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On convex metric spaces V

by

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§ 1. Introduction. Let X be a metric space with a metric ϱ. By a *hyperspace of* X we shall mean any family $\mathfrak{H}(X)$ consisting of non-empty compact subsets of X. Of particular importance will be the hyperspace $\mathrm{Comp}(X)$ consisting of all non-empty compact subsets of X, the hyperspace $C(X)$ consisting of all non-empty compact and connected subsets (i.e., of all subcontinua) of X, the hyperspace $\mathrm{Conv}(X)$ consisting of all non-empty compact and convex subsets of X, and the hyperspace $X(1)$ consisting of all one-point subsets of X.

Evidently, $X(1)\subset \mathrm{Conv}(X)\subset C(X)\subset \mathrm{Comp}(X)$.

In the case of X being a subset of an n-dimensional Euclidean space E^n of importance will be also the hyperspace

$$\overset{+}{\mathrm{Conv}}(X) = \{A\in \mathrm{Conv}(X)\colon \mathrm{Int}(A)\neq 0\}.$$

Recall that a subset A of X is called *convex* if, given any two points a and b of A, the set A contains a metric segment joining a and b (not necessarily one only). If for every two points a and b of A, the set A contains exactly one metric segment joining a and b, A is called *strongly convex*.

As is well known (cf. [13], pp. 87–89), a complete metric space X is convex if and only if for every two points a and b of A there exists a point $c\in X$ which lies between a and b (i.e., $\varrho(a,c)+\varrho(c,b)=\varrho(a,b)$) and is distinct from both a and b. This result will be applied below several times without reference.

Each hyperspace of X will be considered as a metric space with the Hausdorff metric ϱ^1 defined by the formula ([7], p. 291, see also [11], I, p. 106).

$$\varrho^1(A,B) = \max[\sup_{a\in A}\varrho(a,B),\ \sup_{b\in B}\varrho(A,b)], \tag{1}$$

where $\varrho(x,Z)=\inf_{z\in Z}\varrho(x,z)$.

Formula (1) is easily equivalent to the formula

$$\varrho^1(A,B) = \inf\{\eta\geqslant 0\colon A\subset Q(B,\eta) \text{ and } B\subset Q(A,\eta)\}, \tag{2}$$

where $Q(Z, \eta)$ denotes a generalized metric ball in X, i.e.

$$Q(Z, \eta) = \{x \in X\colon\ \varrho(x, Z) \leqslant \eta\}.$$

Similarly, $Q^1(C, \varepsilon)$ will denote a metric ball in $\mathrm{Comp}(X)$ with the center $C \in \mathrm{Comp}(X)$ and radius $\varepsilon \geqslant 0$, i.e.

$$Q^1(C, \varepsilon) = \{D \in \mathrm{Comp}(X)\colon\ \varrho^1(C, D) \leqslant \varepsilon\}.$$

As follows from (1), $X(1)$ is isometric to X itself and it is known that if X is a continuum, then so are $\mathrm{Comp}(X)$ and $C(X)$ (see [9], p. 25). However, it is not the case with $\mathrm{Conv}(X)$ (see § 4 below).

The paper is devoted to some questions concerning convexity of hyperspaces. In § 2 we define a certain property of hyperspaces, related to convexity and called property (S). Some relations between that property and convexity of hyperspaces are established in § 3, and theorems of § 3 yield a sequence of corollaries. Thus, for instance, it is shown in § 4 that the hyperspace $\mathrm{Comp}(X)$ is convex if and only if the underlying compact metric X is convex, the hyperspace $C(X)$ is convex if and only if the underlying metric continuum X is a dendrite with a convex metric, and topological properties of the underlying metric space X like completeness, compactness and local compactness are preserved by the hyperspace $\mathrm{Conv}(X)$. Results of § 5 pertain to the hyperspaces $\mathrm{Conv}(X)$ and $\overset{+}{\mathrm{Conv}}(X)$ in the case of X being a subspace of a Euclidean space E^n.

Theorems of § 5 seem to form a natural topological background for considering convex subsets of E^n. It turns out that some known theorems on convex sets in E^n proved so far by direct metric considerations (like Auswahlsatz of Blaschke, theorem of Hadwiger, etc.) follow easily and some other (like Minkowski theorem on approximation by polyhedra) receive a clear topological formulation.

Notions and notation not defined in the paper come from [11] and [12].

§ 2. Property (S). Let A and B be two compact subsets of a metric space X such that for each two points $a \in A$ and $b \in B$ there is at least one segment $\overline{ab}$ in X (for $a = b$ this segment is reduced to one point). We call a *bridge in X between A and B*, and denote it by $P(A, B)$, any compact union of segments $\overline{ab}$, i.e. a compact set containing at least one segment $\overline{ab}$ with each pair of points $a \in A$ and $b \in B$ and such that each point of it lies on a segment $\overline{ab}$, where $a \in A$ and $b \in B$ ([3], p. 24).

The following proposition is known to be true (see [3], 2.1).

2.1. *If X is a compact convex space, then for every pair $A, B \in \mathrm{Comp}(X)$ a bridge $P(A, B)$ does exist.*

Remarks. If the space X is not strongly convex, bridge $P(A, B)$ need not be unique.

And if X is a complete convex space, Proposition 2.1 is false. Indeed, consider Cartensian product $C \times I$ of the Cantor ternary set C and the segment $I = [0, 1]$, and identify each pair of points (c_1, i) and (c_2, i) such that c_1 and c_2 are end-points of a segment complementary to C and $i = 0$ or $i = 1$. After identification the set $C \times 0$ becomes a segment, denote it by A, and the set $C \times 1$ becomes a segment disjoint with A, denote it by B; the set X obtained in this way from $C \times I$ consists of uncountably many arcs joining points of A to points of B and otherwise disjoint. If we metrize now X according to the length of an arc, assuming that each segment $c \times I$ is now an arc of length 1, then X becomes a complete metric space and $\varrho^1(A, B) = 1$. Evidently, any bridge between A and B must contain each "arc $c \times I$" and thus the only bridge between A and B is the whole space X which is not compact.

A hyperspace $\mathfrak{H}(X)$ of X will be said to have *property* (S) if for each pair of sets $A, B \in \mathfrak{H}(X)$ a bridge $P(A, B)$ in X does exist (however, we do not assume it to be an element of $\mathfrak{H}(X)$) and for some ε satisfying inequalities $0 < \varepsilon < \varrho^1(A, B)$ the set

$$(3)\qquad P(A, B) \cap Q(A, \varepsilon) \cap Q\big(B, \varrho^1(A, B) - \varepsilon\big)$$

belongs to $\mathfrak{H}(X)$.

Some examples of hyperspaces possessing property (S):

1. If X is a convex continuum, there are in general many hyperspaces of X which have property (S). Such is, for instance, each hyperspace $\mathfrak{H}(X)$ which satisfies the following three conditions:

(i) if $A \in \mathfrak{H}(X)$, then also $Q(A, \eta) \in \mathfrak{H}(X)$ for each $\eta \geqslant 0$,

(ii) if $A, B \in \mathfrak{H}(X)$, then there exists a bridge $P(A, B) \subset X$,

(iii) if $A, B \in \mathfrak{H}(X)$, then also $A \cap B \in \mathfrak{H}(X)$.

For each family $\mathfrak{M} \subset \mathrm{Comp}(X)$ one may consider the least hyperspace $\mathfrak{H}_{\mathfrak{M}}(X)$ containing $\mathfrak{M}$ and satisfying conditions (i)–(iii). In view of 2.1 such a hyperspace does exist and it has property (S), of course. It can be shown without much difficulty that if $\mathfrak{M}$ is closed in $\mathrm{Comp}(X)$, the hyperspace $\mathfrak{H}_{\mathfrak{M}}(X)$ is closed in $\mathrm{Comp}(X)$, hence compact.

The hyperspace $X(1)$ and the hyperspace $\mathrm{Comp}(X)$ both have property (S).

2. If X is a convex continuum such that each subcontinuum of X is also convex (i.e., if $C(X) = \mathrm{Conv}(X)$), then the hyperspace $C(X)$ has property (S).

In fact, if $A, B \in C(X)$, then a bridge $P(A, B)$ in X does exist by virtue of 2.1, and all three sets $P(A, B)$, $Q(A, \varepsilon)$ and $Q\big(B, \varrho^1(A, B) - \varepsilon\big)$ are strongly convex for each $0 \leqslant \varepsilon \leqslant \varrho^1(A, B)$ (see [3], 2.5). Hence the common part (3) is compact convex and so it must be an element of $C(X)$.

3. If X is a convex subspace of an n-dimensional Euclidean space E^n, then the two hyperspaces $\mathrm{Conv}(X)$ and $\overset{+}{\mathrm{Conv}}(X)$ both have property (S).

In fact, if A is compact convex, then $Q(A, \varepsilon)$ is compact convex for each $\varepsilon \geqslant 0$, because $Q(A, \varepsilon)$ is equal to the Minkowski union, $Q(A, \varepsilon) = A + Q(p, \varepsilon)$, of two compact convex sets and such a union is known to be compact convex ([6], p. 12–13). Therefore the common part (3) is compact convex, and since $P(A, B)$ is contained in X by the convexity of X, then (3) is an element of $\mathrm{Conv}(X)$. Hence the hyperspace $\mathrm{Conv}(X)$ has property (S).

To prove that also the hyperspace $\overset{+}{\mathrm{Conv}}(X)$ has property (S) it remains to show that if $A, B \in \overset{+}{\mathrm{Conv}}(X)$, then the common part (3) has non-empty interior. To that end show first that

(4) if $A, B \in \overset{+}{\mathrm{Conv}}(X)$, then there exist points $a \in \mathrm{Int}(A)$ and $b \in \mathrm{Int}(B)$ such that $\varrho(a, b) < \varrho^1(A, B)$.

Indeed, let a be any point of $\mathrm{Int}(A)$. Since, in view of formula (2), $A \subset Q\big(B, \varrho^1(A, B)\big)$, then each point of A lying on the boundary of $Q\big(B, \varrho^1(A, B)\big)$ belongs to the boundary of A. Hence a is not on the boundary of $Q\big(B, \varrho^1(A, B)\big)$ and so $\varrho(a, B) < \varrho^1(A, B)$. Consequently, there must exist a point $b_1 \in B$ such that $\varrho(a, b_1) < \varrho^1(A, B)$. Assumption $B \in \overset{+}{\mathrm{Conv}}(X)$ implies that $\mathrm{Int}(B)$ is dense in B and so there exists a point $b \in \mathrm{Int}(B)$ such that $\varrho(b_1, b) < \varrho^1(A, B) - \varrho(a, b_1)$. Therefore, by the triangle inequality, $\varrho(a, b) \leqslant \varrho(a, b_1) + \varrho(b_1, b) < \varrho^1(A, B)$.

Hence (4) is established. Assume that a and b satisfy (4). Since $a \in \mathrm{Int}(A)$ and $b \in \mathrm{Int}(B)$, there exists an $\eta > 0$ such that $Q(a, \eta) \subset A$ and $Q(b, \eta) \subset B$. Let c be a point of the segment $\overline{ab}$ such that $\varrho(a, c) < \varepsilon$ and $\varrho(c, b) < \varrho^1(A, B) - \varepsilon$, where $0 < \varepsilon < \varrho^1(A, B)$. Obviouosly, $Q(c, \eta) \subset P\big(Q(a, \eta), Q(b, \eta)\big) \subset P(A, B)$. Since $Q(c, \eta)$ is a translation of $Q(a, \eta)$ for less than ε, then $Q(c, \eta) \subset Q(A, \varepsilon)$. Similarly, $Q(c, \eta) \subset Q\big(B, \varrho^1(A, B) - \varepsilon\big)$. Hence the common part (3) contains $Q(c, \eta)$ and, consequently, belongs to $\overset{+}{\mathrm{Conv}}(X)$. Thus it is shown that also the hyperspace $\overset{+}{\mathrm{Conv}}(X)$ has property (S).

§ 3. Convexity of hyperspaces. We shall now show several theorems on relations between convexity of hyperspaces and convexity of the underlying space.

To this purpose recall first a lemma ([3], 2.3).

3.1. *Let X be a metric space and let A and B be two compact subsets of X such that there exists a bridge $P(A, B)$ in X between A and B. If ε is a number such that $0 \leqslant \varepsilon \leqslant \varrho^1(A, B)$, then the set*

$$H = P(A, B) \cap Q(A, \varepsilon) \cap Q\big(B, \varrho^1(A, B) - \varepsilon\big)$$

satisfies the conditions

$$\varrho^1(A, H) = \varepsilon \quad \text{and} \quad \varrho^1(H, B) = \varrho^1(A, B) - \varepsilon .$$

This lemma leads to the following

THEOREM 3.2. *Let $\mathfrak{H}(X)$ be a hyperspace of a metric space X. If $\mathfrak{H}(X)$ is complete and has property (S), then $\mathfrak{H}(X)$ is convex.*

Note, however, that for some metric spaces X there exist hyperspaces which are complete and convex but do not possess property (S). Such is, for instance, the hyperspace of E^2 consisting of all circles of diameter 1.

Theorem 3.2 yields a sequence of corollaries on particular hyperspaces of particular spaces (cf. § 4 and § 5 below). Here, however, we shall yet prove two more theorems.

In the proof of the first theorem we shall need a simple lemma which states that a set lies between some two points (in the sense of metric ϱ^1) if and only if each point of it lies between them (in the sense of metric ϱ). More precisely,

3.3. *Let X be a metric space, p and q two points of X, and Z a subset of X. If ε is a number such that $0 \leqslant \varepsilon \leqslant \varrho(p, q)$, then the following two conditions are equivalent*

(5) $$\varrho^1\big((p), Z\big) = \varepsilon \quad \text{and} \quad \varrho^1\big(Z, (q)\big) = \varrho^1\big((p), (q)\big) - \varepsilon ,$$

(6) $$\varrho(p, z) = \varepsilon \quad \text{and} \quad \varrho(z, q) = \varrho(p, q) - \varepsilon \quad \text{for each} \quad z \in Z .$$

Proof. Assume (5) and let $z \in Z$. By the definition of Hausdorff metric (see (1) or (2)), $\varrho^1\big((p), Z\big) = \varepsilon$ implies $\varrho(p, z) \leqslant \varepsilon$ and, in view of $\varrho^1\big((p), (q)\big) = \varrho(p, q)$, the second equality of (5) implies $\varrho(z, q) \leqslant \varrho(p, q) - \varepsilon$. And if for some $z \in Z$ we would have $\varrho(p, z) < \varepsilon$ or $\varrho(z, q) < \varrho(p, q) - \varepsilon$, then, by the triangle inequality, $\varrho(p, q) \leqslant \varrho(p, z) + \varrho(z, q) < \varepsilon + \varrho(p, q) - \varepsilon = \varrho(p, q)$ which is clearly impossible.

Hence (5) implies (6).

The converse implication follows easily by the definition of Hausdorff metric.

THEOREM 3.4. *If X is a complete metric space, then the following three conditions are equivalent:*

(a) *X is convex,*

(b) *there exists a hyperspace of X which is a complete metric space, has property (S) and contains $X(1)$,*

(c) *there exists a hyperspace of X which is convex and contains $X(1)$.*

Proof. (a) $\Rightarrow$ (b). If X is convex, then the hyperspace $X(1)$ has property (S). And since $X(1)$ is isometric to X itself, $X(1)$ is complete.

(b) $\Rightarrow$ (c). This implication follows by Theorem 3.2.

(c) $\Rightarrow$ (a). Let $\mathfrak{H}(X)$ be a hyperspace of X which is convex and contains $X(1)$, and let p and q be any two points of X. Since (p) and (q) are both in $\mathfrak{H}(X)$, the hyperspace $\mathfrak{H}(X)$ contains a segment between (p) and (q) composed of subsets of X. It means that the inequality $0 < \varepsilon$

$< \varrho^1((p),(q))$ implies existence of a set $Z \subset X$ such that (5) holds. By virtue of 3.3, there exists then a point $z \in X$ which lies between p and q and is distinct from both. Hence X is convex.

THEOREM 3.5. *Let X be a complete metric space. If $\mathfrak{H}(X)$ is a convex hyperspace of X, contained in $C(X)$, and containing all subarcs of X, then the underlying space X contains no simple closed curve, is convex, and each element of $\mathfrak{H}(X)$ is convex.*

Proof. By virtue of Theorem 3.4 space X is convex.

If we shall show that each element of $\mathfrak{H}(X)$ is convex, it will imply that X cannot contain any simple closed curve. In fact, for if X would contain a simple closed curve S, there would also exist an arc $L \subset S$ which is not convex ([3], 2.4). But since $L \in \mathfrak{H}(X)$ by hypothesis, L must be convex. A contradiction.

It remains then to show convexity of each element of $\mathfrak{H}(X)$. Suppose, *a contrario*, that there exists an element $A \in \mathfrak{H}(X)$ which is not convex. Since X is convex, there exist two points $p, q \in A$ such that for each metric segment $\overline{pq}$ there is $\overline{pq} \setminus A \neq 0$. If there are two such segments, choose a simple closed curve in their union. If there is only one segment $\overline{pq}$, join p to q with an arc in a sufficiently small (not to enclose $\overline{pq}$) ball $Q(A, \eta)$ (A itself may *a priori* not contain such an arc, but since X is locally convex, each ball $Q(A, \eta)$ does), and choose a simple closed curve in the union of that arc and of the segment pq. In any of the two cases we receive a simple closed curve S which contains a rectilinear segment.

Choose two distinct points, p_1 and p_2, inside a segment of S and denote by P_i^{ε} the subsegment of that segment which has length $\varepsilon > 0$ and the middle point of which is p_i, $i = 1, 2$. Let α_0 be a positive number such that segments $P_1^{\alpha_0}$ and $P_2^{\alpha_0}$ exist and are disjoint. The set $S \setminus (P_1^{\alpha_0} \cup P_2^{\alpha_0})$ consists of two components; choose a point s_i in each of them, $i = 1, 2$. If $0 \leqslant \alpha \leqslant \alpha_0$, then by S_i^{α} we shall mean the component of $S \setminus (P_1^{\alpha} \cup P_2^{\alpha})$ which contains s_i, $i = 1, 2$.

First we show that there exists a number α such that $0 < \alpha \leqslant \alpha_0$ and

$$\varrho(P_i, \overline{S \setminus P_i^{\alpha_0}}) > \frac{\alpha}{2} \quad \text{for both } i = 1 \text{ and } i = 2\,, \tag{7}$$

and

$$\varrho(S_1^{\alpha}, S_2^{\alpha}) > \frac{\alpha}{2}\,. \tag{8}$$

Indeed, for if, contrary to (7), we would have $\varrho(P_i^{\alpha}, \overline{S \setminus P_i^{\alpha_0}}) \leqslant \frac{\alpha}{2}$ for each $\alpha \leqslant \alpha_0$ and some $i = 1$ or 2, then taking a sequence $\{\alpha_n\}_{n=1,2,\ldots}$ of positive numbers $\alpha_n < \alpha_0$ converging to 0, we would be able to choose

sequences of points

$$x_n \in P_i \tag{9}$$

$$y_n \in \overline{S \setminus P_i^{\alpha_0}} \qquad n = 1, 2, \ldots \tag{10}$$

such that

$$\varrho(x_n, y_n) \leqslant \frac{\alpha_n}{2} \quad \text{for } n = 1, 2, \ldots \tag{11}$$

By virtue of (9) and of the definition of $P_i^{\alpha_n}$ we infer that $\varrho(x_n, p_i) \leqslant \frac{\alpha_n}{2}$ for $n = 1, 2, \ldots$, whence and from (11) it follows, by virtue of the triangle inequality, that $\varrho(p_i, y_n) \leqslant \alpha_n$ for $n = 1, 2, \ldots$ Hence the sequence of points $\{y_n\}_{n=1,2,\ldots}$ is convergent to p_i, and this implies, in view of (10), that $p_i \in \overline{S \setminus P_i^{\alpha_0}}$. A contradiction to p_i being the middle point of the segment P_i.

To prove (8) suppose, *a contrario* again, that $\varrho(S_1^{\alpha}, S_2^{\alpha}) \leqslant \frac{\alpha}{2}$ for each $\alpha \leqslant \alpha_0$. Taking now, as before, a sequence $\{\alpha_n\}_{n=1,2,\ldots}$ converging to 0 and consisting of positive numbers $\alpha_n < \alpha_0$, we would be able to choose sequences of points

$$z_1^n \in S_1^{\alpha_n} \quad \text{and} \quad z_2^n \in S_2^{\alpha_n}\,, \quad n = 1, 2, \ldots\,, \tag{12}$$

such that

$$\varrho(z_1^n, z_2^n) \leqslant \frac{\alpha_n}{2} \quad \text{for } n = 1, 2, \ldots \tag{13}$$

Since S is compact, sequences $\{z_1^n\}_{n=1,2,\ldots}$ and $\{z_2^n\}_{n=1,2,\ldots}$ contain subsequences convergent, in view of (13), to a common limit point $z \in S$. Without loss of generality we may assume that $\lim z_1^n = z = \lim z_2^n$.

Let S_i be that subarc of S which contains s_i and has end-points p_1 and p_2, $i = 1, 2$. From (12) and from obvious inclusions $S_i^{\alpha_n} \subset S_i$, where $\alpha_n \leqslant \alpha_0$, we infer that $z_i^n \in S_i$ for $n = 1, 2, \ldots$ and for $i = 1, 2$. Since S_i is closed in S, the limit point z must belong to both S_1 and S_2. Thus $z = p_1$ or $z = p_2$.

Suppose $z = p_1$ (case $z = p_2$ is analogous). Since both sequences $\{z_1^n\}_{n=1,2,\ldots}$ and $\{z_2^n\}_{n=1,2,\ldots}$ are convergent to $z = p_1$ by supposition, there must exist an index n_0 such that

$$z_i^n \in P_1^{\alpha_0} \cap S_i^{\alpha_n} \text{ for each } n > n_0 \text{ and for } i = 1, 2\,. \tag{14}$$

However, the segment $P_i^{\alpha_0}$ is rectilinear and so (14) implies that

$$\varrho(z_1^n, z_2^n) \geqslant \delta[P_1^{\alpha_0} \setminus (S_1^{\alpha_n} \cup S_2^{\alpha_n})] = \delta(P_1^{\alpha_n}) = \alpha_n$$

for each $n > n_0$, contrary to (13).

Hence both (7) and (8) are proved, and in the sequel we assume that a is fixed and satisfies (7) and (8).

Put

$$A = S_1^a \cup P_1^a \cup S_2^a \quad \text{and} \quad B = S_1^a \cup P_2^a \cup S_2^a .$$

Both A and B are arcs and so elements of $\mathfrak{H}(X)$, and we shall proceed to show that there is no subcontinuum H of X which lies in the middle between A and B. In view of the hypothesis $\mathfrak{H}(X) \subset C(X)$ this will be a contradiction to the convexity of $\mathfrak{H}(X)$.

With this end in mind we show first that

(15)
$$\varrho^1(A, B) = \frac{a}{2} .$$

Indeed, for if $a \in A$ and $\varrho(a, B) > 0$, then $a \in P_1^a$, and so

$$\sup_{a \in A} \varrho(a, B) = \sup_{a \in P_1^a} \varrho(a, B) .$$

Since B can be written in the form $B = \overline{P_1^{a_0} \setminus P_1^a} \cup \overline{S \setminus P_1^{a_0}}$, then

(16)
$$\sup_{a \in A} \varrho(a, B) = \min[\sup_{a \in P_1^a} \varrho(a, \overline{P_1^{a_0} \setminus P_1^a}), \sup_{a \in P_1^a} \varrho(a, \overline{S \setminus P_1^{a_0}})] .$$

And since, in view of $a < a_0$ and the rectilinearity of $P_1^{a_0}$, we have

(17)
$$\sup_{a \in P_1^a} \varrho(a, \overline{P_1^{a_0} \setminus P_1^a}) = \frac{a}{2} ,$$

and, by virtue of (7), also

(18)
$$\sup_{a \in P_1^a} \varrho(a, \overline{S \setminus P_1^{a_0}}) > \frac{a}{2} ,$$

then taking into account both (17) and (18) we infer from (16) that

(19)
$$\sup_{a \in A} \varrho(a, B) = \frac{a}{2} .$$

Analogous argument works to the effect that

(20)
$$\sup_{b \in B} \varrho(A, b) = \frac{a}{2} ,$$

and the two equalities, (19) and (20), yield (15).

Now we show that

(21)
$$Q\Big(A, \frac{a}{4}\Big) \cap Q\Big(B, \frac{a}{4}\Big) = Q\Big(S_1^a, \frac{a}{4}\Big) \cup Q\Big(S_2^a, \frac{a}{4}\Big) .$$

In fact, in view of the definitions of A and B we have

$$Q\Big(A, \frac{a}{4}\Big) = Q\Big(S_1^a, \frac{a}{4}\Big) \cup Q\Big(P_1^a, \frac{a}{4}\Big) \cup Q\Big(S_2^a, \frac{a}{4}\Big) ,$$

$$Q\Big(B, \frac{a}{4}\Big) = Q\Big(S_1^a, \frac{a}{4}\Big) \cup Q\Big(P_2^a, \frac{a}{4}\Big) \cup Q\Big(S_2^a, \frac{a}{4}\Big) .$$

Hence, if there would exist a point

$$p \in \Big[Q\Big(A, \frac{a}{4}\Big) \cap Q\Big(B, \frac{a}{4}\Big)\Big] \setminus \Big[Q\Big(S_1^a, \frac{a}{4}\Big) \cup Q\Big(S_2^a, \frac{a}{4}\Big)\Big] ,$$

then we would have $p \in Q\Big(P_1^a, \frac{a}{4}\Big) \cap Q\Big(P_2^a, \frac{a}{4}\Big)$, whence, consequently,

(22)
$$\varrho(P_1^a, P_2^a) \leqslant \frac{a}{2} .$$

But this is impossible, because $P_2^a \subset P_2^{a_0} \subset \overline{S \setminus P_1^{a_0}}$ and so, in view of (7), $\varrho(P_1^a, P_2^a) > \frac{a}{2}$ contrary to (22). Hence (21) holds.

Note also that

(23)
$$Q\Big(S_1^a, \frac{a}{4}\Big) \cap Q\Big(S_2^a, \frac{a}{4}\Big) = 0 ,$$

since otherwise there would be $\varrho(S_1^a, S_2^a) \leqslant \frac{a}{4}$, contrary to (8).

Let H be an arbitrary subcontinuum of X. We shall yet show that if

(24)
$$H \subset Q\Big(S_1^a, \frac{a}{4}\Big) \quad \text{or} \quad H \subset Q\Big(S_2^a, \frac{a}{4}\Big) ,$$

then simultaneously

(25)
$$\varrho^1(A, H) > \frac{a}{4} \quad \text{and} \quad \varrho^1(H, B) > \frac{a}{4} .$$

In view of the symmetry we may assume that $H \subset Q\Big(S_1^a, \frac{a}{4}\Big)$. This assumption implies $\varrho(H, S_2^a) > \frac{a}{4}$ (for otherwise $\varrho(S_1^a, S_2^a) \leqslant \frac{a}{2}$, contrary to (8)) and therefore, in view of the inclusions $S_2^a \subset A$ and $S_2^a \subset B$, we have

$$\sup_{a \in A} \varrho(H, a) > \frac{a}{4} \quad \text{and} \quad \sup_{b \in B} \varrho(H, b) > \frac{a}{4} .$$

The two inequalities yield (25).

Now we may complete the proof. Suppose then that there exists a continuum $H \subset X$ such that

(26) $$\varrho^1(A, H) = \varrho^1(H, B) = \frac{a}{4}.$$

By the formula (2) it follows then that $H \subset Q\left(A, \frac{a}{4}\right)$ and $H \subset Q\left(B, \frac{a}{4}\right)$, whence $H \subset Q\left(A, \frac{a}{4}\right) \cap Q\left(B, \frac{a}{4}\right)$. Hence, in view of (21) and (23), we infer that one of the inclusions in (24) holds. However, any inclusion in (24) implies both inequalities (25). A contradiction with (26).

§ 4. Applications to hyperspaces Comp (X), $C(X)$ **and** Conv (X). The first theorem below is known ([3], Theorem 4.1), but inserting it here for the sake of completeness we supply it with a new proof.

THEOREM 4.1. *Let X be a compact metric space. Then X is convex if and only if* Comp(X) *is convex.*

In fact, if X is convex, then the hyperspace Comp(X) is compact (see [11], II, p. 21) and has property (S), and so, by Theorem 3.2, it must be convex. Conversely, if Comp(X) is convex, then by virtue of Theorem 3.4 the underlying space X is convex too.

Remarks. If X is a complete convex space, the hyperspace Comp(X) need not be convex. An example is provided by the space X constructed in remarks following Proposition 2.1. In fact, if there were a compact set $C \subset X$ such that $\varrho^1(A, C) = \varrho^1(C, B) = \frac{1}{2}$, then, in view of formula (2), we would have $C \subset Q(A, \frac{1}{2}) \cap Q(B, \frac{1}{2})$, and since the common part $Q(A, \frac{1}{2}) \cap Q(B, \frac{1}{2})$ consists of middle points of all "arcs $c \times I$", hence is uncountable with the discrete topology, then C were a finite subset of that common part. It is easy to check that in such a case there would be $\sup_{a \in A} \varrho(a, C) > \frac{1}{2}$ and $\sup_{b \in B} \varrho(b, C) > \frac{1}{2}$, whence, by the formula (1), $\varrho^1(A, C) > \frac{1}{2}$ and $\varrho^1(C, B) > \frac{1}{2}$; a contradiction. Hence the hyperspace Comp(X) is not convex.

Note, however, that if X is a complete metric space and the hyperspace Comp(X) is convex, then the space X is convex by virtue of Theorem 3.4.

Before proceed to the hyperspace $C(X)$ we shall need some simple lemmas.

4.2. *Let X be a metric space and $A_0, A_1, A_2, \ldots$ a sequence of subsets of X. If*

(27) $$\lim \varrho^1(A_n, A_0) = 0,$$

A_0 is closed in X, and each set $A_1, A_2, \ldots$ is connected, then A_0 is connected.

Proof. Assume, *a contrario*, that each set $A_1, A_2, \ldots$ is connected but A_0 is not. Then there are two non-empty closed subsets F_1 and F_2 of X such that $A_0 = F_1 \cup F_2$ and $F_1 \cap F_2 = 0$, and, consequently, there are two open subsets G_1 and G_2 of X such that $F_1 \subset G_1$, $F_2 \subset G_2$ and $G_1 \cap G_2 = 0$.

Since $A_0 \subset G_1 \cup G_2$ and $G_1 \cup G_2$ is open, there is $A_n \subset G_1 \cup G_2$ for n sufficiently large. And since $F_1 \neq 0 \neq F_2$, there is also $A_n \cap G_1 \neq 0 \neq A_n \cap G_2$ for n sufficiently large. But then A_n cannot be connected. A contradiction.

4.3. *Let X be a metric space and $A_0, A_1, A_2, \ldots$ a sequence of subsets of X. If* (27) *holds true and each set $A_1, A_2, \ldots$ is bounded (respectively, totally bounded), then the set A_0 is bounded (respectively, totally bounded).*

Proof. Condition (27) implies that

(28) for each $\varepsilon > 0$ there exists $n(\varepsilon)$ such that $\varrho^1(A_{n(\varepsilon)}, A_0) < \varepsilon$.

Hence and from the formula (2) it follows that $A_0 \subset Q(A_{n(\varepsilon)}, \varepsilon)$, and therefore $\delta(A_0) \leqslant \delta(A_{n(\varepsilon)}) + 2\varepsilon$. Boundedness of $A_{n(\varepsilon)}$ implies that of A_0.

Now suppose that each set $A_1, A_2, \ldots$ is totally bounded and A_0 is not. Hence for some $\eta > 0$ there exists a sequence $c_1, c_2, \ldots$ of points of A_0 such that $\varrho(c_k, c_l) \geqslant 3\eta$ for $k \neq l$. Balls $Q(c_k, \eta)$ are then pairwise disjoint and if we choose for each $k = 1, 2, \ldots$ a point $a_k \in Q(c_k, \eta)$, then

(29) $$\varrho(a_k, a_l) \geqslant \eta \quad \text{for } k \neq l.$$

By virtue of (28) we have now $\varrho^1(A_{n(\eta)}, A_0) < \eta$, whence and from the formula (2) it follows that $A_{n(\eta)} \cap Q(c_k, \eta) \neq 0$ for $k = 1, 2, \ldots$, because $c_k \in A_0$. Choosing now a point $a_k \in A_{n(\eta)} \cap Q(c_k, \eta)$ for $k = 1, 2, \ldots$ we come, in view of (29), to a contradiction with the total boundedness of A_n.

4.4. *Let X be a complete metric space and $A_0, A_1, A_2, \ldots$ a sequence of closed subsets of X. If* (27) *holds true and each set $A_1, A_2, \ldots$ is compact, then A_0 is compact.*

Indeed, as a closed subspace of a complete space, A_0 is a complete space itself (see [11], I, p. 315) and by virtue of 4.3, A_0 is totally bounded. Hence A_0 must be compact (see [11], II, p. 2).

Recall that a *dendrite* is a locally connected metric continuum containing no simple closed curve.

THEOREM 4.5. *Let X be a complete metric space. Then the following conditions are equivalent:*

(a) *X is convex and contains no simple closed curve,*

(b) *every subcontinuum of X is a convex dendrite,*

(c) *$C(X)$ is convex.*

Proof. (a)$\Rightarrow$(b). Assume, *a contrario*, that X contains a subcontinuum C which is not a dendrite. Were C locally connected, it would contain a simple closed curve and this is impossible, because X does not contain any simple closed curve. Thus C is not locally connected and so it must contain a sequence of pairwise disjoint subcontinua $\{A_n\}$ convergent to a subcontinuum A disjoint with each A_n (cf. [11], II, p. 176). Choose n such that

$$2\cdot\varrho^1(A, A_n) < \min[\delta(A), \delta(A_n)]$$

and choose points $a, b \in A$ and $a_n, b_n \in A_n$ such that

$$2\cdot\varrho(a, a_n) < \min[\varrho(a, b), \varrho(a_n, b_n)], \tag{30}$$

$$2\cdot\varrho(b, b_n) < \min[\varrho(a, b), \varrho(a_n, b_n)]. \tag{31}$$

Covering now A with a finite number of convex sets, each disjoint with A_n, we can choose an arc $L \subset X$ joining a to b and disjoint with A_n. Similarly we can choose an arc $L_n \subset X$ joining a_n to b_n and disjoint with L.

Space X is convex by hypothesis (a), there exists a segment $\overline{bb_n} \subset X$. The union $L \cup \overline{bb_n} \cup L_n$ is locally connected (cf. theorem of Hahn–Mazurkiewicz–Sierpiński, [11], II, p. 185) and so it contains an arc M of ends a and a_n. In view of (31), $\delta(M) > \frac{1}{2}\min[\varrho(a, b), \varrho(a_n, b_n)]$. Hence in view of (30) a segment $\overline{aa_n}$ is not contained in M and so the union $M \cup \overline{aa_n}$ contains a simple closed curve. A contradiction.

Thus it is shown that each subcontinuum of X is a dendrite. Were such a dendrite not convex, we would come, in view of the assumed convexity of X, to a contradiction with the hypothesis that X contains no simple closed curve.

Hence the proof of implication (a)$\Rightarrow$(b) is completed.

(b)$\Rightarrow$(c). Let A and B be two subcontinua of X. Take two arbitrary points $a_0 \in A$ and $b_0 \in B$ and consider the union $A \cup \overline{a_0b_0} \cup B$. In view of (b), this union is a convex dendrite, hence a strongly convex continuum and a bridge between A and B. Since moreover, as is easy to check (cf. [3], 2.5), balls $Q(A, \varepsilon)$ and $Q(B, \varrho^1(A, B)-\varepsilon)$ are both strongly convex for each $0 \leqslant \varepsilon \leqslant \varrho^1(A, B)$, then the common part

$$(A \cup \overline{a_0b_0} \cup B) \cap Q(A, \varepsilon) \cap Q(B, \varrho^1(A, B)-\varepsilon)$$

is strongly convex, hence a continuum. Thus it is shown that the hyperspace $C(X)$ has property (S).

Since X is complete by hypothesis, the space 2^X consisting of all non-empty bounded subsets of X and metrized by Hausdorff metric is also complete (see [11], I, p. 314), and by virtue of 4.2 and 4.4 the hyperspace $C(X)$ is closed in 2^X, hence complete itself (see [11], I, p. 315).

Now (c) follows by virtue of Theorem 3.2.

(c)$\Rightarrow$(a). Follows by Theorem 3.5.

The just proved Theorem 4.5 yields immediately

COROLLARY 4.6. *Let X be a metric continuum. Then the following conditions are equivalent:*

(a) *X is a dendrite with a convex metric,*

(b) *every subcontinuum of X is convex,*

(c) *$C(X)$ is convex.*

Remarks. As is known (see [1] or [15]), every locally connected metric continuum can be supplied with a convex metric. It follows then, in view of Corollary 4.6, that dendrites can be distinguished in the class of all metric continua X by the following characteristic property: there exists a metric in X such that the hyperspace $C(X)$ is convex.

Implication (b)$\Rightarrow$(c) of Corollary 4.6 has been known ([3], Theorem 3.2) and implication (c)$\Rightarrow$(b) is a positive answer to the problem P2 stated in [3]. Equivalence (a)$\Leftrightarrow$(c) is then an answer to the problem P1 from [3].

Corollary 4.6 says little on the topological structure of the hyperspace $C(X)$ for X being a dendrite. It is then perhaps worth to mention here that hyperspace $C(X)$ for X being *finite* dendrites were investigated in [5]. In particular, a topological characterization of these hyperspaces has been obtained there.

As Menger has proved (see [13], p. 92), if in a compact metric space X there is a convergent sequence of segments $\{\overline{a_nb_n}\}$ whose end-points a_n and b_n converge, then its limit is a segment in X between $\lim a_n$ and $\lim b_n$. The following proposition generalizes Menger's result to the case of a complete metric space.

4.7. *Let X be a complete metric space and let $\{a_n\}$ and $\{b_n\}$ be two sequences of points of X such that for each $n = 1, 2, \ldots$ there is a segment $\overline{a_nb_n} \subset X$. If $\lim a_n = a$, $\lim b_n = b$ and C is a closed subset of X such that*

$$\lim \varrho^1(\overline{a_nb_n}, C) = 0,$$

then C is a segment in X between a and b [1].

Proof. Since C is compact by 4.4, it suffices to apply Menger's result to the space $C \cup \bigcup \overline{a_nb_n}$.

4.8. *Let X be a complete metric space and let $A_0, A_1, A_2, \ldots$ be a sequence of compact subsets of X. If* (27) *holds true and each set $A_1, A_2, \ldots$ is convex, then A_0 is convex.*

---

(1) The lemma remains true after replacing hypothesis $\lim \varrho^1(\overline{a_nb_n}, C) = 0$ by a weaker one, $\operatorname{Lim} a_nb_n = C$, but the proof is different and somewhat more lengthy.

Proof. Take two arbitrary points $a, b \in A_0$ and choose two sequences of points $a_n, b_n \in A_n$ such that $a = \lim a_n$ and $b = \lim b_n$ (such sequences exist by virtue of (27) and formulas for ϱ^1).

By convexity of A_n there is a segment $\overline{a_n b_n} \subset A_n$, and since compactness of $A_0 \cup \bigcup A_n$ implies that of $C(A_0 \cup \bigcup A_n)$ (cf. [9]), then we may assume that the sequence of segments $\{\overline{a_n b_n}\}$ is convergent to, say, C. By virtue of 4.7, C is a segment between a and b, and since $\overline{a_n b_n} \subset A_n$ for each $n = 1, 2, \ldots$ and $\operatorname{Lim} A_n = A_0$ (see [7], p. 149; cf. also [11], I, p. 248), then $C \subset A_0$. Hence A_0 is convex.

THEOREM 4.9. *Let X be a metric space. Then*

(a) *X is complete if and only if* $\operatorname{Conv}(X)$ *is complete,*

(b) *X is compact if only if* $\operatorname{Conv}(X)$ *is compact,*

(c) *X is locally compact if and only if* $\operatorname{Conv}(X)$ *is locally compact.*

Proof. (a) If X is complete, then the space 2^X consisting of all non-empty bounded subsets of X and metrized by Hausdorff metric is also complete (see [11], I, p. 314). By virtue of 4.4 the hyperspace $\operatorname{Comp}(X)$ is closed in 2^X, hence complete (see [11], I, p. 315). And by virtue of 4.8 the hyperspace $\operatorname{Conv}(X)$ is closed in $\operatorname{Comp}(X)$, hence also complete.

The converse implication holds, because the hyperspace $X(1)$, isometric to X itself, is a closed subspace of the hyperspace $\operatorname{Conv}(X)$.

(b) If X is compact, then the hyperspace $\operatorname{Comp}(X)$ is compact (see [11], II, p. 21) and by 4.8 the hyperspace $\operatorname{Conv}(X)$ is closed in $\operatorname{Comp}(X)$, hence also compact.

The converse implication holds, because the hyperspace $X(1)$, isometric of X itself, is a closed subspace of the hyperspace $\operatorname{Conv}(X)$.

(c) If X is locally compact, then for every $A \in \operatorname{Conv}(X)$ there exists an $\varepsilon > 0$ such that $X \cap Q(A, \varepsilon)$ is compact. By virtue of formula (2), $X \cap Q(A, \varepsilon)$ contains all subsets B of X with the property $\varrho^1(A, B) \leqslant \varepsilon$, and so $\operatorname{Conv}(X \cap Q(A, \varepsilon))$ is a neighbourhood of A in $\operatorname{Conv}(X)$. In view of (b), $\operatorname{Conv}(X \cap Q(A, \varepsilon))$ is compact. Hence the hyperspace $\operatorname{Conv}(X)$ is locally compact.

Since the hyperspace $X(1)$, isometric to X itself, is a closed subspace of the hyperspace $\operatorname{Conv}(X)$, and a closed subspace of a locally compact space is locally compact itself (see [10], p. 146), the converse implication holds too.

Thus the proof of Theorem 4.9 is completed.

Remarks. The proof of Theorem 4.9 does not depend on results of § 2 and § 3.

Strange enough, the hyperspace $\operatorname{Conv}(X)$ does not preserve convexity. To show this consider first an example of the unit circle, $X = \{(x, y)\colon x^2 + y^2 = 1\}$, with the geodesic metric. Since any convex subcontinuum of X is either an arc of length not greater than π or X itself,

then one can show (cf. [4], § 3, example 2) that the hyperspace $\operatorname{Conv}(X)$ is here topologically equivalent to the union of an annulus $X \times I$ and of an isolated point. Hence $\operatorname{Conv}(X)$ need not be connected (to say nothing of convexity) even although X itself is convex. And the example of a continuum $X = (0, 0) \cup \bigcup_{k=0}^{\infty} S_k$, where S_k is the circle in the plane of center $(3/2^{k-2}, 0)$ and radius $1/2^{k+1}$ provided with the geodesic metric (X is again a convex continuum) shows that $\operatorname{Conv}(X)$ need not be even locally connected.

The problem of characterization of those metric spaces for which the hyperspace $\operatorname{Conv}(X)$ is convex can be fully answered, as will be shown in the next section, in the case of X being a subspace of a Euclidean space.

§ 5. Hyperspaces $\operatorname{Conv}(X)$ and $\overset{+}{\operatorname{Conv}}(X)$ in the Euclidean case. One of the most interesting is perhaps the case of X being a subset of a Euclidean space E^n and $\mathfrak{H}(X)$ being the hyperspace $\operatorname{Conv}(X)$ or $\overset{+}{\operatorname{Conv}}(X)$. The two hyperspaces have to-day a well developped theory going back to J. Steiner, H. Brunn and H. Minkowski. In its present shape the theory appears to refrain from any topological reference (cf. a neat presentation of it in the Hadwiger's book [6]), but it seems that it can profit even by a little of topologisation.

Some topological results on the hyperspace $\operatorname{Conv}(X)$ have been collected in the following

THEOREM 5.1. *Let X be a subspace of a Euclidean space E^n. Then*

(a) *X is complete if and only if* $\operatorname{Conv}(X)$ *is complete,*

(b) *X is compact if and only if* $\operatorname{Conv}(X)$ *is compact,*

(c) *X is locally compact if and only if* $\operatorname{Conv}(X)$ *is locally compact,*

(d) *X is convex if and only if* $\operatorname{Conv}(X)$ *is convex.*

Proof. (a), (b), and (c) — see Theorem 4.9 above.

(d) Let X be convex, and let A and B be any two elements of the hyperspace $\operatorname{Conv}(X)$. In the considered here Euclidean case the bridge $P(A, B)$ does exist (and is unique), and since X is convex by assumption, $P(A, B) \subset X$. Both A and B are convex, so is $P(A, B)$. As we have shown in § 2, $\operatorname{Conv}(P(A, B))$ has property (S), and by virtue of (b), $\operatorname{Conv}(P(A, B))$ is compact. Hence, in view of Theorem 3.2, it follows that $\operatorname{Conv}(P(A, B))$ is convex. In particular, it does contain a metric segment between A and B. Since inclusion $P(A, B) \subset X$ implies inclusion $\operatorname{Conv}(P(A, B)) \subset \operatorname{Conv}(X)$, this segment is contained in the hyperspace $\operatorname{Conv}(X)$. Hence the hyperspace $\operatorname{Conv}(X)$ is convex.

To prove the converse note that if p and q are any two points of E^n, then the only subsets of E^n which lie between (p) and (q) (in the sense of Hausdorff metric ϱ^1) are points. Indeed, for if $0 < \varepsilon < d = \varrho(p, q)$, and if Z is a subset of E^n such that $\varrho^1\big((p), Z\big) = \varepsilon$ and $\varrho^1\big(Z, (q)\big) = d-\varepsilon$, then by virtue of 3.3 there is $\varrho(p, z) = \varepsilon$ and $\varrho(z, q) = d-\varepsilon$ for each $z \in Z$. But since E^n is strongly convex, there is only one point $z \in E^n$ for which the last two equalities hold. Hence it must be $Z = (z)$. Therefore, if $p, q \in X$ and the hyperspace $\mathrm{Conv}(X)$ is convex, then the space X contains a metric segment between p and q. Thus it is shown that convexity of $\mathrm{Conv}(X)$ implies that of X.

Hence the proof of Theorem 5.1 is completed.

The hyperspace $\overset{+}{\mathrm{Conv}}(X)$ is clearly a subspace of the hyperspace $\mathrm{Conv}(X)$ and it is non-empty if and only if X has non-empty interior. But topological properties of $\overset{+}{\mathrm{Conv}}(X)$ are not so good as those of $\mathrm{Conv}(X)$. For instance, as an example of a sequence of concentric balls in X with diameters tending vers 0 shows, the hyperspace $\overset{+}{\mathrm{Conv}}(X)$ is neither compact nor complete independently of whether X is such or not. Nevertheless, we have the following

THEOREM 5.2. *Let X be a subspace of a Euclidean space E^n with the non-empty interior. Then*

(a) *$\overset{+}{\mathrm{Conv}}(X)$ is open in $\mathrm{Conv}(X)$,*

(b) *if X is locally compact, then so is $\overset{+}{\mathrm{Conv}}(X)$,*

(c) *if X is convex, then so is $\overset{+}{\mathrm{Conv}}(X)$,*

(d) *if X is open or convex, then $\overset{+}{\mathrm{Conv}}(X)$ is dense in $\mathrm{Conv}(X)$.*

Proof. (a) To show that $\overset{+}{\mathrm{Conv}}(X)$ is open in $\mathrm{Conv}(X)$ take an arbitrary $A \in \overset{+}{\mathrm{Conv}}(X)$ and for each straight line p in E^n passing through the origin O denote by $\omega(p)$ the width of A in the direction of p. Treating then such a p as a point of the projective space P^{n-1} and taking into account that $\omega(p)$ is a continuous function (see [6], p. 10) and that $\omega(p) > 0$ for each $p \in P^{n-1}$ we infer by the theorem of Weierstrass (see [11], II, p. 15) that

$$\omega_0 = \inf_{p \in P^{n-1}} \omega(p) > 0 . \tag{32}$$

The proof will be completed when we show that if

$$0 < \varepsilon < \tfrac{1}{2}\omega_0 , \tag{33}$$

then the ball $Q^1(A, \varepsilon)$ in $\mathrm{Conv}(X)$ with the center A and of the radius ε lies entirely in $\overset{+}{\mathrm{Conv}}(X)$. For that purpose take an arbitrary element B of $Q^1(A, \varepsilon)$, i.e., a compact convex subset of X satisfying

$$\varrho^1(A, B) \leqslant \varepsilon , \tag{34}$$

and suppose, *a contrario*, that $\mathrm{Int}(B) = 0$. This means that there exists an $(n-1)$-dimensional hyperplane H with

$$B \subset H . \tag{35}$$

Let $p_0 \in P^{n-1}$ be the straight line perpendicular to H, and let H_0 and H_1 be planes parallel to H and realizing the width $\omega(p_0)$. In view of (32) and (33),

$$\varepsilon < \tfrac{1}{2}\omega(p_0) . \tag{36}$$

By the definition of width there exist points $c_0 \in A \cap H_0$ and $c_1 \in A \cap H_1$, and since the three hyperplanes H, H_0 and H_1 are all parallel, then either $\varrho(c_0, H) \geqslant \frac{1}{2}\omega(p_0)$ or $\varrho(c_1, H) \geqslant \frac{1}{2}\omega(p_0)$. Whatever of the two inequalities holds, there is $\sup_{a \in A} \varrho(a, H) \geqslant \frac{1}{2}\omega(p_0)$, whence, in view of (35), $\sup_{a \in A} \varrho(a, B) \geqslant \frac{1}{2}\omega(p_0)$, and, consequently, $\varrho^1(A, B) \geqslant \frac{1}{2}\omega(p_0)$, contrary to (34) and (36).

(b) Since, as we have just proved, the hyperspace $\overset{+}{\mathrm{Conv}}(X)$ is an open subspace of the hyperspace $\mathrm{Conv}(X)$, and in the considered case the latter is, by virtue of Theorem 5.1 (c), locally compact, the former must be locally compact too ([10], p. 146).

(c) Since the hyperspace $\overset{+}{\mathrm{Conv}}(X)$ is not a complete metric space, then to prove its convexity we need more than mere statement that this hyperspace has property (S). Namely, for any two elements A and B of $\overset{+}{\mathrm{Conv}}(X)$ we shall construct a family $\{H_\gamma\}_{\gamma \in \Gamma}$, where Γ denotes the set of dyadic rationals of the real segment $[0, 1]$, of sets H_γ such that

$$H_0 = A \quad \text{and} \quad H_1 = B , \tag{37}$$

(38) if $\gamma \in \Gamma$, then H_γ is a compact convex subset of $P(A, B)$,

(39) if $\gamma_1, \gamma_2 \in \Gamma$, then $\varrho^1(H_{\gamma_1}, H_{\gamma_2}) = |\gamma_1 - \gamma_2| \cdot \varrho^1(A, B)$,

(40) there exists an $\alpha > 0$ such that the n-dimensional volume $v(H_\gamma) \geqslant \alpha$ for each $\gamma \in \Gamma$.

Put $A = H_0$ and $B = H_1$. By virtue of (4) there exist points $h_0 \in \mathrm{Int}(H_0)$ and $h_1 \in \mathrm{Int}(H_1)$ such that $\varrho(h_0, h_1) < \varrho^1(H_0, H_1)$. Since $h_0 \in \mathrm{Int}(H_0)$ and $h_1 \in \mathrm{Int}(H_1)$, there exists an $\eta > 0$ such that $Q(h_0, \eta) \subset H_0$ and $Q(h_1, \eta) \subset H_1$. Putting $\alpha = v\big(Q(h_0, \eta)\big)$ we infer that $v(H_\gamma) \geqslant \alpha$ for $\gamma = 1, 2$.

Let $h_{\frac{1}{2}}$ be the middle point of the segment $\overline{h_0 h_1}$. Obviously, $Q(h_{\frac{1}{2}}, \eta) \subset P\big(Q(h_0, \eta), Q(h_1, \eta)\big) \subset P(H_0, H_1)$. Since $Q(h_{\frac{1}{2}}, \eta)$ is a translation of $Q(h_0, \eta)$ for $\varrho(h_0, h_{\frac{1}{2}}) < \frac{1}{2}\varrho^1(H_0, H_1)$, then $Q(h_{\frac{1}{2}}, \eta) \subset Q\big(H_0, \frac{1}{2}\varrho^1(H_0, H_1)\big)$. Similarly, $Q(h_{\frac{1}{2}}, \eta) \subset Q\big(H_1, \frac{1}{2}\varrho^1(H_0, H_1)\big)$. Hence the compact convex set defined by the formula

$$H_{\frac{1}{2}} = P(H_0, H_1) \cap Q\big(H_0, \tfrac{1}{2}\varrho^1(H_0, H_1)\big) \cap Q\big(H_1, \tfrac{1}{2}\varrho^1(H_0, H_1)\big)$$

lies, in view of 3.1, in the middle between H_0 and H_1, and contains $Q(h_{\frac{1}{2}}, \eta)$. Consequently, $v(H_{\frac{1}{2}}) \geqslant \alpha$. Moreover, $\varrho(h_0, h_{\frac{1}{2}}) < \varrho^1(H_0, H_{\frac{1}{2}})$ and $\varrho(h_{\frac{1}{2}}, h_1) < \varrho^1(H_{\frac{1}{2}}, H_1)$.

In analogous manner one defines now sets $H_{\frac{1}{4}}$ and $H_{\frac{3}{4}}$, and by an easy induction the whole family $\{H_\gamma\}_{\gamma\in\Gamma}$ of compact convex sets satisfying (37)–(40).

Now, since the bridge $P(A, B)$ is compact convex, so is, in view of Theorem 5.1 (b) and (d), its hyperspace $\mathrm{Conv}(P(A, B))$. Hence in view of (37)–(39) the closure of the family $\{H_\gamma\}_{\gamma\in\Gamma}$ in the hyperspace $\mathrm{Conv}(P(A, B))$ is a metric segment between A and B ([13], p. 87–89). However, if C is an element of this segment, then in view of (40) and of the continuity of volume there must be $v(C) \geqslant \alpha$. In other words, $C \in \overset{+}{\mathrm{Conv}}(P(A, B))$. And since $P(A, B) \subset X$ by the convexity of X, then $\overset{+}{\mathrm{Conv}}(P(A, B)) \subset \overset{+}{\mathrm{Conv}}(X)$, and so this metric segment between A and B lies in $\overset{+}{\mathrm{Conv}}(X)$. Hence the hyperspace $\overset{+}{\mathrm{Conv}}(X)$ is convex.

(d) Let X be an open subset of E^n. If A is any given element of the hyperspace $\mathrm{Conv}(X)$, then $\varepsilon_A = \varrho(A, X\setminus A) > 0$. Consequently, $0 < \varepsilon < \varepsilon_A$ implies $Q(A, \varepsilon) \subset X$. Hence $Q(A, \varepsilon) \in \overset{+}{\mathrm{Conv}}(X)$ and, as follows from (2), $\varrho^1(A, Q(A, \varepsilon)) = \varepsilon$.

And if X is convex and A is again any element of $\mathrm{Conv}(X)$, then take a point p in the interior of X at a distance from A, say ε (such a point surely exists since $\mathrm{Int}(X)$ is dense in X), and a ball $Q(p, \eta)$ of radius $\eta \leqslant \varepsilon$. Then the bridge $P(A, Q(p, \eta))$ is compact convex, lies inside X, and has non-empty interior. Hence $P(A, Q(p, \eta)) \in \overset{+}{\mathrm{Conv}}(X)$ and it is easy to check that $\varrho^1(A, P(A, Q(p, \eta))) \leqslant 2\varepsilon$.

Remarks. None of the implications of Theorem 5.2 can be reversed. In fact, consider in the plane E^2 a sequence of pairwise disjoint quadrangles converging to a segment and let X be the union of all these quadrangles and of one point of the segment to which these quadrangles converge. It is easy to check that $\overset{+}{\mathrm{Conv}}(X)$ is locally compact (but X is not, hence the converse of (b) does not hold) and that $\overset{+}{\mathrm{Conv}}(X)$ is dense in $\mathrm{Conv}(X)$ (but X is neither open nor convex, hence the converse of (d) does not hold either). And the example of an open half-plane with two points of its boundary adjoined shows that also the converse of (c) does not hold.

Part (c) of Theorem 5.2 has been known for $X = E^n$. Namely, Shephard and Webster have pointed out that the segment in $\overset{+}{\mathrm{Conv}}(X)$ between its two elements A and B with the distance $\varrho^1(A, B) = 1$ is defined by the Minkowski linear system $f\colon [0, 1] \to \overset{+}{\mathrm{Conv}}(X)$, where $f(t) = (1-t)A + tB$ ([17], theorem (23)). It is to be noted, however, that

$$(1-t)A + tB = P(A, B) \cap Q(A, 1-t) \cap Q(B, t)\,,$$

and so in this particular case their idea overlaps with that of ours (cf. 3.1 above).

An easy consequence of Theorem 5.1 (b) is the following well known (see [2], § 18) and for the theory of convex sets most important

5.3. Auswahlsatz of Blaschke. *Any sequence of compact convex sets lying in a bounded region of E^n contains a subsequence convergent to a compact convex subset of E^n.*

In fact, to infer it we need compactness of the hyperspace $\mathrm{Conv}(X)$ only, where X is a sufficiently large ball in E^n containing that sequence.

Note that Blaschke's Auswahlsatz holds in a much more general setting: in view of Theorem 4.9 (b) it remains true for a sequence of compact convex sets lying in an *arbitrary* compact space (compare this to [18] and [8], 5.2).

Similarly easy consequence of Theorem 5.1 (c) and of Theorem 5.2 (b) is the following theorem proved first by Hadwiger in the case of $\mathrm{Conv}(E^n)$ for real continuous functions (see [6], p. 21).

5.4. *Let X be a locally compact subspace of a Euclidean space E^n. If f is a continuous mapping from either* $\mathrm{Conv}(X)$ *or* $\overset{+}{\mathrm{Conv}}(X)$ *into a metric space, then f is locally uniformly continuous.*

In fact, a continuous mapping from a compact metric space into a metric space is by a theorem of Heine uniformly continuous ([11], II, p. 16), and in the considered case each element of $\mathrm{Conv}(X)$ and of $\overset{+}{\mathrm{Conv}}(X)$ has a compact neighbourhood.

Minkowski theorem on approximation of a convex set in E^n by polyhedra ([14], § 5; see also [2], § 17, and [6], p. 23) can be stated in the form

5.5. Minkowski theorem. *If X is a convex subspace of a Euclidean space E^n, then the family $\mathfrak{P}(X)$ consisting of convex polyhedra contained in X is dense in the hyperspace* $\mathrm{Conv}(X)$.

This statement implies more precise statements of Minkowski theorem as those given in [2], § 17 (note that $\mathrm{Conv}(X)$ can be replaced here by $\overset{+}{\mathrm{Conv}}(X)$ and that the hyperspace $\mathfrak{P}(X)$ is convex itself). Also other theorems on approximation (cf. [16], [8], etc.) can be stated in a like form.

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Concerning the ordering of shapes of compacta

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The purpose of this note is to give answers to one question concerning the existence of maximal shapes (majorants) and to another question, concerning the existence of continuously ordered families of shapes.

§ 1. Basic definitions. Let X, Y be two compacta lying in the Hilbert cube Q. A sequence of maps $f_k\colon Q\to Q$ is said to be a *fundamental sequence from X to Y* (notation: $\{f_k, X, Y\}$, or $\underline{f}\colon X\to Y$. Compare [2], p. 225) if for every neighborhood V of Y there is a neighborhood U of X such that

$$f_k/U \simeq f_{k+1}/U \quad \text{in } V \text{ for almost all } k\,.$$

In particular, if f_k is the identity map of Q onto itself for every $k=1,2,\ldots$, then $\{f_k, X, X\}$ is said to be the *fundamental identity sequence $\underline{i}_X$*. Two fundamental sequences $\underline{f}=\{f_k, X, Y\}$ and $\underline{g}=\{g_k, X, Y\}$ are said to be *homotopic* (notation: $\underline{f}\simeq\underline{g}$) if for every neighborhood V of Y there is a neighborhood U of X such that

$$f_k/U \simeq g_k/U \quad \text{in } V \text{ for almost all } k\,.$$

The family of all fundamental sequences homotopic to a given fundamental sequence $\underline{f}\colon X\to Y$ is said to be the *fundamental class* $[\underline{f}]$ *from X to Y*.

If X, Y, Z are compacta lying in Q and if $\underline{f}=\{f_k, X, Y\}$, $\underline{g}=\{g_k, Y, Z\}$ are fundamental sequences, then $\{g_kf_k, X, Z\}$ is a fundamental sequence, called the *composition* of $\underline{f}$ and $\underline{g}$; it is denoted by $\underline{g}\underline{f}$.

Two compacta X, Y (not necessarily lying in Q) are said to be of the same *shape* (notation: $\mathrm{Sh}(X)=\mathrm{Sh}(Y)$ (see [4]) if there are two compacta $X', Y'\subset Q$, homeomorphic to X and Y respectively, and two fundamental sequences $\underline{f}\colon X'\to Y'$, $\underline{g}\colon Y'\to X'$ such that $\underline{g}\underline{f}\simeq\underline{i}_{X'}$ and $\underline{f}\underline{g}\simeq\underline{i}_{Y'}$. If we assume only that $\underline{f}$ and $\underline{g}$ satisfy the first of those homotopies, then $\mathrm{Sh}(X)$ is said to be *not greater* than $\mathrm{Sh}(Y)$ (notation: $\mathrm{Sh}(X)\leqslant\mathrm{Sh}(Y)$). If $\mathrm{Sh}(X)\leqslant\mathrm{Sh}(Y)$, but the relation $\mathrm{Sh}(Y)\leqslant\mathrm{Sh}(X)$ does not hold, then we say that $\mathrm{Sh}(X)$ is *less* than $\mathrm{Sh}(Y)$ and we write $\mathrm{Sh}(X)<\mathrm{Sh}(Y)$.