§ 1. Introduction. Let $X$ be a metric space with a metric $q$. By a hyperspace of $X$ we shall mean any family $\mathcal{S}(X)$ consisting of non-empty compact subsets of $X$. Of particular importance will be the hyperspace $\mathcal{C}(X)$ consisting of all non-empty compact subsets of $X$, the hyperspace $\mathcal{O}(X)$ consisting of all non-empty compact and connected subsets (i.e., of all subcontinua) of $X$, the hyperspace $\mathcal{Conv}(X)$ consisting of all non-empty compact and convex subsets of $X$, and the hyperspace $\mathcal{X}(1)$ consisting of all one-point subsets of $X$.

Evidently, $\mathcal{X}(1) \subseteq \mathcal{Conv}(X) \subseteq \mathcal{O}(X) \subseteq \mathcal{C}(X)$.

In the case of $X$ being a subset of an $n$-dimensional Euclidean space $E^n$ of importance will be also the hyperspace $\mathcal{Conv}(X) = \{A \in \mathcal{Conv}(X) : \text{Int}(A) \neq \emptyset\}$.

Recall that a subset $A$ of $X$ is called convex if, given any two points $a$ and $b$ of $A$, the set $A$ contains a metric segment $a$ and $b$ (not necessarily one only). If for every two points $a$ and $b$ of $A$, the set $A$ contains exactly one metric segment $a$ and $b$, $A$ is called strongly convex.

As is well known (cf. [13], pp. 87-89), a complete metric space $X$ is convex if and only if for every two points $a$ and $b$ of $A$ there exists a point $c \in X$ which lies between $a$ and $b$ (i.e., $q(a,c)+q(c,b) = q(a,b)$) and is distinct from both $a$ and $b$. This result will be applied below several times without reference.

Each hyperspace of $X$ will be considered as a metric space with the Hausdorff metric $q^h$ defined by the formula ([7], p. 291, see also [11], I, p. 106).

\begin{align*}
q^h(A,B) &= \max_{a \in A, b \in B} \{ \sup_{x \in A} q(a,x) + \sup_{y \in B} q(y,b) \},
\end{align*}
where $q(a,z) = \inf_{z \in x} q(x,z)$.

Formula (1) is easily equivalent to the formula

\begin{align*}
q^h(A,B) &= \inf_{\eta \geq 0} \{ A \subseteq Q(B,\eta) \text{ and } B \subseteq Q(A,\eta) \},
\end{align*}
where $Q(Z, \eta)$ denotes a generalized metric ball in $X$, i.e.,

$$Q(Z, \eta) = \{x \in X : d(x, Z) \leq \eta\}.$$ 

Similarly, $Q(C, \epsilon)$ will denote a metric ball in $\text{Comp}(X)$ with the center $C \in \text{Comp}(X)$ and radius $\epsilon \geq 0$, i.e.,

$$Q(C, \epsilon) = \{D \in \text{Comp}(X) : d(C, D) \leq \epsilon\}.$$ 

As follows from (1), $X(1)$ is isometric to $X$ itself and it is known that if $X$ is a continuum, then so are $\text{Comp}(X)$ and $O(X)$ (see [9], p. 25). However, it is not the case with $\text{Conv}(X)$ (see § 4 below).

The paper is devoted to some questions concerning convexity of hyperspaces. In § 2 we define a certain property of hyperspaces, related to convexity and called property (S). Some relations between that property and convexity of hyperspaces are established in § 3, and theorems of § 3 yield a sequence of corollaries. Thus, for instance, it is shown in § 4 that the hyperspace $\text{Conv}(X)$ is a compact $X$; if and only if the underlying compact metric space $X$ is a convex $X$; if and only if the underlying metric continuum $X$ is a dendrite with a convex metric, and topological properties of the underlying metric space $X$ like completeness, compactness, and local compactness are preserved by the hyperspace $\text{Conv}(X)$. Results of § 5 pertain to the hyperspaces $\text{Comp}(X)$ and $\text{Conv}(X)$ in the cases of $X$ being a subspace of a Euclidean space $E^n$.

Theorems of § 6 seem to form a natural topological background for considering convex subsets of $E^n$. It turns out that some known theorems of convex sets in $E^n$ proved so far by direct metric considerations (like Ambiaht's theorem of Blaschke, theorem of Hadwiger, etc.) follow easily and some other (like Minkowski's theorem on approximation by polyhedra) receive a clear topological formulation.

Notations and notation not defined in the paper come from [11] and [12].

§ 2. Property (S). Let $A$ and $B$ be two compact subsets of a metric space $X$ such that for each two points $a \in A$ and $b \in B$ there is at least one segment $a b$ in $X$ (for $a = b$ this segment is reduced to one point). We call a bridge in $X$ between $A$ and $B$, and denote it by $P(A, B)$, any compact union of segments $a b$, i.e., a compact set containing at least one segment $a b$ with each pair of points $a \in A$ and $b \in B$ such that each point of it lies on a segment $a b$, where $a \in A$ and $b \in B$ (§ 3), p. 24).

The following proposition is known to be true (see [3], 2.1).

2.1. If $X$ is a compact convex space, then for every pair $A, B \in \text{Comp}(X)$ a bridge $P(A, B)$ does exist.

Remarks. If the space $X$ is not strongly convex, bridge $P(A, B)$ need not be unique.

And if $X$ is a complete convex space, Proposition 2.1 is false. Indeed, consider Cartesian product $O \times X$ of the Cantor ternary set $O$ and the segment $I = [0, 1]$, and identify each pair of points $(c_1, i)$ and $(c_2, i)$ such that $c_1$ and $c_2$ are endpoints of a segment complementary to $O$ and $i = 0$ or $i = 1$. After identification the set $O \times 0$ becomes a segment, denote it by $A$, and the set $O \times 1$ becomes a segment disjoint with $A$, denote it by $B$; the set $X$ obtained in this way from $O \times I$ consists of uncountably many arcs joining points of $A$ to points of $B$ and otherwise disjoint. If we metricize now $X$ according to the length of an arc, assuming that each segment $x \times I$ is now an arc of length 1, then $X$ becomes a complete metric space and $d(A, B) = 1$. Evidently, any bridge between $A$ and $B$ must contain each "arc $x \times I'$ and thus the only bridge between $A$ and $B$ is the whole space $X$ which is not compact.

A hyperspace $\mathfrak{S}(X)$ of $X$ will be said to have property (S) if for each pair of sets $A, B \in \mathfrak{S}(X)$ a bridge $P(A, B)$ in $X$ does exist (however, we do not assume it to be an element of $\mathfrak{S}(X)$) and for some $\epsilon$ satisfying inequalities $0 < \epsilon < d(A, B)$ the set

$$P(A, B) \cap Q(A, \epsilon) \cap Q(B, d(A, B) - \epsilon)$$

belongs to $\mathfrak{S}(X)$.

Some examples of hyperspaces possessing property (S):

1. If $X$ is a convex continuum, there are in general many hyperspaces of $X$ which have property (S). Such is, for instance, each hyperspace $\mathfrak{S}(X)$ which satisfies the following three conditions:

(i) if $A$ and $B$ are in $\mathfrak{S}(X)$, then also $Q(A, \eta) \cup Q(B, \eta)$ for each $\eta \geq 0$,

(ii) if $A, B \in \mathfrak{S}(X)$, then there exists a bridge $P(A, B) \subseteq X$,

(iii) if $A, B \in \mathfrak{S}(X)$, then also $A \cup B \in \mathfrak{S}(X)$.

For each family $\mathfrak{S} \subseteq \text{Comp}(X)$ one may consider the least hyperspace $\mathfrak{S}(X)$ containing $\mathfrak{S}$ and satisfying conditions (i)-(iii). In view of 2.1 such a hyperspace does exist and it has property (S), of course. It can be shown without much difficulty that if $\mathfrak{S}$ is closed in $\text{Comp}(X)$, the hyperspace $\mathfrak{S}(X)$ is closed in $\text{Comp}(X)$, hence compact.

The hyperspace $X(1)$ and the hyperspace $\text{Comp}(X)$ both have property (S).

2. If $X$ is a convex continuum such that each subcontinuum of $X$ is also convex (i.e., if $O \subseteq X = \text{Conv}(X)$), then the hyperspace $X(1)$ has property (S).

In fact, if $A, B \in O \subseteq \text{Conv}(X)$, then a bridge $P(A, B)$ in $X$ does exist by virtue of 2.1, and all three sets $P(A, B), Q(A, \epsilon)$ and $Q(B, d(A, B))$ are strongly convex for each $0 < \epsilon < d(A, B)$ (see [3], 3.5). Hence the common part (S) is compact convex and so it must be an element of $O(X)$.

3. If $X$ is a convex subspace of an $n$-dimensional Euclidean space $E^n$, then the two hyperspaces $\text{Conv}(X)$ and $\text{Conv}(X)$ both have property (S).
In fact, if \( A \) is compact convex, then \( Q(A, \varepsilon) \) is compact convex for each \( \varepsilon > 0 \), because \( Q(A, \varepsilon) \) is equal to the Minkowski union, \( Q(A, \varepsilon) = A + Q(p, \varepsilon) \), of two compact convex sets and such a union is known to be compact convex ([6], p. 12-13). Therefore the common part (3) is compact convex, and since \( P(A, B) \) is contained in \( X \) by the convexity of \( X \), then (3) is an element of \( \text{Conv}(X) \). Hence the hyperspace \( \text{Conv}(X) \) has property (S).

To prove that also the hyperspace \( \text{Conv}(X) \) has property (S) it remains to show that if \( A, B \in \text{Conv}(X) \), then the common part (3) has non-empty interior. To that end show first that

\[
\text{(4)} \quad \text{if } A, B \in \text{Conv}(X), \text{ then there exist points } a \in \text{Int}(A) \text{ and } b \in \text{Int}(B) \text{ such that } g(a, b) < \varphi(A, B).
\]

Indeed, let \( a \) be any point of \( \text{Int}(A) \). Since, in view of formula (2), \( A \subset Q(B, \varphi(A, B)) \), then each point of \( A \) lying on the boundary of \( Q(B, \varphi(A, B)) \) belongs to the boundary of \( A \). Hence \( a \) is not on the boundary of \( Q(B, \varphi(A, B)) \) and so \( g(a, b) < \varphi(A, B) \). Consequently, there must exist a point \( b \in \text{Int}(B) \) such that \( g(a, b) < \varphi(A, B) \). Assumption \( B \in \text{Conv}(X) \) implies that \( \text{Int}(B) \) is dense in \( B \) and so there exists a point \( b \in \text{Int}(B) \) such that \( g(a, b) < \varphi(A, B) \). Therefore, by the triangle inequality, \( g(a, b) < g(a, b_0) + g(b_0, b) < \varphi(A, B) \).

Hence (4) is established. Assume that \( a \) and \( b \) satisfy (4). Since \( a \in \text{Int}(A) \) and \( b \in \text{Int}(B) \), there exists a \( \eta > 0 \) such that \( Q(a, \eta) \subset A \) and \( Q(b, \eta) \subset B \). Let \( c \in \text{Int}(A) \) be a point of the segment \( ab \) such that \( g(a, c) < \varepsilon \) and \( g(c, b) < \varphi(A, B) - \varepsilon \), where \( 0 < \varepsilon < \varphi(A, B) \). Obviously, \( Q(c, \eta) \subset P(Q(a, \eta), Q(b, \eta)) \subset P(A, B) \). Since \( Q(c, \eta) \) is a translation of \( Q(a, \eta) \) for less than \( \varepsilon \), then \( Q(c, \eta) \subset Q(A, \varepsilon) \). Similarly, \( Q(c, \eta) \subset Q(B, \varphi(A, B) - \varepsilon) \).

Hence the common part (3) contains \( Q(c, \eta) \) and, consequently, belongs to \( \text{Conv}(X) \). Thus it is shown that also the hyperspace \( \text{Conv}(X) \) has property (S).

§ 3. Convexity of hyperspaces. We shall now show several theorems on relations between convexity of hyperspaces and convexity of the underlying space.

To this purpose recall first a lemma ([32], 2.3).

3.1. Let \( X \) be a metric space and let \( A \) and \( B \) be two compact subsets of \( X \) such that there exists a bridge \( P(A, B) \) in \( X \) between \( A \) and \( B \). If \( \varepsilon \) is a number such that \( 0 < \varepsilon < \varphi(A, B) \), then the set \( H = P(A, B) \cap Q(A, \varepsilon) \cap Q(B, \varphi(A, B) - \varepsilon) \) satisfies the conditions

\[
\varphi(A, H) = \varepsilon \quad \text{and} \quad \varphi(H, B) = \varphi(A, B) - \varepsilon.
\]

This lemma leads to the following

**Theorem 3.2.** Let \( S(X) \) be a hyperspace of a metric space \( X \). If \( S(X) \) is complete and has property (S), then \( S(X) \) is convex.

Note, however, that for some metric spaces \( X \) there exist hyperspaces which are complete and convex but do not possess property (S). Such is, for instance, the hyperspace of \( P^2 \) consisting of all circles of diameter 1.

Theorem 3.2 yields a sequence of corollaries on particular hyperspaces of particular spaces (cf. § 4 and § 5 below). Here, however, we shall yet prove two more theorems.

If the proof of the first theorem we shall need a simple lemma which states that a set lies between two points (in the sense of metric \( g \)) if and only if each point of it lies between them (in the sense of metric \( g \)).

More precisely,

3.3. Let \( X \) be a metric space, \( p \) and \( q \) two points of \( X \), and \( Z \) a subset of \( X \). If \( \varepsilon \) is a number such that \( 0 < \varepsilon < g(p, q) \), then the following two conditions are equivalent:

\[
\text{(5)} \quad \varphi(p, Z) = \varepsilon \quad \text{and} \quad \varphi(Z, q) = \varphi(p, q) - \varepsilon,
\]

\[
\text{(6)} \quad g(p, Z) = \varepsilon \quad \text{and} \quad g(Z, q) = g(p, q) - \varepsilon \text{ for each } z \in Z.
\]

**Proof.** Assume (5) and let \( z \in Z \). By the definition of Hausdorff metric (see (1) or (2)), \( \varphi(p, Z) = \varepsilon \) implies \( g(p, z) < \varepsilon \) and, in view of \( \varphi(p, q) = g(p, q) \), the second equality of (5) implies \( g(z, q) < g(p, q) - \varepsilon \).

And if for some \( z \in Z \) we would have \( g(p, z) < \varepsilon \) or \( g(z, q) < g(p, q) - \varepsilon \), then, by the triangle inequality, \( g(p, g) < g(p, z) + g(z, q) < \varepsilon + g(p, q) - \varepsilon \), which is clearly impossible.

Hence (5) implies (6).

The converse implication follows easily by the definition of Hausdorff metric.

**Theorem 3.4.** If \( X \) is a complete metric space, then the following three conditions are equivalent:

(a) \( X \) is convex,

(b) there exists a hyperspace of \( X \) which is a complete metric space, has property (S) and contains \( X(1) \),

(c) there exists a hyperspace of \( X \) which is convex and contains \( X(1) \).

**Proof.** (a) \( \Rightarrow \) (b). If \( X \) is convex, then the hyperspace \( X(1) \) has property (S) and since \( X(1) \) is isometric to \( X \) itself, \( X(1) \) is complete.

(b) \( \Rightarrow \) (c). This implication follows by Theorem 3.2.

(c) \( \Rightarrow \) (a). Let \( S(X) \) be a hyperspace of \( X \) which is convex and contains \( X(1) \), and let \( p \) and \( q \) be any two points of \( X \). Since \( g(p, q) \) are both in \( S(X) \), the hyperspace \( S(X) \) contains a segment between \( p \) and \( q \) composed of subsets of \( X \). It means that the inequality \( 0 < \varepsilon \)
< \phi(\psi(q, q)) \) implies existence of a set \( Z \subseteq X \) such that (5) holds. By virtue of 3.3, there exists then a point \( x \in X \) which lies between \( p \) and \( q \) and is distinct from both. Hence \( X \) is convex.

**Theorem 3.5.** Let \( X \) be a complete metric space. If \( \bar{S}(X) \) is a convex hyperspace of \( X \), contained in \( G(X) \), and containing all subharmonics of \( X \), then the underlying space \( X \) contains no simple closed curve, is convex, and each element of \( \bar{S}(X) \) is convex.

**Proof.** By virtue of Theorem 3.4 space \( X \) is convex.

If we shall show that each element of \( \bar{S}(X) \) is convex, it will imply that \( X \) cannot contain any simple closed curve. In fact, for if \( X \) would contain a simple closed curve \( S \), there would also exist an arc \( L \subseteq S \) which is not convex (3.3, 2.4). But since \( L \subseteq \bar{S}(X) \) by hypothesis, \( L \) must be convex. A contradiction.

It remains then to show convexity of each element of \( \bar{S}(X) \). Suppose, by \( A \), that there exists an element \( A \subseteq \bar{S}(X) \) which is not convex. Since \( X \) is convex, there exist two points \( p, q \in A \) such that for each metric segment \( pq \) there is \( pq \subseteq A = \emptyset \). If there are two such segments, choose a simple closed curve in their union. If there is only one segment \( pq \), join \( p \) to \( q \) with an arc in a sufficiently small (not to enclose \( pq \)) ball \( (A, \eta) \) (it itself may `a priori' not contain such an arc, but since \( X \) is locally convex, each ball \( (A, \eta) \) does), and choose a simple closed curve in the union of that arc and of the segment \( pq \). In any of the two cases we receive a simple closed curve \( S \) which contains a rectilinear segment.

Choose two distinct points, \( p_1 \) and \( p_2 \), inside a segment of \( S \) and denote by \( P_1 \) the segment of that segment which has length \( z > 0 \) and the middle point of which is \( p_1 \), \( i = 1, 2 \). Let \( a \) be a positive number such that segments \( P_1 \) and \( P_2 \) exist and are disjoint. The set \( S, (P_1 \cup P_2) \) consists of two components; choose a point \( s_1 \) in each of them, \( i = 1, 2 \).

If \( 0 < a < a_1 \), then by \( S \) we shall mean the component of \( S, (P_1 \cup P_2) \) which contains \( s_1 \).

First we show that there exists a number \( a \) such that \( 0 < a < a_1 \) and

\[
(7) \quad \phi(P_1, S \setminus P_2) > \frac{\alpha}{2} \quad \text{for both} \quad i = 1 \quad \text{and} \quad i = 2,
\]

and

\[
(8) \quad \phi(S_1, S_2) > \frac{\alpha}{2}.
\]

Indeed, for if, contrary to (7), we would have \( \phi(P_1, S \setminus P_2) < \frac{\alpha}{2} \) for each \( a < a_1 \) and some \( i = 1 \) or 2, then taking a sequence \( \{a_n\}_{n=1}^{\infty} \) of positive numbers \( a_n < a_1 \) converging to 0, we would be able to choose sequences of points

\[
(9) \quad \begin{cases} x_n \in P_1, & n = 1, 2, \ldots \\ y_n \in S \setminus P_1, & n = 1, 2, \ldots \end{cases}
\]

such that

\[
(10) \quad \phi(x_n, y_n) \leq \frac{\alpha_n}{2} \quad \text{for} \quad n = 1, 2, \ldots
\]

By virtue of (9) and of the definition of \( P_1 \) we infer that \( \phi(x_n, y_n) \leq \frac{\alpha_n}{2} \) for \( n = 1, 2, \ldots \). Whence and from (11) it follows, by virtue of the triangle inequality, that \( \phi(p_1, p_2) \leq \alpha_n \) for \( n = 1, 2, \ldots \). Hence the sequence of points \( \{p_n\}_{n=1}^{\infty} \) is convergent to \( p_1 \) and, this implies, in view of (10), that \( p_1 \in S \setminus P_1 \). A contradiction to \( p_1 \) being the middle point of the segment \( P_1 \).

To prove (8), suppose, by \( A \), again, that \( \phi(S_1, S_2) < \frac{\alpha}{2} \) for each \( a < a_2 \). Taking now, as before, a sequence \( \{a_n\}_{n=1}^{\infty} \) converging to 0 and consisting of positive numbers \( a_n < a_2 \), we would be able to choose sequences of points

\[
(12) \quad \begin{cases} x_i \in S_1, & n = 1, 2, \ldots \\ y_i \in S_2, & n = 1, 2, \ldots \end{cases}
\]

such that

\[
(13) \quad \phi(x_i, y_i) < \frac{\alpha_n}{2} \quad \text{for} \quad n = 1, 2, \ldots
\]

Since \( S \) is compact, sequences \( \{x_i\}_{n=1}^{\infty} \) and \( \{y_i\}_{n=1}^{\infty} \) contain subsequences convergent, in view of (13), to a common limit point \( z \in S \).

Without loss of generality we may assume that \( \lim x_i = \lim y_i = z \).

Let \( S_i \) be that subarc of \( S \) which contains \( x_i \) and has end-points \( p_1 \) and \( p_2 \), \( i = 1, 2 \). From (12) and from obvious inclusions \( S_1 \subset S_i \), where \( a_i < a_2 \), we infer that \( z \in S_i \) for \( n = 1, 2, \ldots \) and for \( i = 1, 2 \). Since \( S_i \) is closed in \( S \), the limit point \( z \) must belong to both \( S_i \) and \( S_i \). Thus \( z = p_1 \) or \( z = p_2 \).

Suppose \( z = p_1 \) (case \( z = p_2 \) is analogous). Since both sequences \( \{x_i\}_{n=1}^{\infty} \) and \( \{y_i\}_{n=1}^{\infty} \) are convergent to \( z = p_1 \) by exposition, there must exist an index \( n_0 \) such that

\[
(14) \quad x_i \in P_1 \cap S \setminus P_2 \quad \text{for each} \quad n > n_0 \quad \text{and for} \quad i = 1, 2.
\]

However, the segment \( P_1 \) is rectilinear and so (14) implies that

\[
\phi(x_i, y_i) \geq \delta(P_1 \cap (S_1 \cup S_2)) = \delta(P_1) = a_0
\]

for each \( n > n_0 \), contrary to (13).
Hence both (7) and (8) are proved, and in the sequel we assume that \( a \) is fixed and satisfies (7) and (8).

Put

\[
A = \overline{S'} \cup P_1 \cup S'' \quad \text{and} \quad B = \overline{S'} \cup P_2 \cup S''.
\]

Both \( A \) and \( B \) are arcs and so elements of \( \mathcal{C}(X) \), and we shall proceed to show that there is no subcontinuum \( H \) of \( X \) which lies in the middle between \( A \) and \( B \). In view of the hypotheses \( \mathcal{C}(X) \subset \mathcal{C}(X) \) this will be a contradiction to the convexity of \( \mathcal{C}(X) \).

With this end in mind we show first that

\[
\tag{15}
q'(A, B) = \frac{a}{2}.
\]

Indeed, for if \( a \cdot A \) and \( g(a, B) > 0 \), then \( a \in P_1 \), and so

\[
\sup_{a \in A} g(a, B) = \sup_{a \in P_1} g(a, B).
\]

Since \( B \) can be written in the form \( B = \overline{P_1} \cup \overline{S'} \cup P_2 \), then

\[
\sup_{a \in A} g(a, B) = \min\{\sup_{a \in P_1} g(a, P_1), \sup_{a \in \overline{S'}} g(a, \overline{S'}), \sup_{a \in P_2} g(a, P_2)\}. \tag{16}
\]

And since, in view of \( a < a_0 \) and the rectilinearity of \( P_1 \), we have

\[
\sup_{a \in P_1} g(a, P_1) = \frac{a}{2}, \tag{17}
\]

d and, by virtue of (7), also

\[
\sup_{a \in P_2} g(a, P_2) > \frac{a}{2}. \tag{18}
\]

then taking into account both (17) and (18) we infer from (16) that

\[
\sup_{a \in A} g(a, B) = \frac{a}{2}. \tag{19}
\]

Analogous argument works to the effect that

\[
\sup_{a \in \overline{S'}} g(a, \overline{S'}) > \frac{a}{2}. \tag{20}
\]

and the two equalities, (19) and (20), yield (15).

Now we show that

\[
\tag{21}
q\left(\frac{a}{2}, \frac{a}{4}\right) = q\left(\frac{a}{4}, \frac{a}{2}\right) = q\left(\frac{a}{4}, \frac{a}{2}\right) \cup q\left(\frac{a}{4}, \frac{a}{2}\right).
\]

In fact, in view of the definitions of \( A \) and \( B \) we have

\[
q\left(\frac{a}{2}, \frac{a}{4}\right) = q\left(\frac{a}{4}, \frac{a}{2}\right) \cup q\left(\frac{a}{2}, \frac{a}{4}\right) \cup q\left(\frac{a}{4}, \frac{a}{2}\right),
\]

\[
q\left(\frac{a}{4}, \frac{a}{2}\right) = q\left(\frac{a}{2}, \frac{a}{4}\right) \cup q\left(\frac{a}{4}, \frac{a}{2}\right) \cup q\left(\frac{a}{4}, \frac{a}{2}\right).
\]

Hence, if there would exist a point

\[
p \in q\left(\frac{a}{4}, \frac{a}{2}\right) \cap q\left(\frac{a}{4}, \frac{a}{2}\right) \cap q\left(\frac{a}{4}, \frac{a}{2}\right),
\]

then we would have \( p \in q\left(\frac{a}{4}, \frac{a}{2}\right) \cap q\left(\frac{a}{4}, \frac{a}{2}\right) \), whence, consequently,

\[
\tag{22}
q\left(\frac{a}{4}, \frac{a}{2}\right) = \frac{a}{2}.
\]

But this is impossible, because \( P_1 \subset P_2 \subset \overline{S'} \cup P_2 \) and so, in view of (7), \( q\left(\frac{a}{4}, \frac{a}{2}\right) = \frac{a}{4} \) contrary to (22). Hence (21) holds.

Note also that

\[
q\left(\frac{a}{4}, \frac{a}{2}\right) \cap q\left(\frac{a}{4}, \frac{a}{2}\right) = 0,
\]

since otherwise there would be \( q\left(\frac{a}{4}, \frac{a}{2}\right) = \frac{a}{4} \), contrary to (8).

Let \( H \) be an arbitrary subcontinuum of \( X \). We shall yet show that if

\[
\tag{24}
H \subset q\left(\frac{a}{4}, \frac{a}{2}\right) \quad \text{or} \quad H \subset q\left(\frac{a}{4}, \frac{a}{2}\right),
\]

then simultaneously

\[
\tag{25}
q'(H, a) > \frac{a}{4} \quad \text{and} \quad q'(H, b) > \frac{a}{4}.
\]

In view of the symmetry we may assume that \( H \subset q\left(\frac{a}{4}, \frac{a}{2}\right) \). This assumption implies \( q(H, a) > \frac{a}{4} \) (for otherwise \( q\left(\frac{a}{4}, \frac{a}{2}\right) = \frac{a}{4} \), contrary to (8)) and therefore, in view of the inclusions \( S' \subset A \) and \( S' \subset B \), we have

\[
\sup_{a \in H} g(a, a) > \frac{a}{4} \quad \text{and} \quad \sup_{a \in H} g(H, b) > \frac{a}{4}.
\]

The two inequalities yield (25).
Now we may complete the proof. Suppose then that there exists a continuum $H \subset X$ such that

$$
\phi'(A, H) = \phi'(B, H) = 0.
$$

By the formula (2) it follows then that $H \subset Q\left(A_1, \frac{d}{2}\right)$ and $H \subset Q\left(B_1, \frac{d}{4}\right)$, whence $H \subset Q\left(A_1, \frac{d}{4}\right) \cap Q\left(B_1, \frac{d}{4}\right)$. Hence, in view of (21) and (23), we infer that one of the inclusions in (24) holds. However, any inclusion in (24) implies both inequalities (25). A contradiction with (26).

§ 4. Applications to hyperspaces $\text{Comp} (X)$, $C(X)$ and Conv $(X)$. The first theorem below is known ([3], Theorem 4.1), but inserting it here for the sake of completeness we supply it with a new proof.

**Theorem 4.1.** Let $X$ be a compact metric space. Then $X$ is convex if and only if $\text{Comp}(X)$ is convex.

In fact, if $X$ is convex, then the hyperspace $\text{Comp}(X)$ is compact (see [11], II, p. 21) and has property $(S)$, and so, by Theorem 3.2, it must be convex. Conversely, if $\text{Comp}(X)$ is convex, then by virtue of Theorem 3.4 the underlying space $X$ is convex too.

Remarks. If $X$ is a complete convex space, the hyperspace $\text{Comp}(X)$ need not be convex. An example is provided by the space $X$ constructed in remarks following Proposition 2.1. In fact, if there were a compact set $C \subset X$ such that $\phi'(A, C) = \phi'(B, C) = \frac{1}{2}$, then, in view of formula (2), we would have $C \subset Q\left(A, \frac{d}{4}\right) \cap Q\left(B, \frac{d}{4}\right)$, and since the common part $Q\left(A, \frac{d}{4}\right) \cap Q\left(B, \frac{d}{4}\right)$ consists of middle points of all $\text{arc } x \times y$, hence is uncountable with the discrete topology, then $A$ were a finite subset of this common part. It is easy to check that in such a case there would be sup $\phi'(x, C) > \frac{1}{2}$ and sup $\phi'(x, C) > \frac{1}{2}$, whence, by the formula (1), $\phi'(A, C) > \frac{1}{2}$ and $\phi'(B, C) > \frac{1}{2}$; a contradiction. Hence the hyperspace $\text{Comp}(X)$ is not convex.

Note, however, that if $X$ is a complete metric space and the hyperspace $\text{Comp}(X)$ is convex, then the space $X$ is convex by virtue of Theorem 3.4.

Before proceed to the hyperspace $\text{Comp}(X)$ we shall need some simple lemmas.

4.2. Let $X$ be a metric space and $A_0, A_1, A_2, \ldots$ be a sequence of subsets of $X$. If

$$
\lim \phi'(A_n, A_0) = 0,
$$

$A_0$ is closed in $X$, and each set $A_n$, $A_{n+1}$, is connected, then $A_0$ is connected.

Proof. Assume, to the contrary, that each set $A_1, A_2, \ldots$ is connected but $A_0$ is not. Then there are two non-empty closed subsets $F_1$ and $F_2$ of $X$ such that $A_0 = F_1 \cup F_2$ and $F_1 \cap F_2 = \emptyset$, and, consequently, there are two open subsets $G_1$ and $G_2$ of $X$ such that $F_1 \subset G_1$, $F_2 \subset G_2$ and $G_1 \cap G_2 = \emptyset$.

Since $A_0 \subset G_1 \cup G_2$ and $G_1 \cup G_2$ is open, there is $A_0 \subset G_1 \cup G_2$ for $n$ sufficiently large. And since $F_1 \neq G_1 \cup G_2$, there is also $A_0 \subset G_1 \cup G_2$ for $n$ sufficiently large. But then $A_0$ cannot be connected. A contradiction.

4.3. Let $X$ be a metric space and $A_0, A_1, A_2, \ldots$ be a sequence of subsets of $X$. If (27) holds true and each set $A_1, A_2, \ldots$ is bounded (respectively, totally bounded), then the set $A_0$ is bounded (respectively, totally bounded).

Proof. Condition (27) implies that

$$
(28)
$$

for each $\varepsilon > 0$ there exists $n(\varepsilon)$ such that $\phi'(A_0, A_n) < \varepsilon$.

Hence and from the formula (2) it follows that $A_0 \subset Q(A_0, \varepsilon)$ and therefore $\Gamma(A_0) \subset Q(A_0, 5\varepsilon)$. Boundedness of $A_0$ implies that of $A_0$.

Now suppose that each set $A_1, A_2, \ldots$ is totally bounded and $A_0$ is not. Hence for some $n > 0$ there exists a sequence $a_1, a_2, \ldots$ of points of $A_0$ such that $d(a_k, a_l) > 5\varepsilon$ for $k \neq l$. Balls $Q(a_k, \eta)$ are then pairwise disjoint and if we choose for each $k = 1, 2, \ldots$ a point $a_k \in Q(a_k, \eta)$, then

$$
(29)
$$

for $k \neq l$.

By virtue of (28) we have now $\phi'(A_0, A_n) < \varepsilon$, whence and from the formula (2) it follows that $A_0 \subset Q(A_0, \varepsilon)$, and $Q(A_0, \varepsilon)$ contains $Q(a_k, \eta)$ for $k = 1, 2, \ldots$, because $a_k \in A_k$. Choosing now a point $a_k \in A_0 \cap Q(a_k, \eta)$ for $k = 1, 2, \ldots$ we come, in view of (29), to a contradiction with the total boundedness of $A_0$.

4.4. Let $X$ be a complete metric space and $A_0, A_1, A_2, \ldots$ be a sequence of closed subsets of $X$. If (27) holds true and each set $A_1, A_2, \ldots$ is compact, then $A_0$ is compact.

Indeed, as a closed subspace of a complete space, $A_0$ is a complete space itself (see [11], II, p. 315) and by virtue of 4.3, $A_0$ is totally bounded. Hence $A_0$ must be compact (see [11], II, p. 3).

Recall that a **dendrite** is a locally connected metric continuum containing no simple closed curve.

**Theorem 4.5.** Let $X$ be a complete metric space. Then the following conditions are equivalent:

(a) $X$ is convex and contains no simple closed curve,

(b) every subcontinuum of $X$ is a convex dendrite,

(c) $\text{conv}(X)$ is convex.

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Proof. (a) = (b). Assume, a contrario, that \( X \) contains a subcontinuum \( C \) which is not a dendrite. Were \( C \) locally connected, it would contain a simple closed curve and this is impossible, because \( X \) does not contain any simple closed curve. Thus \( C \) is not locally connected and so it must contain a sequence of pairwise disjoint subcontinua \( \{A_n\} \) convergent to a subcontinuum \( A \) disjoint with each \( A_n \) (cf. [11], I, p. 176).

Choose \( \delta \) such that

\[ 2 \cdot \varrho(A, A_n) < \min\{\delta(A), \delta(A_n)\} \]

and choose points \( a, b \in A \) and \( a_n, b_n \in A_n \) such that

\[ 2 \cdot \varrho(a, a_n) < \min\{\varrho(a, b), \varrho(a_n, b_n)\} \quad (30) \]

\[ 2 \cdot \varrho(b, b_n) < \min\{\varrho(a, b), \varrho(a_n, b_n)\} \quad (31) \]

Covering now \( A \) with a finite number of convex sets, each disjoint with \( A_n \), we can choose an arc \( L \subset X \) joining \( a \) to \( b \) and disjoint with \( A_n \). Similarly we can choose an arc \( L_n \subset X \) joining \( a_n \) to \( b_n \) and disjoint with \( L \).

The space \( X \) is convex by hypothesis (a), there exists a segment \( b_n \subset L \). The union \( L \cup b_n \cup L_n \) is locally connected (cf. theorem of Hahn–Mazurkiewicz–Sierpiński, [11], I, p. 185) and so it contains an arc \( M \) of ends \( a \) and \( a_n \). In view of (31), \( \delta(M) \geq 2 \min\{\varrho(a, b), \varrho(a_n, b_n)\} \). Hence in view of (30) a segment \( a_n a \) is not contained in \( M \) and so the union \( M \cup a_n a \) contains a simple closed curve \( A \), a contradiction.

Thus it is shown that each subcontinuum of \( X \) is a dendrite. Were such a dendrite not convex, we would come, in view of the assumed convexity of \( X \), to a contradiction with the hypothesis that \( X \) contains no simple closed curve.

Hence the proof of implication (a) \( \Rightarrow \) (b) is completed.

(b) \( \Rightarrow \) (c). Let \( A \) and \( B \) be two subcontinua of \( X \). Take two arbitrary points \( a, b \in A \) and \( b', b \in B \) and consider the union \( A \cup a_n b \cup B \). If \( a_n b \cup A \cup B \), this union is a convex dendrite, hence a strongly convex continuum and a bridge between \( A \) and \( B \). Since moreover, as is easy to check (cf. [9], 2.3), balls \( Q(A, e) \) and \( Q(B, f(A, B) - e) \) are both strongly convex for each \( 0 < e < \varrho(A, B) \), then the common part

\[ (A \cup a_n b \cup B) \cap Q(A, e) \cap Q(B, f(A, B) - e) \]

is strongly convex, hence a continuum. Thus it is shown that the hyperspace \( C(X) \) has property (S).

Since \( X \) is complete by hypothesis, the space \( 2^X \) consisting of all non-empty bounded subsets of \( X \) and metrized by Hausdorff metric is also complete (see [11], I, p. 314), and by virtue of 4.2 and 4.4 the hyperspace \( C(X) \) is closed in \( 2^X \), hence complete itself (see [11], I, p. 315).

Now (c) follows by virtue of Theorem 3.2.

(c) \( \Rightarrow \) (a). Follows by Theorem 3.5.

The just proved Theorem 4.5 yields immediately

**Corollary 4.6.** Let \( X \) be a metric continuum. Then the following conditions are equivalent:

(a) \( X \) is a dendrite with a convex metric,

(b) every subcontinuum of \( X \) is convex,

(c) \( C(X) \) is convex.

Remarks. As is known (see [1] or [15]), every locally connected metric continuum can be supplied with a convex metric. It follows then, in view of Corollary 4.6, that dendrites can be distinguished in the class of all metric continua \( X \) by the following characteristic property: there exists a metric in \( X \) such that the hyperspace \( C(X) \) is convex.

Implication (b) \( \Rightarrow \) (c) of Corollary 4.6 has been known ([3], Theorem 3.2) and implication (c) \( \Rightarrow \) (b) is a positive answer to the problem P2 stated in [3]. Equivalence (a) \( \Leftrightarrow \) (c) is then an answer to the problem P1 from [3].

Corollary 4.6 says little on the topological structure of the hyperspace \( C(X) \) for \( X \) being a dendrite. It is then perhaps worth to mention here that hyperspace \( C(X) \) for \( X \) being finite dendrites were investigated in [5]. In particular, a topological characterization of these hyperspaces has been obtained there.

As Menger has proved (see [13], p. 92), if in a compact metric space \( X \) there is a convergent sequence of segments \( \{a_n b_n\} \) whose end-points \( a_n \) and \( b_n \) converge, then its limit is a segment in \( X \) between \( \lim a_n \) and \( \lim b_n \).

The following proposition generalizes Menger's result to the case of a complete metric space.

4.7. Let \( X \) be a complete metric space and let \( \{a_n\} \) and \( \{b_n\} \) be two sequences of points of \( X \) such that for each \( n = 1, 2, \ldots \) there is a segment \( a_n b_n \subset X \). If \( \lim a_n = a \), \( \lim b_n = b \) and \( C \) is a closed subset of \( X \) such that

\[ \lim \varrho(a_n b_n, C) = 0, \]

then \( C \) is a segment in \( X \) between \( a \) and \( b \).

Proof. Since \( C \) is compact by 4.4, it suffices to apply Menger's result to the space \( C \cup \{a_n b_n\} \).

4.8. Let \( X \) be a complete metric space and let \( A_1, A_2, A_3, \ldots \) be a sequence of compact subsets of \( X \). If (27) holds true and each set \( A_1, A_2, \ldots \) is convex, then \( A_n \) is convex.

(*) The lemma remains true after replacing hypothesis \( \lim \varrho(a_n b_n, C) = 0 \) by a weaker one, \( \lim a_n = a \), but the proof is different and somewhat more lengthy.
Proof. Take any two points $a, b \in A$, and choose two sequences of points $a_n, b_n \in A_n$ such that $a = \lim a_n$ and $b = \lim b_n$ (such sequences exist by virtue of (27) and formulas for $q$).

By convexity of $A$, there is a segment $a_n b_n \in A$, and since compactness of $A_n \cup A$ implies that of $C(A_n \cup A)$ (cf. [9]), we may assume that the sequence of segments $(a_n b_n)$ is convergent to say $C$. By virtue of 4.7, $C$ is a segment between $a$ and $b$, and since $a_n b_n \in A_n$ for each $n = 1, 2, \ldots$, and $\lim A_n = A_n$ (see [7], p. 149; cf. also [11], I, p. 248), then $C \in A_n$. Hence $A_n$ is convex.

Theorem 4.9. Let $X$ be a metric space. Then

(a) $X$ is complete if and only if $\text{Conv}(X)$ is complete,

(b) $X$ is compact if and only if $\text{Conv}(X)$ is compact,

(c) $X$ is locally compact if and only if $\text{Conv}(X)$ is locally compact.

Proof. (a) If $X$ is complete, then the space $2^X$ consisting of all non-empty bounded subsets of $X$ and metrized by Hausdorff metric is also complete (see [11], I, p. 314). By virtue of 4.4 the hyperspace $\text{Comp}(X)$ is closed in $2^X$, hence complete (see [11], I, p. 315). And by virtue of 4.8 the hyperspace $\text{Conv}(X)$ is closed in $\text{Comp}(X)$, hence also complete.

The converse implication holds, because the hyperspace $X(1)$, isometric to $X$ itself, is a closed subspace of the hyperspace $\text{Conv}(X)$.

(b) If $X$ is compact, then the hyperspace $\text{Comp}(X)$ is compact (see [11], II, p. 21) and by 4.8 the hyperspace $\text{Conv}(X)$ is closed in $\text{Comp}(X)$, hence also compact.

The converse implication holds, because the hyperspace $X(1)$, isometric to $X$ itself, is a closed subspace of the hyperspace $\text{Conv}(X)$.

(c) If $X$ is locally compact, then for every $A \in \text{Conv}(X)$ there exists an $\varepsilon > 0$ such that $X \cap Q(A, \varepsilon)$ is compact. By virtue of formula (2), $X \cap Q(A, \varepsilon)$ contains all subsets $B$ of $X$ with the property $q(A, B) \leq \varepsilon$, and so $\text{Conv}(X \cap Q(A, \varepsilon))$ is a neighbourhood of $A$ in $\text{Conv}(X)$. In view of (b), $\text{Conv}(X \cap Q(A, \varepsilon))$ is compact. Hence the hyperspace $\text{Conv}(X)$ is locally compact.

Since the hyperspace $X(1)$, isometric to $X$ itself, is a closed subspace of the hyperspace $\text{Conv}(X)$, and a closed subspace of a locally compact space is locally compact itself (see [10], p. 146), the converse implication holds too.

Thus the proof of Theorem 4.9 is completed.

Remarks. The proof of Theorem 4.9 does not depend on results of § 2 and § 3.

Strange enough, the hyperspace $\text{Conv}(X)$ does not preserve convexity. To show this consider first an example of the unit circle, $X = \{(a, b) : a^2 + b^2 = 1\}$, with the geodesic metric. Since any convex subcontinuum of $X$ is either an arc of length not greater than $\pi$ or $X$ itself, then one can show (cf. [4], § 3, example 2) that the hyperspace $\text{Conv}(X)$ is here topologically equivalent to the union of an annulus $X \times I$ and of an isolated point. Hence $\text{Conv}(X)$ need not be connected (to say nothing of convexity) even although $X$ itself is convex. And the example of a continuum $X = (0, 0) \cup \cup S_n$, where $S_n$ is the circle in the plane of center $(3/2n^2, 0)$ and radius $1/2n^{3/2}$ provided with the geodesic metric (X is again a convex continuum) shows that $\text{Conv}(X)$ need not even be locally connected.

The problem of characterization of those metric spaces for which the hyperspace $\text{Conv}(X)$ is convex can be fully answered, as will be shown in the next section, in the case of $X$ being a subspace of a Euclidean space.

§ 5. Hyperspaces $\text{Conv}(X)$ and $\text{Conv}(X)$ in the Euclidean case.

One of the most interesting is perhaps the case of $X$ being a subset of a Euclidean space $E^n$ and $X$ being the hyperspace $\text{Conv}(X)$ or $\text{Conv}(X)$. The two hyperspaces have to-day a well developed theory going back to J. Steiner, H. Brunn and H. Minkowski. In its present shape the theory appears to refrain from any topological reference (cf. a neat presentation of it in the Hadwiger’s book [9]), but it seems that it can profit even by a little of topologisation.

Some topological results on the hyperspace $\text{Conv}(X)$ have been collected in the following

Theorem 5.1. Let $X$ be a subspace of a Euclidean space $E^n$. Then

(a) $X$ is complete if and only if $\text{Conv}(X)$ is complete,

(b) $X$ is compact if and only if $\text{Conv}(X)$ is compact,

(c) $X$ is locally compact if and only if $\text{Conv}(X)$ is locally compact,

(d) $X$ is convex if and only if $\text{Conv}(X)$ is convex.

Proof. (a), (b), and (c) — see Theorem 4.9 above.

(d) Let $X$ be convex, and let $A$ and $B$ be any two elements of the hyperspace $\text{Conv}(X)$. In the considered here Euclidean case the bridge $P(A, B)$ does exist (and is unique), and since $X$ is convex by assumption, $P(A, B) \subset X$. Both $A$ and $B$ are convex, so is $P(A, B)$. As we have shown in § 2, Conv($P(A, B)$) has property (B), and by virtue of (b), Conv($P(A, B)$) is compact. Hence, in view of Theorem 3.2, it follows that Conv($P(A, B)$) is compact. In particular, it does contain a metric segment between $A$ and $B$. Since inclusion $P(A, B) \subset X$ implies inclusion Conv($P(A, B)$) $\subset \text{Conv}(X)$, this segment is contained in the hyperspace $\text{Conv}(X)$. Hence the hyperspace $\text{Conv}(X)$ is convex.

To prove the converse note that if $p$ and $q$ are any two points of $E^n$, then the only subsets of $E^n$ which lie between $(p)$ and $(q)$ (in the sense of Hausdorff metric $q'$) are points. Indeed, for $0 < \varepsilon < d = q(p, q)$,
and if $Z$ is a subset of $E^n$ such that $\phi^1(p, Z) = \varepsilon$ and $\phi^1(Z, q) = d - \varepsilon$, then by virtue of 3.3 there is $q(p, \varepsilon) = \varepsilon$ and $q(x, q) = d - \varepsilon$ for each $x \in Z$. But since $E^n$ is strongly convex, there is only one point $x \in E^n$ for which the last two equalities hold. Hence it must be $Z = \{x\}$. Therefore, if $p, q \in X$ and the hyperspace $\text{Conv}(X)$ is convex, then the space $X$ contains a metric segment between $p$ and $q$. Thus it is shown that convexity of $\text{Conv}(X)$ implies that of $X$.

Hence the proof of Theorem 5.1 is completed.

The hyperspace $\text{Conv}(X)$ is clearly a subspace of the hyperspace $\text{Conv}(E^n)$ and it is non-empty if and only if $X$ has non-empty interior. But topological properties of $\text{Conv}(X)$ are not so good as those of $\text{Conv}(E^n)$. For instance, as an example of a sequence of concentric balls in $X$ with diameters tending to 0 shows, the hyperspace $\text{Conv}(X)$ is neither compact nor complete independently of whether $X$ is such or not. Nevertheless, we have the following

**Theorem 5.2.** Let $X$ be a subspace of a Euclidean space $E^n$ with the non-empty interior. Then

(a) $\text{Conv}(X)$ is open in $\text{Conv}(E^n)$;

(b) if $X$ is locally compact, then so is $\text{Conv}(X)$;

(c) if $X$ is convex, then so is $\text{Conv}(X)$;

(d) if $X$ is open or convex, then $\text{Conv}(X)$ is dense in $\text{Conv}(E^n)$.

Proof. (a) To show that $\text{Conv}(X)$ is open in $\text{Conv}(E^n)$ take an arbitrary $A \in \text{Conv}(X)$ and for each straight line $p$ in $E^n$ passing through the origin $O$ denote by $\omega(p)$ the width of $A$ in the direction of $p$. Treating then such a $p$ as a projective line $P^{-1}$ and taking into account that $\omega(p)$ is a continuous function (see [9], p. 12) and that $\omega(p) > 0$ for each $p \in P^{-1}$ we infer by the theorem of Weierstrass (see [11], p. 15) that

$$\omega_0 = \inf_{p \in P^{-1}} \omega(p) > 0.$$  

(32)

The proof will be completed when we show that if

$$\omega_0 < \varepsilon < \frac{1}{2} \omega_0,$$

then the ball $Q(A, \varepsilon)$ in $\text{Conv}(X)$ with the center $A$ and of the radius $\varepsilon$ lies entirely in $\text{Conv}(X)$. For that purpose take an arbitrary element $B$ of $Q(A, \varepsilon)$, i.e., a compact convex subset of $X$ satisfying

$$\phi^2(B, A) < \varepsilon,$$

and suppose, a contrario, that $\text{Int}(B) = \emptyset$. This means that there exists an $(n-1)$-dimensional hyperplane $H$ with

$$B \subset H.$$  

(35)

Let $p_k \in P^{-1}$ be the straight line perpendicular to $H$, and let $H_0$ and $H_1$ be planes parallel to $H$ and realizing the width $\omega(p_k)$. In view of (32) and (33),

$$\varepsilon < \frac{1}{2} \omega(p_k).$$  

(36)

By the definition of width there exist points $x_0 \in A \cap H_0$ and $x_k \in A \cap H_1$, and since the three hyperplanes $H_0, H_1, H$ are all parallel, then either $\omega(x_0, H) \geq \frac{1}{2} \omega(p_k)$ or $\omega(x_k, H) \geq \frac{1}{2} \omega(p_k)$. Whatever of the two inequalities holds, there is sup $\omega(x, H) \geq \frac{1}{2} \omega(p_k)$, whence, in view of (35),

$$\omega(x, H) \geq \frac{1}{2} \omega(p_k),$$

and, consequently, $\omega(B, A) \geq \frac{1}{2} \omega(p_k)$, contrary to (34) and (36).

(b) Since, as we have just proved, the hyperspace $\text{Conv}(X)$ is an open subspace of the hyperspace $\text{Conv}(E^n)$, and in the considered case the latter is, by virtue of Theorem 5.1 (c), locally compact, the former must be locally compact too ([10], p. 116).

(c) Since the hyperspace $\text{Conv}(X)$ is not a complete metric space, then to prove its convexity we need more than mere statement that this hyperspace has property $(S)$. Namely, for any two elements $A$ and $B$ of $\text{Conv}(X)$ we shall construct a family $\{H_\alpha\}_{\alpha \in \Gamma}$, where $\Gamma$ denotes the set of dyadic rationals of the real segment $[0, 1]$, of sets $H_\alpha$ such that

$$H_0 = A \cap B,$$

$$H_1 = B \cap B,$$

$$\text{if } \gamma \in \Gamma, \text{ then } H_\alpha = A \cap B.$$

(38)

If $\gamma = \Gamma$, then $H_\alpha$ is a compact convex subset of $P(A, B)$.

$$\text{if } \gamma_0, \gamma_1 \in \Gamma, \text{ then } x_\alpha \in H_\alpha,$$

(39)

there exist an $\alpha > 0$ such that the $n$-dimensional volume $v(H_\alpha) > \alpha$ for each $\gamma \in \Gamma$.

Put $A = H_0$ and $B = H_1$. By virtue of (4) there exist points $h_0 \in \text{Int}(H_0)$ and $h_1 \in \text{Int}(H_1)$ such that $q(h_0, h_1) < q(H_0, H_1)$. Since $h_0 \in \text{Int}(H_0)$ and $h_1 \in \text{Int}(H_1)$, there exists an $\eta > 0$ such that $Q(h_0, \eta) \subset H_0$ and $Q(h_1, \eta) \subset H_1$. Putting $\alpha = v(Q(h_0, \eta))$ we infer that $v(H_\alpha) > \alpha$ for $\gamma = 1, 2$.

Let $h_2$ be the middle point of the segment $h_0h_1$. Obviously, $Q(h_2, \eta) \subset P(Q(h_0, \eta), Q(h_1, \eta)) \subset P(H_0, H_1)$. Since $Q(h_2, \eta)$ is a translation of $Q(h_0, \eta)$, for $q(h_0, h_2) < q(H_0, H_1)$, then $Q(h_2, \eta) \subset Q(H_0, \frac{1}{2} q(H_0, H_1))$. Similarly, $Q(h_2, \eta) \subset Q(H_1, \frac{1}{2} q(H_1, H_1))$. Hence the compact convex set $H_2$ defined by the formula

$$H_2 = \{P(H_0, H_1) \cap Q(H_0, \frac{1}{2} q(H_0, H_1)) \cap Q(H_1, \frac{1}{2} q(H_1, H_1)) \}
$$

lies, in view of 3.1, in the middle between $H_0$ and $H_1$, and contains $Q(h_2, \eta)$. Consequently, $v(H_2) > \alpha$. Moreover, $q(h_0, h_2) < q(H_0, H_1)$ and $q(h_2, h_1) < q(H_1, H_1)$. Therefore, $v(H_2) > \alpha$ and $v(H_\alpha) > \alpha$ for each $\gamma \in \Gamma$. Therefore, $v(H_\alpha) > \alpha$ for each $\gamma \in \Gamma$.
In analogous manner one defines now sets $H_1$ and $H_2$, and by an easy induction the whole family $(H_{n_r})_{r=1}^t$ of compact convex sets satisfying (37)-(40).

Now, since the bridge $P(A, B)$ is compact convex, so is, in view of Theorem 5.1 (b) and (d), its hyperspace $Conv(P(A, B))$. Hence in view of (37)-(39) the closure of the family $(H_{n_r})_{r=1}^t$ in the hyperspace $Conv(P(A, B))$ is a metric segment between $A$ and $B$ ([13], p. 87-89). However, if $C$ is an element of this segment, then in view of (40) and of the continuity of volume there must be $v(C) \geq \alpha$. In other words, $C \in Conv(P(A, B))$. And since $P(A, B) \subseteq X$ by the convexity of $X$, then $Conv(P(A, B)) \subseteq Conv(X)$, and so this metric segment between $A$ and $B$ lies in $Conv(X)$. Hence the hyperspace $Conv(X)$ is convex.

(d) Let $X$ be an open subset of $E^n$. If $A$ is any given element of the hyperspace $Conv(X)$, then $e_\varepsilon = e(A, X \setminus A) > 0$. Consequently, $0 < e < e_\varepsilon$ implies $Q(A, e) \subseteq X$. Hence $Q(A, e) \subseteq Conv(X)$ and, as follows from (2), $Q(A, Q(A, e)) = e$.

And if $X$ is convex and $A$ is again any element of $Conv(X)$, then take a point $p$ in the interior of $A$ at a distance from $A$, say $e$ (such a point surely exists since Int($X$) is dense in $X$), and a ball $Q(p, \eta)$ of radius $\eta < e$. Then the bridge $P(A, Q(p, \eta))$ is compact convex, lies inside $X$, and has non-empty interior. Hence $P(A, Q(p, \eta)) \subseteq Conv(X)$ and it is easy to check that $Q(A, P(A, Q(p, \eta))) \leq 2\varepsilon$.

Remarks. None of the implications of Theorem 5.2 can be reversed. In fact, consider in the plane $E^2$ a sequence of pairwise disjoint quadrangles converging to a segment and let $X$ be the union of all these quadrangles and of one point of the segment to which these quadrangles converge. It is easy to check that $Conv(X)$ is locally compact (but $X$ is not, hence the converse of (b) does not hold) and that $Conv(X)$ is dense in $Conv(X)$ (but $X$ is neither open nor convex; hence the converse of (d) does not hold either). And the example of an open half plane with two points of its boundary adjoined shows that the converse of (c) does not hold.

Part (c) of Theorem 5.2 has been known for $X = E^n$. Namely, Shephard and Webster have pointed out that the segment in $Conv(X)$ between its two elements $A$ and $B$ with the distance $d(A, B) = 1$ is defined by the Minkowski linear system $f: [0, 1] \rightarrow Conv(X)$, where $f(t) = (1-t)A + tB$ ([17], theorem (23)). It is to be noted, however, that

$$(1-t)A + tB = P(A, B) \cap Q(A, 1-t) \cap Q(B, t),$$

and so in this particular case their idea overlaps with that of ours (cf. 3.1 above).
Concerning the ordering of shapes of compacta

by

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The purpose of this note is to give answers to one question concerning the existence of maximal shapes (majorants) and to another question, concerning the existence of continuously ordered families of shapes.

§ 1. Basic definitions. Let \( X, Y \) be two compacta lying in the Hilbert cube \( Q \). A sequence of maps \( f_k : Q \to Q \) is said to be a fundamental sequence from \( X \) to \( Y \) (notation: \( f_k : X \to Y \)), or \( f : X \to Y \). Compare [3], p. 225) if for every neighborhood \( V \) of \( Y \) there is a neighborhood \( U \) of \( X \) such that

\[
\exists_k \{ f_k \} \in V \quad \text{for almost all } k.
\]

In particular, if \( f_k \) is the identity map of \( Q \) onto itself for every \( k = 1, 2, \ldots \), then \( \{ f_k : X \to X \} \) is said to be the fundamental identity sequence from \( X \) to \( Y \). Two fundamental sequences \( f = \{ f_k : X \to Y \} \) and \( g = \{ g_k : X \to Y \} \) are said to be homotopic (notation: \( f \approx g \)) if for every neighborhood \( V \) of \( Y \) there is a neighborhood \( U \) of \( X \) such that

\[
\exists_k \{ f_k \} \in V \quad \text{for almost all } k.
\]

The family of all fundamental sequences isomorphic to a given fundamental sequence \( f : X \to Y \) is said to be the fundamental class \([f]\) from \( X \) to \( Y \).

If \( X, Y, Z \) are compacta lying in \( Q \) and if \( f = \{ f_k : X \to Y \} \), \( g = \{ g_k : X \to Z \} \) are fundamental sequences, then \( \{ f_k \times g_k : X \times Z \} \) is a fundamental sequence, called the composition of \( f \) and \( g \); it is denoted by \( fg \).

Two compacta \( X, Y \) (not necessarily lying in \( Q \)) are said to be of the same shape (notation: \( Sh(X) = Sh(Y) \)) (see [4]) if there are two compacta \( X', Y' \subset Q \), homeomorphic to \( X \) and \( Y \) respectively, and two fundamental sequences \( f : X' \to Y' \), \( g : Y' \to X' \) such that \( fg \approx f \) and \( gf \approx 1' \). If we assume only that \( f \) and \( g \) satisfy the first of these homotopies, then \( Sh(X) \) is said to be less than \( Sh(Y) \) (notation: \( Sh(X) < Sh(Y) \)). If \( Sh(X) < Sh(Y) \), but the relation \( Sh(X) \leq Sh(Y) \) does not hold, then we say that \( Sh(X) \) is less than \( Sh(Y) \) and we write \( Sh(X) < Sh(Y) \).