1) $x_i \in N_i$ and (2) each of the first stage solid tori used in describing $N_i$ has diameter at most 1.

If $i = 1, 2, \ldots, m_{m_0}$, there are no $A_{m_0}$ lying in Int $T_{m_0}$, containing $N_i \cap T_{m_0}$ and constructed as described in Section 2. The area $A_{m_0}, A_{m_1}, \ldots, A_{m_{m_0}}$ are mutually disjoint, and each has diameter less than 1. Let $A_i$ denote $\bigcup_{j=i}^{m} A_{j}$. Note that if $i = 1, 2, \ldots, m_{m_0}$ and $j = 1, 2, \ldots, m_{m_0}$, $A_i$ and $A_j$ are disjoint.

Let this process be continued. There results a sequence $N_1, N_2, N_3, \ldots$ of Antoine's necklaces of type $A$ in Int $\Sigma_0$, each circling $\Sigma_0$, and a sequence $A_{m_1}, A_{m_2}, \ldots, A_{m_{m_0}}, A_{m_{m_0+1}}, \ldots$ of mutually disjoint arcs in Int $\Sigma_0$, such that for each positive integer $n$, the following hold:

(1) $N_n \subseteq A_{m_0} \cup A_{m_0+1} \cup \ldots \cup A_{m_{m_0}}$.

(2) $x_n \in \bigcup_{i=1}^{m_0} (A_i \cap A_{i+1} \cup \ldots \cup A_{m_{m_0}})$.

(3) If $j = 1, 2, \ldots, m_{m_0}$, then $(\text{diam}(A_j)) < 1/2^n$.

Let $a$ denote the collection $(A_1, A_2, \ldots, A_{m_1}, A_{m_2}, \ldots, A_{m_{m_0}}, \ldots)$. Then $a$ is a null collection, i.e., for each positive number $\epsilon$, at most finitely many sets of a have diameters greater than $\epsilon$.

Let $G$ denote the collection consisting of the arcs of the family $a$ together with the singleton subsets of $S^3 - \bigcup \{A : A \in a\}$. Since $a$ is a null collection, it follows that $G$ is an upper semicontinuous decomposition of $S^3$.

Throughout the remainder of the paper, we shall let $X$ denote the decomposition space associated with $G$, and we shall let $P$ denote the projection map from $S^3$ onto $X$.

Each nondegenerate element of $G$ lies in Int $\Sigma_0$, and thus $P(\text{Int} \Sigma_0)$ is open in the associated decomposition space. Further, since $(x_1, x_2, \ldots)$ a dense in Int $\Sigma_0$, it follows that if $U$ is any open set in the decomposition space intersecting $P(\Sigma_0)$, then for some $\lambda$ of $a$, $P[\lambda] \subseteq U$.

An open set $W$ in $S^3$ is saturated if and only if $W$ is a union of elements of $G$.

4. Preliminary lemmas. If $T$ is a solid torus, then $D$ is a meridional disc in $T$ if and only if $D$ is a disc in $T$ such that $D \cap D' = \emptyset$, $D \cap D' = \{p\}$ on $\partial D$, and $D \cap D' = \{p\}$ Int $T$. We shall use the projection map from $T'$ onto $T$. Let $p_1$ be a copy of $D$ in $T'$. It is easily seen that $p_1$ intersects $\Sigma_1$. Let $p_2$ be a point of $D_1 \cap \Sigma_1$. Let $p_3$ be an integer such that $\phi(p_3) \in T_{m_0}$. Let $T_{m_0}$ be the copy of $T_{m_0}$ in $T'$ containing $p_2$.

Let $T_{m_0, t_1, t_2, \ldots, t_{m_0}}$ and $T_{m_0, t_1, t_2, \ldots, t_{m_0}}$ be copies in $T'$ of $T_{m_0}$, $T_{m_0}$, $T_{m_0}$, respectively, so that $T_{m_0, t_1, t_2, \ldots, t_{m_0}}$ forms a (linear) chain. Let $T_{m_0, t_1, t_2, \ldots, t_{m_0}}$ and $T_{m_0, t_1, t_2, \ldots, t_{m_0}}$ be copies in $T'$ of $T_{m_0}$, $T_{m_0}$, $T_{m_0}$, respectively, so that $T_{m_0, t_1, t_2, \ldots, t_{m_0}}$ forms a (linear) chain and (2) $T_{m_0}$ links $T_{m_0}$.

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Let $D_1$ and $D_2$ be copies of $D$ in $T^*$ such that $D_1$ is adjacent to $D_2$, $D_2$ is adjacent to $D_1$, $D_2$ separates $D_1$ and $D_3$ in $T^*$, and $D_2$ intersects $T_r^*$. Let $p_r$ be the point of $D_2$ such that $\varphi(p_r) = \varphi(p_1)$. Finally, let $A^*$ and $A_2^*$ denote the copies in $T^*$ of $A_1^*$ containing $p_r$ and $p_1$, respectively. We shall establish the following:

**Proposition 1.** Every point of $\varphi^{-1}(N_k) \cap \bigcap_{r=1}^n (T_r^* \cup T_r^*)$ lies either in $A^*$ or in $A_2^*$.

Proof. First consider $\varphi^{-1}(N_k) \cap \bigcap_{r=1}^n (T_r^* \cup T_r^*)$. Clearly, if $r = 1, 2, \ldots, m_s$, or $m_s + 1$, the subarc $a_{rs}$ of $A_1^*$ lying in $T_{rs}$ and containing $N_k \cap T_{rs}$ lifts to an arc $x_{rs}$ lying in $T_r^*$. Let $x_{rs}$ and $y_{rs}$ be the points of $x_{rs}$ such that $\varphi(x_{rs})$ and $\varphi(y_{rs})$ are the endpoints $x_{rs}$ and $y_{rs}$, respectively, of $a_{rs}$.

We regard $T^*$ as $D^2 \times E^2$ where $D^2$ is a disc and $E^2$ is the real line. We may suppose that if $t \in E^2$, $\varphi(D^2 \times \{t\})$ is a cross-section of $T_r$. Further, we suppose that the positive direction on $E^2$ corresponds to the clockwise orientation on $S^1$. Suppose $t \in E^2$. Then it follows from the construction of $b_{rs}$ that there is a copy of $b_{rs}$ in $T^*$ with endpoints $x_{rs}$ and $y_{rs}$. Clearly then, $A^* = (\bigcup_{r=1}^n a_{rs}) \cup (\bigcup_{r=1}^n b_{rs})$. Thus each point of $\varphi^{-1}(N_k) \cap \bigcap_{r=1}^n (T_r^* \cup T_r^*)$ lies in $A^*$, and by a similar argument, $\varphi^{-1}(N_k) \cap \bigcap_{r=1}^n (T_r^* \cup T_r^*) \subseteq A_2^*$. This establishes Proposition 1.

It is easily seen that there is a point $p_r$ of $D_1 \cap \varphi^{-1}(N_k) \cap \bigcap_{r=1}^n (T_r^* \cup T_r^*)$. Thus $p_r \subset A^*$ or $p_r \subset A_2^*$. Hence one of $A^*$ and $A_2^*$ intersects adjacent ones of $D_1$, $D_2$, and $D_3$. Thus there is a subarc $B^*$ of $A^*$ or $A_2^*$ with endpoints on adjacent ones of $D_1$, $D_2$, and $D_3$ and so that $\varphi(B^*)$ misses $D_1 \cup D_2 \cup D_3$. Let $B_3^*$ denote $\varphi(B^*)$. It is clear that $B_3^*$ satisfies the conclusion of Lemma 1.

Suppose that $M$ is a polyhedral 2-manifold in $S^3$, $A$ is a polyhedral singular disc in $S^3$ such that $Bd A$ misses $M$, and $M$ and $A$ are in relative general position. Let $D_3$ be a 2-simplex, and let $f$ be a piecewise linear map from $D_3$ onto $A$ such that at each point of $f^{-1}(A \cap M)$, $f$ is locally a homeomorphism. It follows that each component of $f^{-1}(A \cap M)$ is a simple closed curve. The statement that $\gamma$ is a curve of intersection of $A$ with $M$ means that for some component $y_1$ of $f^{-1}(A \cap M)$, $\gamma = \varphi(y_1)$.

**Lemma 2.** Suppose that $k$ is a positive integer, $i = 2, \ldots, m_m$, and $U$ is a saturated open set in $S^3$ containing a singular disc $A$ such that $Bd A \subset T_{m_m+1}$ and $Bd A = 0$ in $T_{m_m+1}$. Then $U$ contains a loop $\gamma$ such that $\gamma \subset T_{m_m}$ and $\gamma = 0$ in $T_{m_m}$.

**Proof.** We may suppose that $A$ is a polyhedral singular disc, in general position relative to $Bd T_{m_m}$. If there exists a curve of intersection $\gamma$ of $A$ with $Bd T_{m_m}$ such that $\gamma = 0$ in $T_{m_m}$, then the Lemma is established. Hence we shall suppose that each such curve of intersection is homotopic to 0 in $T_{m_m}$.

If every curve of intersection of $A$ with $Bd T_{m_m}$ is homotopic to 0 on $Bd T_{m_m}$, it would follow that $T_{m_m}$ and $T_{m_m+1}$ are not linked, a contradiction. Thus for some curve of intersection $\gamma$ of $A$ with $Bd T_{m_m}$, $\delta = 0$ on $Bd T_{m_m}$. It follows that there exists a curve of intersection $\gamma$ of $A$ with $Bd T_{m_m}$ such that $\gamma = 0$ on $Bd T_{m_m}$ but (2) if $\alpha$ is a subdisc of $A$ bounded by $\lambda$ and $\gamma$ is any curve of intersection of $A$ with $Bd T_{m_m}$ lying in $T_{m_m}$, then $\gamma = 0$ on $Bd T_{m_m}$.

Let $T_{m_m}$ be a polyhedral solid torus in $Int T_{m_m}$, concentric with $T_{m_m}$, and such that $A_3 \cap (T_{m_m} \cup T_{m_m+1}) \subset Int T_{m_m}$. For each curve of intersection $\gamma$ of $A$ with $Bd T_{m_m}$ such that $\gamma = 0$ on $Bd T_{m_m}$, replace the subdisc of $A$ bounded by $\gamma$ by a singular disc on $Bd T_{m_m}$, and deform this new singular disc slightly into $(Int T_{m_m}) \backslash T_{m_m}$. This yields a singular disc $A'$ such that $\alpha' \subset (Int T_{m_m}) \backslash T_{m_m}$.

By the loop theorem [10, 12], there is a polynomial disc $D$ in $T_{m_m}$ such that $D \subset Bd T_{m_m}$, $D \cap 0$ on $Bd T_{m_m}$, and $D$ lies in a small neighborhood of $A'$. Indeed, we may assume that $D \subset T_{m_m} \subset U$. We suppose $D$ and $Bd T_{m_m}$ to be in relative general position.

Since $D$ is a meridional disc in $T_{m_m}$, it follows that $D$ contains a punctured disc $D_{m_m}$ such that $Bd D_{m_m} \subset Bd T_{m_m}$, and $Int D_{m_m} \subset Int T_{m_m}$, one boundary curve $\mu_0$ of $D_{m_m}$ is not homotopic to 0 on $Bd T_{m_m}$, and every other boundary curve is homotopic to 0 on $Bd T_{m_m}$. Note that $Bd D_{m_m} \subset U$. Now we may construct a polyhedral meridional disc $F$ in $T_{m_m}$ by (1) attaching to $D_{m_m}$ an annulus in $T_{m_m} \backslash Int T_{m_m}$ having $\mu_0$ as one boundary curve and having as its other a simple closed curve $\mu$ on $Bd T_{m_m}$ such that $\mu = 0$ on $Bd T_{m_m}$, and (2) capping every other boundary curve of $D_{m_m}$ with a disc lying, except for its boundary, in $(Int T_{m_m}) \backslash T_{m_m}$. We may suppose that $F$ is constructed so that $\gamma \cap T_{m_m} = D_{m_m}$.

By Lemma 1, there is a subarc $B_3$ of $A_3$ such that (1) the endpoints of $B_3$ lie on $F$ and $Int B_3$ misses $F$, and (2) the two ends of $B_3$ abut on $F$ from opposite sides. Clearly the endpoints of $B_3$ lie in $D_{m_m}$. Hence $D_{m_m} \cup B_3$ contains a loop $\gamma$ such that $\gamma = 0$ in $T_{m_m}$.

Since $D_{m_m} \subset U$, $A_{m_m}$ intersects $U$. Since $U$ is saturated, $A_{m_m} \subset U$. Hence $\gamma \subset U$. Clearly $\gamma \subset T_{m_m}$. This establishes Lemma 2.

**Lemma 3.** Suppose $k$ and $n$ are positive integers, $U$ is an open set, $T_n$ is a stage $n$ torus in the construction of $X_2$, and if $i = 1, 2, \ldots, m_m$, $T_{m_m}$ contains a polyhedral simple closed curve $\gamma_i$ such that $\gamma_i = 0$ in $T_{m_m}$.
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$y_I \subset V$, and $y_I \sim 0$ in $U$. Then $T_{m}^n$ contains a polygonal simple closed curve $\gamma$ such that $\gamma \subset U$ and $\gamma \sim 0$ in $T_{m}^n$.

Proof. If $i = 1, 2, \ldots$, or $m_{n+1}$, we shall assume that $y_I$ bounds a polyhedral singular disc $A_I$ in general position relative to $Bd T_{m}^n$.

Suppose there is an integer $j$, $1 \leq j \leq m_{n+1}$, such that some curve of intersection of $A_I$ and $Bd T_{m}^n$ is not homotopic to 0 in $T_{m}^n$. If we let $y_I$ be such a curve, then $y_I$ satisfies the conclusion of Lemma 3. Thus we may assume that if $1 \leq j \leq m_{n+1}$, each curve of intersection of $A_I$ with $Bd T_{m}^n$ is homotopic to 0 in $T_{m}^n$.

Let $T^*$ be the universal covering space of $T_{m}^n$, and let $y_I$ be the projection from $T^*$ onto $T_{m}^n$. Let $A_I, A_{I+1}, \ldots, A_{m_{n+1}}$ and $y_I^*, y_I^*, \ldots, y_{m_{n+1}}^*$ be copies in $T^*$ of $y_I, y_{I+1}, \ldots, y_{m_{n+1}}$ and $y_I$, respectively, so that $y_I^*, y_{I+1}^*, \ldots, y_{m_{n+1}}^*, y_I^*$ forms a (linear) chain of loops.

If $j = 1, 2, \ldots$, or $m_{n+1}$, let $f_I$ be a piecewise linear map from a standard $2$-simplex $A_I$ onto $A_I$ such that $f_I(Bd A_I)$ is a homeomorphism onto $y_I$. Some component of $A_I - f_I^{-1}(Bd A_I)$ contains $Bd A_I$, and $f_I$ denote this component. Let $A_I$ denote $f_I[A_I]$. Then $A_I$ is a polyhedral singular punctured disc with $y_I^*$ as one boundary curve and such that every other boundary curve of $A_I$ is on $Bd T_{m}^n$ and is homotopic to 0 there.

If $j = 1, 2, \ldots, m_{n+1}$, there exists a singular punctured disc $A_{m}^*$ in $T^*$ and a piecewise linear map $g_I$ from $A_{m}^*$ onto $A_I$ such that $g_I(Bd A_{m}^*)$ is a homeomorphism from $Bd A_I$ onto $y_I$. This may be seen as follows: Since each boundary curve of $A_I$ other than $y_I$ is homotopic to 0 in $T_{m}^n$, there is an extension $h_I$ of $f_I(Bd A_I)$ to all of $A_I$, so that $h_I[A_I] \subset T_{m}^n$. There is a map $g_I$ from $A_{m}^*$ into $T^*$ such that $g_I(Bd A_{m}^*)$ is a homeomorphism from $Bd A_I$ onto $y_I$. This may be seen as follows: Since each boundary curve of $A_I$ other than $y_I$ is homotopic to 0 in $T_{m}^n$, there is an extension $h_I$ of $f_I(Bd A_I)$ to all of $A_I$, so that $h_I[A_I] \subset T_{m}^n$. There is a map $g_I$ from $A_{m}^*$ into $T^*$ such that $g_I(Bd A_{m}^*)$ is a homeomorphism from $Bd A_I$ onto $y_I$.

By a similar argument, there is a singular punctured disc $A_{m}^*$ in $T^*$ and a piecewise linear map $g_I$ from $A_{m}^*$ onto $A_I$ such that $g_I(Bd A_{m}^*)$ is a homeomorphism from $Bd A_I$ onto $y_I$. This may be seen as follows: Since each boundary curve of $A_I$ other than $y_I$ is homotopic to 0 in $T_{m}^n$, there is an extension $h_I$ of $f_I(Bd A_I)$ to all of $A_I$, so that $h_I[A_I] \subset T_{m}^n$. There is a map $g_I$ from $A_{m}^*$ into $T^*$ such that $g_I(Bd A_{m}^*)$ is a homeomorphism from $Bd A_I$ onto $y_I$.

PROPOSITION 2. If $1 \leq j \leq m_{n+1}$, $y_I^*$ intersects $A_{m}^*$, and if $1 \leq j \leq m_{n+1}$, $y_I^*$ intersects $A_{m}^*$.

Proof. We establish only the first assertion; the second follows by an analogous argument.

Suppose $1 \leq j \leq m_{n+1}$, $y_I^*$ does not intersect $A_{m}^*$, then let $D$ be a cross-sectional disc in $T^*$ such that $D$ misses $y_I^*$. Then $D \cap Bd T^*$ is simply connected. Then since each boundary curve of $A_{m}^*$ distinct from $y_I^*$, lies on $Bd T^*$, there is an extension $d_I$ of $g_I$ to all of $A_I$ so that $d_I$ sends $A_I - \text{Int } A_I$ to $D \cap Bd T^*$. Thus $y_I^*$ bounds a singular disc in $T^*$ missing $y_I^*$. This is a contradiction since $y_I^*$ and $y_{I+1}^*$ are linked in $T^*$. Hence $y_I^*$ intersects $A_{m}^*$. This establishes Proposition 2.

Let $x$ be a point of $y_I^*$, and (1) if $n = 2$, let $y$ be a double translate of $x$ belonging to $y_{I+1}^*$, and (2) if $n \neq 2$, let $y$ be a translate of $x$ belonging to $y_I^*$. (Recall that, in each first stage solid torus, the chain of second stage solid tori circles twice, but for every other $n$, the chain of $(n+1)$-st stage solid tori in a stage $n$ solid torus circles only once.) It is easy to see that $A_1 \cup A_2 \cup \cdots \cup A_{m+1} \cup A^n$ contains a path $\beta$ from $x$ to $y$ such that $\beta(\beta)$ is a loop in $T_{m}^n$ circling $T_{m}^n$ once (if $n \neq 2$) or twice (if $n = 2$). Since each of $A_1, A_2, \ldots$, and $A_{m+1}$ lies in $U$, $\beta(\beta)$ lies in $A$. A slight adjustment of $\beta(\beta)$ yields a polygonal simple closed curve $\gamma$ such that $\gamma \subset U$, $y_I^* \subset T_{m}^n$, and $y_I^* \sim 0$ in $U$. This establishes Lemma 3.

LEMMA 4. Suppose that $U_{i+1} \subset U_i$, ..., is a sequence of open sets in $S^3$ such that for each $i$, $U_{i+1} \subset U_i$ and each loop in $U_{i+1}$ is homotopic to 0 in $U_i$.

Suppose $V$ is an open set, $V \subset \bigcap_{i=1}^{m} U_i$, and for some integers $k$ and $j$, $A_{k+1} \subset V$.

Then there is a polygonal simple closed curve $\gamma$ in $U_{m} \cap T_{m}^n$ such that $\gamma \sim 0$ in $T_{m}^n$.

Proof. Now $X_{k+1} \cap T_{m}^n \subset A_{k+1}$, and since $A_{k+1} \subset V$, there is a positive integer $n$ such that each stage $n$ torus in the construction of $X_{k+1}$ lying in $T_{m}^n$ lies in $V$.

Now consider the set $U_1$. Since $V \subset X_{k+1}$, then each stage $n$ torus in the construction of $X_{k+1}$ lying in $T_{m}^n$ lies in $U_1$. Consider any stage $(n+1)$ solid torus $T_{m}^n$ in the construction of $X_{k+1}$ lying in $T_{m}^n$. Then $T_{m+1}, T_{m+2}, \ldots$, and $T_{m+n}$ are the stage $n$ solid tori in $T_{m}^n$. If $r = 1, 2, \ldots$, or $m_{n+1}$, let $y_I^*$ be a polygonal simple closed curve in $T_{m}^n$ such that $y_I^* \sim 0$ in $T_{m}^n$. Since $T_{m}^n \subset U_{r}$, $y_I^* \subset U_{r}$. Then $y_I^* \sim 0$ in $U_{r}$.

By Lemma 3, there is a polygonal simple closed curve $y_I^*$ in $T_{m}^n \cap U_{r-1}$ such that $y_I^* \sim 0$ in $T_{m}^n$. Thus, if $T_{m}^n$ is any stage $(n+1)$ solid torus lying in $T_{m}^n$, then there is a polygonal simple closed curve $\gamma$ such that $\gamma \subset T_{m}^n \cap U_{r-1}$ and $\gamma \sim 0$ in $T_{m}^n$. Hence the argument above may be repeated, using any stage $(n+1)$ solid torus $T_{m}^n$ and the stage $(n+1)$ solid tori in the construction of $X_{k+1}$ that lie in $T_{m}^n$.

After at most $n$ repetitions of this argument, we obtain a polygonal simple closed curve $\gamma$ lying in some of $U_1, U_2, U_3, \ldots$, and hence in $U_n$ such that $\gamma \subset T_{m}^n$ and $\gamma \sim 0$ in $T_{m}^n$.
Singularity of Mazurkiewicz in absolute neighborhood retracts

5. Additional preliminary results. The following lemma is a consequence of (113), Theorem 4 and the fact that each AR is simply connected.

Lemma 6. Suppose $M$ is a compact absolute retract in an LC$^n$ locally compact metric space, and suppose that $U$ is an open set containing $M$. Then, there is an open set $V$ such that $M \subset V \subset U$ and each loop in $V$ is homotopic to 0 in $U$.

By repeated application of Lemma 6, we may establish the following result.

Lemma 7. Suppose $M$ is a compact absolute retract in an LC$^2$ locally compact metric space, and suppose $U_0$ is an open set containing $M$. Then, there is a sequence $U_0, U_1, U_2, \ldots$ of open sets such that for each $i$, $U_{i+1} \subset U_i$ and each loop in $U_{i+1}$ is homotopic to 0 in $U_i$.

The following lemma is just a restatement of Corollary 12.14 of Chapter V of [9]; it is also established in [4].

Lemma 8. If $X$ is an ANR, $R$ is an upper semicontinuous decomposition of $X$ into compact subsets, and $S'$ is the associated decomposition space with $X$ the projection map from $S'$ onto $X$. Suppose that $U$ and $V$ are open sets in $S'$ such that $V \subset U$ and each loop in $V$ is homotopic to 0 in $U$. Then each loop in $\pi^{-1}(V)$ is homotopic to 0 in $\pi^{-1}(U)$.

Proof. Suppose $g$ is a loop in $\pi^{-1}(V)$. Then $g \pi = g'$ is a loop in $V$ and thus $g \pi^{-1}$ is a loop in $U$. Let $F'$ be a map from a disc $D$ into $U$ such that $F'[D] = g'$. Then $\pi^{-1}(F'[D])$ is a compact set in $\pi^{-1}(U)$, and is a union of elements of $G$.

For each point $x$ of $F'[D]$, let $W_x$ be an open set such that $\pi^{-1}(y) \subset W_x \subset \pi^{-1}(U)$ and each loop in $W_x$ is homotopic to 0 in $\pi^{-1}(U)$; such an open set $W_x$ exists since each element of $G$ is a CAR. We further assume that each such $W_x$ is a union of elements of $G$.

By compactness of $\pi^{-1}(F'[D])$, there is a finite subset $\{s_1, s_2, \ldots, s_r\}$ of $S'$ such that $\{W_{s_1}, W_{s_2}, \ldots, W_{s_r}\}$ covers $\pi^{-1}(F'[D])$. If $1 \leq i \leq r$, we denote $W_i = W_{s_i}$, then $\{W_1, W_2, \ldots, W_r\}$ is an open cover of $\pi^{-1}(F'[D])$. Note that if $x \in F'[D]$, $\pi^{-1}(x)$ lies in some set of $W$. It follows that $\{\pi(W_1), \pi(W_2), \ldots, \pi(W_r)\}$ is an open cover of $F'[D]$. Let $T$ be a triangulation of $D$ such that $F[X] \cap \pi(T)$ refined $W$.

We now construct a certain singular disc in $\pi^{-1}(U)$. We shall do this by "lifting" $F'[D]$ into $\pi^{-1}(U)$. If $x \in \pi^{-1}(F'[D])$ and $y \in \pi^{-1}(T)$, let $x'$ be a point of $\pi^{-1}(F'[D])$.

Since $x'$ is a vertex of $T$ on $D \times I$, let $y' \in \gamma(x')$. If $\pi(x)$ is a 2-simplex in $\pi^{-1}(F'[D])$, then $\gamma' \in \pi^{-1}(U)$ is a 2-simplex in $\pi^{-1}(F'[D])$.

Select open sets $W_i'$ and $W_j'$ in $W_j$ so that $F'[D] \subset \pi(W_i')$ and $F'[D'] \subset \pi(W_j')$. Then $W_i'$ and $W_j'$ belong to the same component of $W_i \cap W_j$, and let $\sigma'$ be an arc in this component of $W_i \cap W_j$. We use $W_i'$ and $W_j'$ to construct $W_i' \cap W_j'$. It is a 1-simplex of $\sigma$ on $D \times I$, and $\sigma'$ denotes $\gamma(x')$.

Suppose $A$ is a 2-simplex of $\pi^{-1}(F'[D])$ with 1-dimensional faces $\partial_0 A$, $\partial_1 A$, and $\partial_2 A$. Let $W_k$ be a set of $W$ so that $F'[D] \subset \pi(W_k)$. In $W_k$, we have constructed arcs $\delta_0$, $\delta_1$, and $\delta_2$ so that $\delta_0 \subset \delta_1 \subset \delta_2$ is a loop $\mu$. Now $\mu \subset \pi^{-1}(U)$ by construction of $W_k$. Let $A'$ be a singular disc in $\pi^{-1}(U)$ bounded by $\mu$.

It is clear that $\bigcup_{D \subset T} (A \cap D)$ is a singular disc in $\pi^{-1}(U)$, and that this singular disc has boundary $\gamma$. Hence $\gamma \subset \pi^{-1}(T)$.

6. The main result.

Theorem 1. The space $X$ described in Section 2 is a compact absolute neighborhood retract with the singularity of Mazurkiewicz but such that $X \times S^2$ is homeomorphic to $S^2 \times S^2$.

To prove Theorem 1 we first establish two lemmas:

Lemma 10. $X \times S^2$ is homeomorphic to $S^2 \times S^2$.

Proof. It follows from Theorem 5 of [9] (and from [5]) that if $G$ is a monotone decomposition of $E^2$ into countably many arcs and points, and $W$ is the associated decomposition space, then $W \times S^2$ is homeomorphic to $E^2 \times S^2$. With no essential change in the proof, an analogous result could be established for $S^2$ (in place of $E^2$).

In the proof of Theorem 5 of [9], the $E^0$-factor of the product $E^0 \times E^2$ is divided into a sequence $L^1, L^2, \ldots, L^1, L^2, \ldots$ of closed intervals of equal length and, corresponding to each interval $I_j$, certain homeomorphisms $h_j$ of the product $E^0 \times E^2$ are defined. Now the homeomorphisms corresponding to different intervals are constructed by "copying" those for one interval, so that for any $I_j$, there is an order-preserving translation $g_j$ from $I_j$ onto $I_j$ such that if $x \in E^0$, $t \in I_j$, and $h_j(x, t) = (x', t')$, then $h_j(x, g_j(t)) = (x', t + g_j(x))$. It follows that such homeomorphisms can be constructed for products of the form $E^2 \times S^2$ and $S^2 \times S^2$, since, in the case of $S^2$, the factors repeat cyclically.

Thus by a similar modification of the proof of Theorem 5 of [9], we have the following result: If $G$ is a monotone decomposition of $E^2$ into countably many arcs and points, and $W$ is the associated decomposition space, then $W \times S^2$ is homeomorphic to $S^2 \times S^2$. Hence Lemma 10 follows.
Corollary. \( X \) has dimension 3.

Proof. It is known that if \( S \) is a compact metric space of finite dimension, then \( \dim (S \times S) = 1 + \dim S \). Thus \( \dim X = 3 \).

Let \( \Omega_k \) denote a polyhedral solid torus in \( S^3 \) such that \( \Omega_k \cap \text{Int} \Omega_1 \), and \( \Omega_2 \) and \( \Omega_3 \) are concentric.

**Lemma 11.** There exists no \( AM \) in \( X \) such that (1) \( M \subset \text{P}[\text{Int} \Omega_3] \), and (2) if \( i(M) \) denotes the (topological) interior (in \( X \)) of \( M \cap \text{P}[\text{Int} \Omega_3] \), then for some integers \( k \) and \( j \), \( \text{P}[\text{A}_2] \subset i(M) \).

Proof. Suppose there is such an \( AM \). By Lemma 8, \( X \) is an ANR, so \( X \) is \( L^0 \). Thus by Lemma 7, there exists a sequence of open sets in \( X \), \( \text{P}[\text{Int} \Omega_3], W_n, W_{n+1}, \ldots \) such that for each \( i \), \( M \subset W_{n+i} \subset W_i \), each loop in \( W_{n+i} \) is homotopic to 0 in \( W_i \), and each loop in \( W_i \) is homotopic to 0 in \( \text{P}[\text{Int} \Omega_3] \). Further, the interior (in \( X \)) of \( i(M) \) of \( M \cap \text{P}[\text{Int} \Omega_3] \) has the property that for each \( i \), \( i(M) \subset W_i \). Since \( \text{P}[\text{Int} \Omega_3] \subset \text{Int} \Omega_n \), \( X \) is an ANR.

Let \( V \) denote \( \text{P}^{-1}(i(M)) \), and for each \( i \), \( U_i \) denote \( \text{P}^{-1}(W_i) \). Then by Lemma 9, for each \( i \), each loop in \( U_{n+i} \) is homotopic to 0 in \( U_i \), and each loop in \( U_i \) is homotopic to 0 in \( \text{Int} \Omega_3 \). Since for some integers \( k \) and \( j \), \( \text{P}[\text{A}_2] \subset i(M) \), then \( \text{P}[\text{A}_2] \subset V \). By Lemma 5, there is a loop \( \gamma \) in \( \text{Int} \Omega_3 \cap U_n \) such that \( \gamma_0 = 0 \). From the construction of \( \Omega_k \) and \( \Omega_3 \), it follows that \( \gamma - 0 \in \Omega_3 \). However, each loop in \( U_3 \) is homotopic to 0 in \( \text{Int} \Omega_3 \). This is a contradiction, and Lemma 11 is established.

We return to the proof of the theorem.

Proof of Theorem 1. By the Corollary to Lemma 10, \( X \) is finite dimensional. Hence by Lemma 8, \( X \) is an ANR.

The referee pointed out the following simple proof that \( X \) is an ANR. \( X \) is a retract of \( X \times S^3 \). But \( X \times S^3 \) is an ANR since, by Lemma 10, \( X \times S^3 \) is homeomorphic to \( S^3 \times S^3 \).

Now we shall show that \( X \) has the singularities of Mazurkiewicz.

We suppose \( X \) is not a retract of \( X \times S^3 \). Then there is a positive \( \epsilon \) such that any set in \( X \) of diameter less than \( \epsilon \) and intersecting \( P[\text{Int} \Omega_3] \) must lie in \( \text{P}[\text{Int} \Omega_3] \). Suppose \( X \) is covered by at most countably many absolute retracts, each of diameter less than \( \epsilon \). Let \( C \) be the family of those that intersect \( P[\text{Int} \Omega_3] \); each set of \( C \) lies in \( \text{P}[\text{Int} \Omega_3] \).

Let \( C' \) be \( \{ c \cap \text{P}[\text{Int} \Omega_3] \mid c \in C \} \); \( C' \) covers \( \text{P}[\text{Int} \Omega_3] \) and is countable. By the Baire theorem, some set of \( C' \) has nonvoid interior relative to the locally compact metric space \( \text{P}[\text{Int} \Omega_3] \). Let \( M \) be a set of \( C \) such that \( M \cap \text{P}[\text{Int} \Omega_3] \) has nonvoid interior in \( \text{P}[\text{Int} \Omega_3] \). Since \( \text{P}[\text{Int} \Omega_3] \) is open in \( X \), then \( M \cap \text{P}[\text{Int} \Omega_3] \) has nonvoid interior \( i(M) \) in \( X \).

By the construction of \( C' \), each open subset of \( \text{P}[\text{Int} \Omega_3] \) contains, for some integers \( k \) and \( j \), \( \text{P}[\text{A}_2] \subset i(M) \). Thus for some \( k \) and \( j \), \( \text{P}[\text{A}_2] \subset i(M) \).

Thus if \( X \) is covered by at most countably many absolute retracts, each of diameter less than \( \epsilon \), then \( X \) contains a subarc \( i(M) \subset \text{P}[\text{Int} \Omega_3] \) for some \( k \) and \( j \), \( \text{P}[\text{A}_2] \subset i(M) \).

7. Local properties of \( X \). The methods of this paper are closely related to those used in other papers studying local properties of decomposition spaces, especially [1]; see [2] and [3] also. In this section we consider further the local structure of \( X \).

The argument of Lemma 11 can be used to establish the following result.

**Lemma 12.** If \( x \in P[\text{Int} \Omega_3] \), there is no compact, locally connected, simply connected neighborhood of \( x \) (in \( X \)) lying in \( P[\text{Int} \Omega_3] \).

The conclusions of Lemmas 11 and 12 hold if, in the hypotheses of those lemmas, “compact AR” is replaced by “compact, locally connected, simply connected set.” For Lemma 6, this follows from ([11], Theorem 4).

**Theorem 2.** It is not true that each point of \( X \) has arbitrarily small compact, locally connected, simply connected neighborhoods.

The following may be established by a modification of the argument given in this paper.

**Lemma 13.** If \( x \in P[\text{Int} \Omega_3] \), there is no simply connected open neighborhood of \( x \) lying in \( P[\text{Int} \Omega_3] \).

We shall say that a topological space is simply locally simply connected if each point of the space has arbitrarily small simply connected open neighborhoods. Thus we have the following result:

**Theorem 3.** The space \( X \) is not simply locally simply connected.

In proving Lemma 13, we use the following consequence of Corollary 5.4 of [5].

**Lemma 14.** If \( U \) is a simply connected open set in \( X \), then \( P^{-1}(U) \) is simply connected.

The proof of Lemma 13 is essentially the same as that of Lemma 11, except that, in place of the sequence \( U_0, U_1, U_2, \ldots \) of open sets, we can use a single open set.

A topological space is locally peripherally spherical if each point of the space has arbitrarily small neighborhoods whose (topological) boundaries are (topological) 2-spheres.

**Theorem 4.** \( X \) is not locally peripherally spherical.

Theorem 4 follows from Lemmas 12 and 15. For a proof of Lemma 15, see [1].
Lemma 15. If W is a compact neighborhood of a point of a simply connected metric space such that the topological boundary of W is a 2-sphere, then W is compact, locally connected, and simply connected.

Thus we have proved that the space X described in Section 2 is a 3-dimensional ANR but (1) X is not strongly locally simply connected, (2) X is not locally peripherally spherical, and (3) it is not true that X has arbitrarily small compact, locally connected, simply connected neighborhoods.

8. Remarks.

1. By representing $S^2$ as the union of two solid tori $T_1$ and $T_2$, and carrying out the construction of Section 3 in each solid torus, we obtain a 3-dimensional totally non-locally compact space $Y$ such that $X = S^3$ is homeomorphic to $S^3$.

2. For each of the results mentioned above, there is a corresponding result obtained by decomposing $F^n$.

3. According to Borsuk [6], a topological property is multiplicative provided that for every two spaces $X_1$ and $X_2$ with the property, their product $X_1 \times X_2$ has that property. Borsuk raises the following question [6]: Is the singularity of Mazurkiewicz multiplicative?

Kwan [8] established the following theorem: Suppose $m$ and $n$ are positive integers, $a$ and $b$ are arcs in $F^m$ and $F^n$, respectively, and $A$ and $B$ denote the spaces obtained by collapsing $a$ and $b$, respectively, to points. Then $A \times B$ is homeomorphic to $F^{m+n}$.

One may conjecture that Kwan’s result holds in the case of upper semi-continuous decompositions of euclidean spaces into at most countably many arcs. If this conjecture is true, then the construction of Section 2, applied to $F^n$, would yield a space $Z$ with the singularity of Mazurkiewicz but such that $Z \times S^2$ is homeomorphic to $F^4$. It seems plausible to conjecture that, for locally compact metric spaces, the singularity of Mazurkiewicz is not multiplicative.

References