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A 2-complex is collapsible if and only if it admits a strongly convex metric

by

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§ 1. Introduction. A metric d on a compact space X is *strongly convex* if, for any two points $x, y \in X$, there is a unique point $m \in X$ such that $d(x, m) = d(m, y) = \frac{1}{2}d(x, y)$. In the last few years, there has been considerable interest in characterizing the spaces which admit convex metrics. Lelek and Nitka [3] and Rolfsen [4] have shown that cells are the only compact 2 and 3-dimensional spaces which admit strongly convex metrics with the property that no midpoint of x and y is a midpoint of x and y' unless $y = y'$. Rolfsen [4] has further shown that the only compact n-manifold, $n \leqslant 3$, admitting a strongly convex metric is the cell.

It is well known (see [2]) that any compact space which admits a strongly convex metric is contractible, but Sieklucki [5] has demonstrated a contractible 2-complex which admits no strongly convex metric. Joseph Martin conjectured in 1966 that the stronger condition of collapsibility does characterize the 2-complexes which admit strongly convex metrics, and a proof of this is the object of this note. It is interesting to note that this theorem also provides, conversely, a topological characterization of collapsibility in 2-complexes, and thus cannot be directly extended to higher dimensions, for a 3-cell can have a non-collapsible triangulation [1].

§ 2. A collapsible 2-complex admits a strongly convex metric.

DEFINITIONS. All simplices are closed simplices. If $a_1, a_2, \ldots, a_k$ are points in a simplex σ, then $a_1 a_2 \ldots a_k$ is their convex hull in the linear structure of σ. A *triangle* is a 2-simplex in E^2 with the regular euclidean metric $\|x-y\|$.

All maps are continuous; if X and Y are spaces, the notation $f\colon X \to Y$ denotes a map from X onto Y. If K is a complex, then $K^{(k)}$ denotes the k-skeleton of K.

Let X be a compact space with a strongly convex metric d. Any two points x, y of X are joined in X by a unique arc, the *segment* $\widehat{xy}$, which is isometric to a closed interval of the real line ([2]). A *concave collection for* d is a finite collection T of segments in X satisfying:

(2.1) *If* ϱ, $\tau \in T$ *and* $x_1, x_2 \in \varrho$, $y_1, y_2 \in \tau$, *then* $d(x_m, y_m) \leqslant \frac{1}{2}[d(x_1, y_1) + d(x_2, y_2)]$, *where* x_m, y_m *are the midpoints of* $\widehat{x_1x_2}$ *and* $\widehat{y_1y_2}$.

(2.2) *If* $\tau \in T$ *and* $x_1, x_2 \in \tau$ *then, for any point* $y \in X$, $d(y, x_m) \leqslant \frac{1}{2}[d(y, x_1) + d(y, x_2)]$, *where* x_m *is the midpoint of* $\widehat{x_1x_2}$.

LEMMA 2. *Suppose that* $X \cup \sigma$ *is a metric space and* $X \cap \sigma = \tau$ *is an arc. Let* d *be a strongly convex metric for* X *and let* T *be a concave collection for* d,

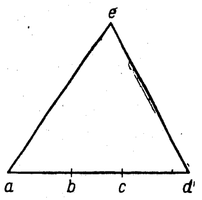

Fig. 1

an element of which contains τ. *Suppose* $abcde$ *(Figure 1) is a triangle with vertices* a, d, *and* e, *and let* $\varphi\colon abcde \to \sigma$ *be a homeomorphism such that* $\varphi(bc) = \tau$ *and* $d\big(\varphi(x), \varphi(y)\big) = \|x-y\|$ *for every* $x, y \in bc$.

Then there is a strongly convex metric d' *for* $X \cup \sigma$ *such that:*

(2.3) $$d'(x, y) = d(x, y) \quad \text{for all} \quad x, y \in X,$$

(2.4) $$d'(x, y) = \|\varphi^{-1}(x) - \varphi^{-1}(y)\| \quad \text{for all} \quad x, y \in \sigma,$$

(2.5) $T \cup \{\varphi(ab), \varphi(cd), \varphi(de), \varphi(ea)\}$ *is a concave collection for* d'.

Proof. Define d' by:

$$d'(x, y) = \begin{cases} d(x, y), & x, y \in X, \\ \|\varphi^{-1}(x) - \varphi^{-1}(y)\|, & x, y \in \sigma, \\ \min\limits_{p \in \tau}\{d'(x, p) + d'(p, y)\}, & x \in \sigma,\ y \in X \text{ or } x \in X,\ y \in \sigma. \end{cases}$$

Checking that d' satisfies 2.3–2.5 is then a straightforward process. ■

THEOREM 2. *Let* L *be a 2-complex with a strongly convex metric* d *on* $|L|$, *and let* T *be a concave collection for* d *which covers* $|L^{(1)}|$. *Suppose that* σ *and* τ *are a simplex and face, respectively, such that* $L' = L \cup \{\sigma, \tau\}$ *is a 2-complex and* $L' \to L$ *is an elementary collapse.*

Then there is a strongly convex metric d' *for* $|L'|$ *and a concave collection* T' *for* d' *satisfying:*

(2.6) T' *covers* $|L'^{(1)}|$,

(2.7) $$d'(x, y) = d(x, y) \quad \text{for} \quad x, y \in |L|.$$

Proof. The case when σ is a 1-simplex is left to the reader to check. Suppose σ is a 2-simplex; then σ meets L in two of its faces, τ_1 and τ_2 (Figure 2). T covers the one-skeleton of L, so we can find a subtriangulation $u_0u_1, u_1u_2, \ldots, u_{k-1}u_k$ of $\{\tau_1, \tau_2\}$ such that each element $u_{i-1}u_i$

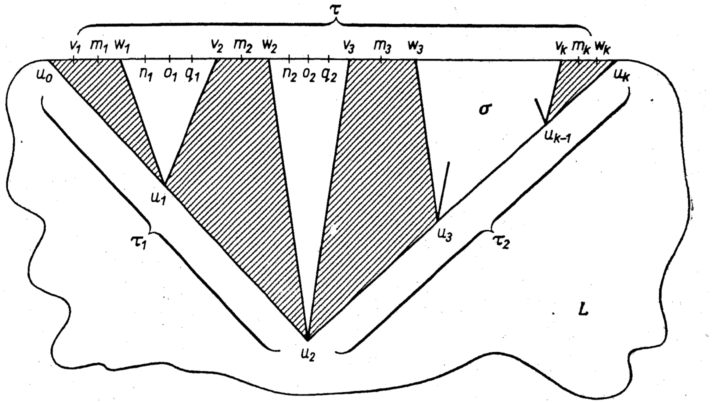

Fig. 2

belongs to T. Choose points v_i, w_i, m_i, $i = 1, \ldots, k$, and n_i, o_i, q_i, $i = 1, \ldots, k-1$, so that the order of points along τ is $u_0, v_1, m_1, w_1, n_1, o_1, q_1, v_2, m_2, w_2, n_2, o_2, q_2, \ldots, v_{k-1}, m_{k-1}, w_{k-1}, n_{k-1}, o_{k-1}, q_{k-1}, v_k, m_k, w_k, u_k$, as illustrated in Figure 2.

PROPOSITION 2A. *There is a strongly convex metric* d_1 *for* $L_1 = |L| \cup u_0u_1w_1v_1 \cup \ldots \cup u_{k-1}u_kw_kv_k$ *(shaded in Figure 2) such that:*

(1) $T_1 = T \cup \{v_1u_0, u_1w_1, w_1m_1, m_1v_1, \ldots, v_ku_{k-1}, u_kw_k, w_km_k, m_kv_k\}$ *is a concave collection for* d_1.

(2) $v_ju_{j-1} \cup u_{j-1}u_j \cup u_jw_j = \widehat{v_jw_j}$, *for each* $j = 1, \ldots, k$.

Proof. The quadrilaterals $u_{j-1}u_jw_jv_j$, $j = 1, \ldots, k$, meet each other only in $|L|$, so we can apply Lemma 2 repeatedly to obtain d_1. ■

We now extend d_1 to the rest of $|L \cup \sigma|$ by induction. Let

$$L_j = L_1 \cup w_1u_1v_2 \cup \ldots \cup w_{j-1}u_{j-1}v_j$$

and

$$T_j = T_1 \cup \{w_1n_1, n_1o_1, o_1q_1, q_1v_2, \ldots, w_{j-1}n_{j-1}, n_{j-1}o_{j-1}, o_{j-1}q_{j-1}, q_{j-1}v_j\}$$

for $j = 2, \ldots, k$, and suppose that, for some $i \in \{1, \ldots, k\}$, we have a strongly convex metric d_i for L_i such that:

(i) $d_i(x, y) = d_1(x, y)$ for all x, y in L_1.

(ii) T_i is a concave collection for d_i.

If $i < k$, consider $w_iu_i \cup u_iv_{i+1}$.

PROPOSITION 2B. $w_iu_i \cup u_iv_{i+1} = \widehat{w_iv_{i+1}}$.

Proof. This is certainly true if $u_i \in \widehat{w_iv_{i+1}}$, since $\widehat{w_iu_i} = w_iu_i$ and $\widehat{u_iv_{i+1}} = u_iv_{i+1}$. But u_i must lie on $\widehat{w_iv_{i+1}}$, since $\widehat{w_iv_{i+1}}$ has to hit u_iu_{i+1} and $v_{i+1}u_i \cup u_iu_{i+1}$ is a segment by Proposition 2A(2) and (i). ■

PROPOSITION 2C. *If* $p \in \overline{L_i \setminus u_{i-1}u_iw_iv_i}$ (*or* $p \in \overline{L_i \setminus u_iu_{i+1}w_{i+1}v_{i+1}}$) *and* $x \in w_iu_i$ ($x \in u_iv_{i+1}$), *then* $d_i(p, x) = d_i(p, u_i) + d_i(u_i, x)$.

Proof. $\widehat{px}$ must hit $v_iu_{i-1} \cup u_{i-1}u_i$ ($u_iu_{i+1} \cup u_{i+1}w_{i+1}$), but $v_iu_{i-1} \cup u_{i-1}u_i \cup u_iw_i$ ($v_{i+1}u_i \cup u_iu_{i+1} \cup u_{i+1}w_{i+1}$) is a segment, so $u_i \in \widehat{px}$. ■

PROPOSITION 2D. $T_i \cup \{\widehat{w_iv_{i+1}}\}$ *is a concave collection for* d_1.

Proof. We will show that $T_i \cup \{\widehat{w_iv_{i+1}}\}$ and d_i satisfy (2.1). Suppose that $\tau \in T_i \setminus \{u_iw_i, w_im_i, m_iv_i\}$, and let $x_1, x_2 \in \widehat{w_iv_{i+1}}$, $y_1, y_2 \in \tau$ (Figure 3).

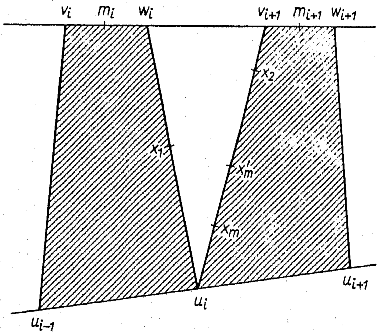

Fig. 3

If $x_1, x_2 \in u_iw_i$ or $x_1, x_2 \in u_iv_{i+1}$ then we are through, since u_iw_i and $u_iv_{i+1} \in T_i$, so assume that $x_1 \in u_iw_i$ and $x_2 \in u_iv_{i+1}$. Let x_m, y_m, and x'_m be the midpoints of $\widehat{x_1x_2}$, $\widehat{y_1y_2}$, and $\widehat{u_ix_2}$ respectively. Now, we know that $d_i(x'_m, y_m) \leqslant \frac{1}{2}[d_i(u_i, y_1) + d_i(x_2, y_2)]$. By Proposition 2C, $d_i(x_1, y_1) = d_i(x_1, u_i) + d_i(u_i, y_1)$. $d_i(x_m, x'_m) = \frac{1}{2}d_i(x_1, u_i)$, so

$$\begin{aligned} d_i(x_m, y_m) &\leqslant d_i(x_m, x'_m) + d_i(x'_m, y_m) = \tfrac{1}{2}d_i(x_1, u_i) + d_i(x'_m, y_m) \\ &\leqslant \tfrac{1}{2}[d_i(x_1, u_i) + d_i(u_i, y_1) + d_i(x_2, y_2)] = \tfrac{1}{2}[d_i(x_1, y_1) + d_i(x_2, y_2)]. \end{aligned}$$

A symmetric argument works if τ is u_iw_i, w_im_i, or m_iv_i, and the proof that $T_i \cup \{\widehat{w_iv_{i+1}}\}$ and d_i satisfy (2.2) is similar. ■

PROPOSITION 2E. *There is a strongly convex metric* d_{i+1} *for* L_{i+1} *which satisfies conditions* (i) *and* (ii) *with* $i+1$ *substituted for* i.

Proof. Propositions 2B and 2D allow us to apply Lemma 2. ■

By induction, there is a strongly convex metric d_k for $L_k = |L'|$ which satisfies conditions (i) and (ii) with k substituted for i. It is easy to check that $d' = d_k$ and $T' = T_k$ then satisfy the conclusions of the theorem. ■

COROLLARY 2. A collapsible 2-complex admits a strongly convex metric.

§ 3. A 2-complex which admits a strongly convex metric is collapsible.

DEFINITIONS. We will consider all complexes to be embedded linearly in some euclidean space, although not, of course, with the inherited metric. If C is a complex and x a point of $|C|$, we define

$\mathrm{St}(x, C) = \{\sigma \in C\colon\ x \in \sigma\}$,

$\mathrm{Lk}(x, C) = \{\tau \in C\colon\ \tau \subset |\mathrm{St}(x, C)|,\ x \notin \tau\}$.

$|\mathrm{St}(x, C)| = x|\mathrm{Lk}(x, C)|$, and we can use this cone structure of $\mathrm{St}(x, C)$ to define the natural projection $\pi(x, C)\colon\ |\mathrm{St}(x, C)| \setminus \{x\} \to |\mathrm{Lk}(x, C)|$. Similarly, if $S^\varepsilon(x) = \{y \in |C|\colon\ \|y - x\| = \varepsilon\}$ is contained in $|\mathrm{St}(x, C|$, we can define a natural homeomorphism $\pi^\varepsilon(x, C)\colon\ |\mathrm{Lk}(x, C)| \to S^\varepsilon(x)$ such that $\pi(x, C)\pi^\varepsilon(x, C) = I$, the identity. A subcomplex C' of C is a *spine* of C if C collapses to C'.

LEMMA 3. *Let* K *be an* n*-complex,* $n = 2$ *or* 3, *with a strongly convex metric* d, *and let* $L \subset K$ *be a subcomplex consisting of* n*-simplices and their faces. Then there is an* n*-simplex of* L *with a face free in* L.

Proof. Fix a point p in the interior of an n-simplex of L. The metric d induces a contraction $H\colon\ |K| \times [0, 1] \to |K|$ such that (see [2]):

(i) $H(x, 0) = x$ for all $x \in |K|$,

(ii) $H(x, 1) = p$ for all $x \in |K|$,

(iii) $H(x, t) \in \widehat{px}$ for all $x \in |K|$, $t \in [0, 1]$.

PROPOSITION 3A. *There is a point* x_0 *of* $|L|$ *such that*:

(1) x_0 *is not a vertex of* K,

(2) *for any* $y \in |L|$, $x_0 \in \widehat{py} \Rightarrow x_0 = y$.

Proof. $S^\varepsilon(p)$ separates the n-cell containing p if ε is small enough, and $S \cap H(K^{(0)} \times [0, 1])$ is a finite set. Choose $x \in S \setminus H(K^{(0)} \times [0, 1])$, and let $F = \{y \in |L|\colon\ x \in \widehat{py}\}$. F is closed and hence compact, and so contains a point x_0 such that $d(p, x) \leqslant d(p, x_0)$ for all $x \in F$. It is easy to check that x_0 satisfies Proposition 3A (1) and Proposition 3A (2). ■

PROPOSITION 3B. $|\mathrm{Lk}(x_0, L)|$ *can be shrunk to a point in* $|\mathrm{Lk}(x_0, K)|$.

Proof. Pick $t_1 \in [0,1]$ so that $H(x_0, t_1) \neq x_0$ and $H(x_0, t) \in \operatorname{int}|\operatorname{St}(x_0, K)|$ for all $t \in [0, t_1]$. Let $\pi = \pi(x_0, K)$. We can find a neighborhood N_0 of x_0 in $|K|$ such that:

(iv) $H(N_0, t) \subset \operatorname{int}|\operatorname{St}(x_0, K)|$ for all $t \in [0, t_1]$,

(v) $\pi[H(N_0, t_1)]$ can be shrunk to a point in $|\operatorname{Lk}(x_0, K)|$.

The second condition is possible because $|\operatorname{Lk}(x_0, K)|$, being a polyhedron, is locally contractible. Let $\pi_1 = \pi^{\varepsilon}(x_0, K)$, where ε is small enough that $S^{\varepsilon}(x_0) \subset N_0$, and consider the function $\pi H(\cdot, t)\pi_1$ on $|\operatorname{Lk}(x_0, L)|$ for $t \in [0, t_1]$. Proposition 3A (2) and (iv) show that it is well-defined, since $\pi_1[|\operatorname{Lk}(x_0, L)|] \subset |L|$. It is a continuous family of mappings from $|\operatorname{Lk}(x_0, L)|$ into $|\operatorname{Lk}(x_0, K)|$, and $\pi H(\cdot, 0)\pi_1 = I$. By (v), $\pi H(\cdot, t_1)\pi_1$ of $[|\operatorname{Lk}(x_0, L)|]$ can be shrunk to a point in $|\operatorname{Lk}(x_0, K)|$, and we are through. ■

PROPOSITION 3C. *x_0 lies on a $(n-1)$-simplex which is the face of exactly one n-simplex of L.*

Proof. We will consider the case when $n = 3$, leaving the case $n = 2$ for the reader to check. Let τ be the simplex of L containing x_0 in its interior. It follows from Proposition 3B that $|\operatorname{Lk}(x_0, L)|$ contains no 2-sphere, so τ is not a 3-simplex and, if τ is a 2-simplex, it is the face of exactly one 3-simplex in L.

If τ is a 1-simplex, then $\operatorname{Lk}(x_0, L)$ is the suspension of $\operatorname{Lk}(\tau, L)$, where $\operatorname{Lk}(\tau, L) = \{\mu\colon \tau\mu \in L\}$. $\operatorname{Lk}(\tau, L)$ contains no simple closed curve, because $\operatorname{Lk}(x_0, L)$ contains no 2-sphere. $\operatorname{Lk}(\tau, L)$ does contain a 1-simplex, since τ is the face of a 3-simplex in L. $\operatorname{Lk}(\tau, L)$ therefore contains a 1-simplex μ with a vertex v free in $\operatorname{Lk}(\tau, L)$. $\mu\tau$ is then a 3-simplex of $\operatorname{St}(x_0, L)$ with a face $v\tau$ free in L. ■

THEOREM 3. *If K is an n-complex, $n = 2$ or 3, with a strongly convex metric, then K has an $(n-1)$-dimensional spine.*

Proof. Suppose we have already collapsed K down to a subcomplex L. If L contains n-simplices, let L' be the collection of n-simplices of L and their faces. Applying Lemma 3 to L', we get an n-simplex of L with a face which must be free in L, since $L - L'$ is $(n-1)$-dimensional. ■

COROLLARY 3. *If K is a 2-complex with a strongly convex metric, then K is collapsible.*

Proof. K is contractible and, by Theorem 3, collapses to a 1-complex, which must also be contractible. Any contractible 1-complex is collapsible. ■

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