Nondegenerately continuous decompositions of 3-manifolds *

by

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The principal result of this paper states essentially that if a countable number of 0-dimensional upper semicontinuous decompositions of a 3-manifold, M, "fit together" properly, and if each decomposition yields M, then the sum of the decompositions will also yield M. We apply this result to certain decompositions of E^3 where the nondegenerate elements of the decompositions lie in various planes.

Let M be a metric space, and suppose K is a collection of mutually disjoint subsets of M. If $g \in K$, then K is said to be *continuous* at g in case for each $\varepsilon > 0$, there exists an open set, V, in M such that (1) $g \subset V$ and (2) if $g' \in K$ and $g' \cap V \neq \emptyset$, then $g \subset S(g', \varepsilon)$ and $g' \subset S(g, \varepsilon)$. A decomposition, G, of a metric space is said to be a *nondegenerately continuous decomposition* in case H_G (the collection of nondegenerate elements) is continuous at g for each $g \in H_G$. In general terms this means that if a sequence of nondegenerate elements converges to a nondegenerate element, g, then the "size" of the elements of the sequence approaches the "size" of g. Although placing obvious limitations on the nature of the decomposition space, the continuity restriction is not so severe as to eliminate such interesting spaces as Bing's dogbone space, which is, in fact a nondegenerately continuous decomposition.

Bing has shown the existence of a point-like decomposition of E^3 with only countably many nondegenerate elements, such that E^3/G is not homeomorphic to E^3 [4]. In addition, these nondegenerate elements may be made to lie in the union of two perpendicular planes. The following theorems and corollaries indicate among other things that similar situations can not exist if the decompositions in question are nondegenerately continuous.

Notation and Definitions. Let G be an upper semicontinuous decomposition (henceforth, referred to simply as a decomposition) of

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a space X. Then the decomposition space associated with G will be denoted by X/G, the natural projection mapping from X onto X/G by P, and H_G will denote the collection of nondegenerate elements of G. An open set, U, in X will be said to be *saturated with respect to* G (or simply *saturated* whenever the context is clear) in case $U = P^{-1}[P[U]]$.

A 3-manifold is a separable metric space with the property that each point has a neighbourhood which is a 3-cell. If M is a 3-manifold, a point p of M is an *interior point* of M if and only if p has an open neighbourhood in M which is an open 3-cell. The interior of M is the set of all interior points and the *boundary* of M is M — interior M. It should perhaps be noted that a 3-manifold as defined in this paper is referred to as a 3-manifold with boundary in the papers of Bing and Armentrout.

Suppose that M is a 3-manifold. G is said to be a *monotone decomposition* of M in case each element of G is compact and connected. Furthermore, we require that each nondegenerate element of G lie in the interior of M. A subset K of M is said to be *cellular* in case there exists a sequence $C_1, C_2, \ldots$ of 3-cells in M such that $C_i \subset$ interior C_{i-1} and $\bigcap_{i=1}^{\infty} C_i = K$. Cellular subsets of a manifold must then lie in the interior of the manifold. A decomposition of a 3-manifold is said to be cellular in case each nondegenerate element is cellular.

If X and Y are topological spaces, then a *homotopy* from X to Y is a map $H\colon X \times [a, b] \to Y$. We denote the restriction of H to $X \times \{t\}$ by H_t, for $a \leqslant t \leqslant b$. If H_t is a homeomorphism for $a \leqslant t < b$, then H is called a *pseudo-isotopy*, and if H_t is a homeomorphism for $a \leqslant t \leqslant b$ then H is called an *isotopy*.

If C is a collection of subsets of a topological space, then C^* will denote $\bigcup \{c\colon c \in C\}$. If M is a set, Cl M denotes the closure of M, Int M denotes the topological interior of M, and Bd M denotes the topological boundary of M. If ε is a positive number and A is a subset of a metric space, then $S(A, \varepsilon)$ denotes the ε-neighbourhood of A. $\emptyset$ denotes the empty set.

σ-weakly continuous decompositions. Suppose $G_1, G_2, \ldots$ is a sequence of decompositions of a 3-manifold, M, such that if $g \in H_{G_i}$, $g' \in H_{G_j}$, and $g \cap g' \neq \emptyset$, then $g = g'$. Define $\sum_{i=1}^{\infty} G_i = \bigcup_{i=1}^{\infty} H_{G_i} \cup (M - \bigcup_{i=1}^{\infty} H_{G_i})$ to be the sum of decompositions G_i. Then $G = \sum_{i=1}^{\infty} G_i$ is said to be a *σ-weakly continuous decomposition* of M in case

(1) G is a decomposition of M,

(2) for each positive integer, k, if $g \in H_{G_k}$, then $\bigcup \{H_{G_i}\colon i \neq k\} \bigcup \{g\}$ is continuous at g.

Let G be a decomposition of a metric space, M. Then G is said to be *weakly shrinkable* in case for each positive number, ε, and each open set, U, containing H_G^*, there exists a homeomorphism, h, from M onto M such that

(1) for each $g \in H_G$, diam $g < \varepsilon$

(2) if $x \in M - U$, then $h(x) = x$.

A proof for Theorem 1 when M is a 3-manifold may be found in [8] and when $M = E^3$ in [2].

THEOREM 1. *Let G be a cellular decomposition of a 3-manifold, M, such that $P[H_G^*]$ is 0-dimensional. Then M/G is homeomorphic to M if and only if G is weakly shrinkable.*

The proof of the first four conclusions of the following theorem may be found in [2]. Part (5) is established in [8].

THEOREM 2. *Suppose G is a monotone decomposition of an n-manifold, M, such that $P[H_G^*]$ is 0-dimensional and $\mathcal{U}$ is a cover in M of H_G^* by saturated open sets. Then there exists an open covering, $\mathcal{V}$, in M of H_G^* such that*

(1) *the sets of $\mathcal{V}$ are mutually disjoint;*

(2) *each set of $\mathcal{V}$ lies in some set of $\mathcal{U}$;*

(3) *if $V \in \mathcal{V}$, then* $(\mathrm{Bd} V) \cap H_G^* = \emptyset$;

(4) *if for each $V \in \mathcal{V}$, f_V is a homeomorphism from M onto M such that for $x \in M - V$, $f_V(x) = x$, then the function, f, defined as follows is a homeomorphism*

(a) *if $x \in V$,* then $f(x) = f_V(x)$,

(b) *if $x \notin \mathcal{U}\{V\colon V \in \mathcal{V}\}$, then $f(x) = x$;*

(5) *if h is a homeomorphism from M onto M with the property that if $V \in \mathcal{V}$ and $x \in \mathrm{Bd} V$, then $h(x) = x$, then for each $W \in \mathcal{V}$, $h[W] = W$.*

We now consider the principal theorem of the paper. It represents a generalization of a result of Lamoreaux [7].

THEOREM 3. *Let $G_1, G_2, \ldots$ be cellular decompositions of a 3-manifold, M, such that for each i, M/G_i is homeomorphic to M. If $G = \sum_{i=1}^{\infty} G_i$ is a σ-weakly continuous decomposition of M such that $P[H_G^*]$ is 0-dimensional, then M/G is homeomorphic to M.*

Proof. During the proof we shall always assume that all open sets are saturated with respect to G. We shall show that G is weakly shrinkable. Suppose that H_G^* is contained in an open set, U, and let ε be a positive number. Henceforth, we assume that all open sets are contained in U.

Step 1. We consider two cases.

A. Suppose $g \in H_{G_1}$.

(i) $\operatorname{diam} g \geqslant \varepsilon$. Choose an open set, U_g^1, such that:

(1) $g \subset U_g^1 \subset S(g, 1)$,

(2) if $g' \subset U_g^1$, and $g' \notin H_{G_1}$, then $\operatorname{diam} g' > \varepsilon/2$ (this possible since G is σ-weakly continuous);

(ii) $\operatorname{diam} g < \varepsilon$. Choose U_g^1 such that:

(1) $g \subset U_g^1 \subset S(g, 1)$,

(2) if $g' \subset U_g^1$ and $g' \in H_G$, then $\operatorname{diam} g' < \varepsilon$.

B. Suppose $g \in H_G$, $g \notin H_{G_1}$.

Choose U_g^1 such that:

(1) $g \subset U_g^1 \subset S(g, 1)$,

(2) $U_g^1 \cap H_{G_1}^* = \emptyset$ (possible since G is σ-weakly continuous).

Let $\mathcal{U}_1 = \{U_g^1\colon g \in H_G\}$. Let $\mathcal{V}_1$ be a refinement of $\mathcal{U}_1$ satisfying the conclusions of Theorem 2. Let $\mathcal{W}_1 = \{V^1 \in \mathcal{V}_1\colon$ there exists a $g \subset V^1$ such that $g \in H_{G_1}\}$. Then $\mathcal{W}_1^*$ is an open set containing $H_{G_1}^*$. But since M/G_1 is homeomorphic to M, we have by Theorem 1 that G_1 is weakly shrinkable. Therefore, there exists a homeomorphism, $\hat{h}_1$, from M onto M such that:

(1) if $g \in H_{G_1}$, $\operatorname{diam} \hat{h}_1[g] < \varepsilon$,

(2) if $x \in M - \mathcal{W}_1^*$, $\hat{h}_1(x) = x$.

Let $\hat{\mathcal{V}}_1 = \{V^1 \in \mathcal{V}_1\colon$ for each $g \subset V^1$, $\operatorname{diam} g < \varepsilon\}$. We define a homeomorphism, h_1, from M onto M as follows.

(1) $h_1(x) = x$, if $x \in \hat{\mathcal{V}}_1^*$,

(2) $h_1(x) = \hat{h}_1(x)$, otherwise.

Step 2. In this step we shrink elements of H_{G_2} without disturbing elements of H_{G_1}.

A. Suppose $g \in H_{G_1}$.

We choose an open set, U_g^2, such that

(1) $g \subset U_g^2 \subset S(g, 1/2) \cap V_{(g)}^1$, where $V_{(g)}^1$ is the unique open set in $\mathcal{V}_1$ containing g,

(2) if $g' \subset U_g^2$, then $\operatorname{diam} h_1[g'] < \varepsilon$. Recall that $\operatorname{diam} h_1[g] < \varepsilon$.

B. Suppose $g \in H_{G_2}$, $g \notin H_{G_1}$.

(i) $\operatorname{diam} h_1[g] \geqslant \varepsilon$. Choose U_g^2 such that:

(1) $g \subset U_g^2 \subset S(g, 1/2) \cap V_{(g)}^1$, where $V_{(g)}^1$ is the unique open set in $\mathcal{V}_1$ containing g.

(2) $U_g^2 \cap H_{G_1}^* = \emptyset$,

(3) if $g' \subset U_g^2$ and $g' \notin H_{G_2}$, then $\operatorname{diam} h_1[g'] > \varepsilon/2$.

(ii) $\operatorname{diam} h_1[g] < \varepsilon$. Choose U_g^2 such that:

(1) same as B. (i) (1),

(2) same as B. (1) (2),

(3) if $g' \subset U_g^2$, $g' \in H_G$, then $\operatorname{diam} h_1[g'] < \varepsilon$.

C. Suppose that $g \in H_G$, $g \notin H_{G_1} \cup H_{G_2}$.

Choose U_g^2 such that:

(1) same as B. (i) (1),

(2) $U_g^2 \cap (H_{G_1}^* \cup H_{G_2}^*) = \emptyset$.

Let $\mathcal{U}_2 = \{U_g^2\colon g \in H_G\}$. Let $\mathcal{V}_2$ be a refinement of $\mathcal{U}_2$ satisfying all the conclusions of Theorem 2. Let $\mathcal{W}_2 = \{V^2 \in \mathcal{V}_2\colon$ there exists an element $g \in H_{G_2}$ such that $g \subset V^2\}$.

Hence, $\mathcal{W}_2^*$ is an open set containing $H_{G_2}^*$. Therefore, by Theorem 1 and ([2], Theorem 1) there exists a homeomorphism, $\hat{h}_2$, from M onto M such that:

(1) if $g \in H_{G_2}$, $\operatorname{diam} \hat{h}_2[g] < \varepsilon$,

(2) if $x \in M - \mathcal{W}_2^*$, $\hat{h}_2(x) = h_1(x)$.

Let $\hat{\mathcal{V}}_2 = \{V^2 \in \mathcal{V}_2\colon$ if $g \subset V^2$, then $\operatorname{diam} h_1[g] < \varepsilon\}$. Define a homeomorphism, h_2, from M onto M by:

(1) $h_2(x) = h_1(x)$, if $x \in M - \hat{\mathcal{V}}_2^*$,

(2) $h_2(x) = \hat{h}_2(x)$, otherwise.

We assume that h_k, $\mathcal{U}_k$, $\mathcal{V}_k$, $\mathcal{W}_k$, and $\hat{\mathcal{V}}_k$ have been defined for $k = 1, 2, \ldots, n-1$.

Step n. At this stage all elements of $H_{G_1}, H_{G_2}, \ldots, H_{G_{n-1}}$ have been shrunk to a diameter of less than ε by h_{n-1}, and now we proceed to shrink the elements of H_{G_n} without disturbing the nondegenerate sets of the first $n-1$ decompositions.

A. Suppose $g \in H_{G_1} \cup H_{G_2} \cup \ldots \cup H_{G_{n-1}}$.

Then $\operatorname{diam} h_{n-1}[g] < \varepsilon$. We choose an open set, U_g^n, such that:

(1) $g \subset U_g^n \subset S(g, 1/n) \cap V_{(g)}^{n-1}$, where $V_{(g)}^{n-1}$ is the unique open set in $\mathcal{W}_{n-1}$ containing g.

(2) if $g' \subset U_g^n$, then $\operatorname{diam} h_{n-1}[g'] < \varepsilon$.

B. Suppose $g \in H_{G_n}$, $g \notin H_{G_1} \cup H_{G_2} \cup \ldots \cup H_{G_{n-1}}$.

(i) $\operatorname{diam} h_{n-1}[g] \geqslant \varepsilon$. Choose U_g^n such that:

(1) $g \subset U_g^n \subset S(g, 1/n) \cap V_{(g)}^{n-1}$, where $V_{(g)}^{n-1}$ is the unique open set in $\mathcal{W}_{n-1}$ containing g,

(2) $U_g^n \cap (H_{G_1}^* \cup H_{G_2}^* \cup \ldots \cup H_{G_{n-1}}^*) = \emptyset$,

(3) if $g' \subset U_g^n$ and $g' \notin H_{G_n}$, then $\operatorname{diam} h_{n-1}[g'] > \varepsilon/2$.

(ii) $\operatorname{diam} h_{n-1}[g'] < \varepsilon$. Choose U_g^n such that

(1) same as B. (i) (1)

(2) same as B. (i) (2)

(3) if $g' \subset U_2^n$ and $g' \in H_G$, then $\operatorname{diam} h_{n-1}[g'] < \varepsilon$.

C. Suppose $g \in H_G$, $g \notin H_{G_1} \cup H_{G_2} \cup \ldots \cup H_{G_n}$.

Choose U_g^n such that:

(1) same as B. (i) (1),

(2) $U_g^n \cap (H_{G_1}^* \cup H_{G_2}^* \cup \ldots \cup H_{G_n}^*) = \emptyset$.

Let $\mathfrak{U}_n = \{U_g^n \colon g \in H_G\}$. Let $\mathfrak{V}_n$ be a refinement of $\mathfrak{U}_n$ satisfying all the conclusions of Theorem 2. Let $\mathfrak{W}_n = \{V^n \in \mathfrak{V}_n$: there exists an element $g \in H_{G_n}$ such that $g \subset V^n\}$.

Hence, $\mathfrak{W}_n^*$ is an open set containing $H_{G_n}^*$. Therefore, as before, there exists a homeomorphism, h_n, from M onto M such that:

(1) $\hat{h}_n(x) = h_{n-1}(x)$, if $x \in M - \mathfrak{W}_n^*$,

(2) if $g \in H_{G_n}$, $\operatorname{diam} \hat{h}_n[g] < \varepsilon$.

Let $\hat{\mathfrak{V}}_n = \{V^n \in \mathfrak{V}_n$: if $g \subset V^n$, $g \in H_G$, then $\operatorname{diam} h_{n-1}[g] < \varepsilon\}$.

Define a homeomorphism, h_n, from M onto M by

(1) $h_n(x) = h_{n-1}(x)$, if $x \in \hat{\mathfrak{V}}_n^*$,

(2) $h_n(x) = \hat{h}_n(x)$, otherwise.

Let $h = \operatorname{Limit} h_n$. We shall now show that h is a homeomorphism from M onto itself.

Claim: *For each $x \in M$, there exists an open neighbourhood, U_x, of x and a positive integer, N, such that if $n \geqslant N$, then $h_n|_{U_x} = h_N|_{U_x}$.*

Proof. Let $x \in M$, and first suppose that $x \in H_G^*$. Let s be the first integer such that $x \in H_{G_s}^*$, and, say $x \in g_s$. Now $\operatorname{diam} h_s[g_s] < \varepsilon$. Suppose g_s is contained in $V^{s+1} \in \mathfrak{V}_{s+1}$. By Step $s+1$ we have that for each $g' \subset V^{s+1}$, $\operatorname{diam} h_s[g'] < \varepsilon$. Therefore, $V^{s+1} \in \hat{\mathfrak{V}}_{s+1}$. Thus, if we let $U_x = V^{s+1}$ and $N = s+1$, the claim will be satisfied.

Now suppose that $x \in M - H_G^*$. Since G is a decomposition of M, $\bigcup \{g \colon g \in H_G$ and $\operatorname{diam} g \geqslant \varepsilon/4\}$ is a closed set. Let U_x be an open neighbourhood of x such that $U_x \cap (\bigcup \{g \colon g \in H_G$ and $\operatorname{diam} g \geqslant \varepsilon/4\}) = \emptyset$. Choose N large enough such that if $n \geqslant N$, then whenever a set $V^n \in \mathfrak{V}_n$ contains an element $g \in H_G$ and $\operatorname{diam} g \geqslant \varepsilon/2$, V^n will not intersect U_x. Suppose that a set $V^N \in \mathfrak{V}_N$ is such that $V^N \cap U_x \neq \emptyset$. Hence, if $g \in H_G$ and $g \subset V^N$, it must be the case that $\operatorname{diam} g < \varepsilon/2$. We wish to show that for all $n \geqslant N$, $h_n|_{V^N} = h_N|_{V^N}$.

In order to do this we need only show that for each $g \in H_G$ such that $g \subset V^N$, $\operatorname{diam} h_N[g] < \varepsilon$. This would imply that if $V^{N+1} \in \mathfrak{V}_{N+1}$, and, $V^{N+1} \subset V^N$, then $V^{N+1} \in \hat{\mathfrak{V}}_{N+1}$, and, hence, $h_{N+1}|_{V^{N+1}} = h_N|_{V^{N+1}}$, or in general, if $V^{N+k} \subset V^N$, then $h_{N+k}|_{V^{N+k}} = h_N|_{V^{N+k}}$.

Let $g \subset V^N$. Let s be the first positive integer such that $g \in H_{G_s}$. If $s = 1$, then $\operatorname{diam} h_1[g] < \varepsilon$, and since g will not be moved by succeeding homeomorphisms, $\operatorname{diam} h_N[g] < \varepsilon$. Suppose $s \neq 1$. If $g \subset V^1 \in \mathfrak{W}_1$, then since $\operatorname{diam} g < \varepsilon/2$, V^1 must belong to $\hat{\mathfrak{V}}_1$ (recall Step 1, A.). Therefore, $h_1[g] = g$. If $g \cap \mathfrak{W}_1^* = \emptyset$, then, of course, we also have that $h_1[g] = g$. Thus, g is not affected by h_1. A similar argument may be used to show that for $i = 1, 2, \ldots, s-1$, $h_i[g] = g$. But in Step s, $\operatorname{diam} h_s[g] < \varepsilon$, and, further-

more, for $p = 1, 2, \ldots$, $h_{s+p}\, g = h_s[g]$. Hence, it is clear that $\operatorname{diam} h[g] < \varepsilon$. Since

$$\bigcup \{V^N \colon V^N \cap U_x \neq \emptyset\} \supset \bigcup \{V^{N+p} \colon V^{N+p} \cap U_x \neq \emptyset\},$$

it follows that for $n \geqslant N$, $h_n|_{U_x} = h_N|_{U_x}$, and the claim is established.

h is then clearly well defined and one to one. h is continuous since if a sequence, $\{x_n\}$, converges to a point, x, then eventually the sequence lies in U_x, and for a suitably large positive integer, N, $h_N|_{U_x} = h|_{U_x}$.

To see that h is onto, we observe the following. Suppose $x \in M$. If $x \in M - \mathfrak{V}_1^*$, then $h(x) = x$. Therefore, let us assume that x belongs to some set V^1 of $\mathfrak{V}_1$. Since for each positive integer, n, $h_n[V^1] = V^1$, we have that for each n, $h_n^{-1}(x) (= y_n)$ lies in V^1. Let y be a cluster point of the sequence, $\{y_n\}$. We claim that $h(y) = x$. There exists a positive integer, N, and an open neighborhood, U_y, of y such that for each $n \geqslant N$, $h_n|_{U_y} = h_N|_{U_y}$. Using subsequences if necessary, we can find an integer, $N' \geqslant N$, such that if $n \geqslant N'$, then $y_n \in U_y$. Then for $n \geqslant N'$, $h(y_n) = h_n(y_n) = x$. Since h is continuous, and the sequence $\{y_n\}$ (or possibly a subsequence of $\{y_n\}$) converges to y, $h(y) = x$.

That h^{-1} is continuous follows easily from the fact that h is onto and that for each x, there exists an open neighborhood of x, U_x, and a positive integer, N, such that $h|_{U_x} = h_N|_{U_x}$.

The function, h, then is a homeomorphism from M onto itself which shrinks elements of H_G to diameter less than ε, and is the identity on the complement of the given open set, U, containing H_G^*. Therefore, G is weakly shrinkable, and by Theorem 1, M/G is homeomorphic to M.

Corollary 3.1. *Let G be a cellular countable non-degenerately continuous decomposition of a 3-manifold, M. Then M/G is homeomorphic to M.*

Proof. Let $g_1, g_2, \ldots$ be the nondegenerate elements of G. For $i = 1, 2, \ldots$, let G_i be the decomposition of M whose only nondegenerate element is g_i. Then for each i, M/G_i is homeomorphic to M (see, for example, Bing [5]). The collection $\{G_i \colon i = 1, 2, \ldots\}$ satisfies the hypotheses of Theorem 3, and Corollary 3.1 follows.

Corollary 3.2. *Suppose G is a cellular nondegenerately continuous decomposition of E^3 such that $P[H_G^*]$ is 0-dimensional. Let $Q_1, Q_2, \ldots$ be a sequence of planes in E^3 such that for each $g \in H_G$, g is contained in at least one of these planes. Then E^3/G is homeomorphic to E^3.*

Proof. We first prove the following assertion.

Claim. *Suppose A is a cellular subset of E^3 which lies in a plane, Q. Then A is cellular in Q.*

Suppose not. Then by well known theorems on the plane, A separates Q. Let x and y be points in different components of $Q - A$. Let C be a circle perpendicular to Q and piercing Q in precisely the two points, x and y. Let c_1 and c_2 be points of C which lie in different components of $C-Q$. Since A is cellular in E^3, we may define a mapping, f, from the standard disk, D, into E^3 such that $f|_{\mathrm{Bd}D}$ is a homeomorphism onto C, and $f[D]\cap A=\emptyset$. $f^{-1}[Q\cap f[D]]$ separates $f^{-1}[c_1]$ and $f^{-1}[c_2]$ in D, and, hence, by unicoherence some component, H, of $f^{-1}[Q\cap f[D]]$ must also separate $f^{-1}[c_1]$ and $f^{-1}[c_2]$ in D. Clearly $f^{-1}[x]$ and $f^{-1}[y]$ belong to H. Therefore, $f[H]$ is connected, lies in Q, misses A, and contains x and y, which contradicts our assumption that A separated x and y. The claim is then established.

For $i=1,2,\ldots$, let $G_i=\{g\in G\colon\ g\subset Q_i\}$. From a result of Hamstrom–Dyer [6], E^3/G_i is homeomorphic to E^3. Since the collection $\{G_i\colon\ i=1,2,\ldots\}$ satisfies the hypotheses of Theorem 3, the proof of Corollary 3.2 is complete.

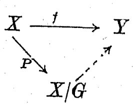
Before proceeding to Theorem 4, we shall state a number of general propositions concerning the relationship between decompositions and certain continuous functions. It is rather difficult to find actual proofs of these propositions in the literature, but these results are regarded as being well known and they have found a secure place in the theory of decomposition spaces.

If X and Y are topological spaces and f is a function from X onto Y, then f is *closed* if and only if for each closed set, M, in X, $f[M]$ is closed in Y. f is said to be compact in case for each compact set, K, in Y, $f^{-1}[K]$ is compact in X.

We shall, henceforth, assume that X and Y are metric spaces.

PROPOSITION 1. *Suppose f is a closed mapping from X onto Y. Then $G=\{f^{-1}[y]\colon\ y\in Y\}$ is an upper semi-continuous decomposition of X. G is referred to as the decomposition of X induced by f.*

Suppose as in Proposition 1, G is the decomposition induced by a *closed* mapping, f, from X onto Y. Let P be the projection map from X onto X/G. There exists a third function which we shall now describe.



Let ψ be the function from X/G into Y defined as follows: If $x\in X/G$, then $\psi(x)$ is that element, y, of Y such that $\{y\}=fP^{-1}[x]$.

PROPOSITION 2. *The function, ψ, is a homeomorphism from X/G onto Y.*

It should be noted that upper semicontinuous decompositions may be induced in some cases by mappings which are not closed. The function, ψ, will then be well defined, onto, one to one, and continuous.

Since a compact map, f, from X onto Y is closed (X and Y are metric spaces), we have the following proposition.

PROPOSITION 3. *If f is a compact mapping from X onto Y, then ψ is a homeomorphism.*

The next proposition is often quite useful.

PROPOSITION 4. *Let X be a metric space, and suppose G is a collection of mutually disjoint compact subsets of X such that $G^*=X$. Then G is an upper semicontinuous decomposition of X if and only if for each sequence, $\{x_i\}$, $x_i\in g_i\in G$, which converges to a point $x\in g\in G$, and for any sequence, $\{y_i\}$, $y_i\in g_i$, there exists a subsequence, $\{y_{n_i}\}$, of $\{y_i\}$ which converges to a point $y\in g$.*

We are now in a position to prove the following lemma.

LEMMA 4.1. *Suppose G is a decomposition of a metric space, X, into compact sets. Let h be a compact mapping from X onto X such that if g_1 and g_2 are distinct elements of G, then $h[g_1]\cap h[g_2]=\emptyset$. Let $G'=\{h[g]\colon g\in G\}$. Then G' is a decomposition of X, and X/G' is homeomorphic to X/G.*

Proof. A fairly direct proof utilizing Proposition 4 and the fact that h is compact may be used to show that G' is a decomposition of X. Let P be the projection map from X onto X/G and P' the projection map from X onto X/G'. Let $f=P'\circ h\circ P^{-1}$. Since the projection mappings and h are compact, it follows that $P'\circ h$ is a compact map from X onto X/G'. Therefore, by Proposition 3, f is a homeomorphism from X/G onto X/G' and the proof of Lemma 4.1 is concluded.

A decomposition, G, of a metric space, M, is said to be *shrinkable* in case for each covering $\mathcal{U}$ of H_G^* by saturated open sets of M, for each $\varepsilon>0$, and for an arbitrary homeomorphism, h, from M onto M, there exists a homeomorphism, f, from M onto itself such that

(1) if $x\in M-\mathcal{U}^*$, then $f(x)=h(x)$

(2) for each $g\in G$: (a) $\operatorname{diam} f[g]<\varepsilon$ and (b) there exists $D\in\mathcal{U}$ such that $h[D]\supset h[g]\cup f[g]$.

The following result is proved in [8].

LEMMA 4.2. *Suppose G is a cellular decomposition of a 3-manifold, M. Then M/G is homeomorphic to M if and only if G is shrinkable.*

We now establish a pseudo-isotopy lemma, which will also be used in the proof of Theorem 5.

LEMMA 4.3. *Let G be a monotone decomposition of E^3 such that each nondegenerate element of G lies on a line, L, formed by the intersection of two planes, Q_1 and Q_2.*

Then there exists a pseudo-isotopy from E^3 onto E^3 which shrinks distinct elements of G to distinct points, i.e., there exists a pseudo-isotopy $H\colon E^3\times[0,1]\to E^3$ such that

(1) $H_0(x) = x$, *for each* $x \in E^3$,

(2) *if* $g \in G$, *then* $H_1[g]$ *is a point in* E^3,

(3) *if* $g, g' \in G$ *and* $g \neq g'$, *then* $H_1[g] \neq H_1[g']$.

Furthermore, we may assume that H *leaves the planes,* Q_1 *and* Q_2, *and the line,* L, *set-wise fixed.*

Proof. We shall use the following notation. If F is a bounded line segment lying in L, then $m(F)$ will denote the length of F, $r(F)$ and $l(F)$ will denote the right and left endpoints, respectively, of F. We may assume that $Q_1 = \{(x, y, z): z = 0\}$ and $Q_2 = \{(x, y, z): y = 0\}$. Therefore, L is the x-axis, and, hence, when the context is clear, we shall identify $r(F)$ and $l(F)$ with the x-coordinates of $r(F)$ and $l(F)$, respectively. During the course of the proof we shall construct a number of boxes (cubes) containing selected nondegenerate elements of the decomposition. It will always be assumed that the ends of each box are perpendicular to L, and the sides are parallel either to Q_1 or Q_2. If B is such a box, then $r(B)$ will denote the right end of B and $l(B)$ the left end.

We let $g_1, g_2, \ldots$ be the elements of H_G. We shall construct the pseudo-isotopy, $H: E^3 \times [0, 1] \to E^3$ "piece by piece", i.e., first by defining it on $[0, 1/2]$, then on $[0, 3/4]$, then on $[0, 7/8]$, etc.

Step 1. In this step we shrink all nondegenerate elements of the decomposition to a length less than or equal to 1/2. Let

$$A_1 = \{g \in H_G: m(g) \geqslant 1/2\} \cup \{g_1\}.$$

We cover the sets of A_1 with a sequence $\{B_i^1\}$ of mutually disjoint boxes with the following properties

(1) For each i, B_i^1 contains precisely one element, g_i^1, of A_1, and $g_1 = g_1^1$.

(2) $g_i^1 \cap l(B_i^1) = l(g_i^1)$.

(3) $r(B_i^1) \cap H_G^* = \emptyset$.

(4) $B_i^1 \subset S(g_i^1, 1)$.

For each i, let p_i^1 be the point lying in g_i^1 such that $d(p_i^1, l(g_i^1)) = 1/2$. If $m(g_1) \leqslant 1/2$, then the isotopy we construct will be the identity on B_1^1, and p_1^1 is not defined. In each box, B_i^1, (with the possible exception of B_1^1) we construct a bisequence

$$\ldots a_{-n}, a_{-n+1}, \quad , a_{-N_i}, \quad , a_{-1}, a_0, a_1, a_2, \ldots$$

with the following properties

(1) $a_{-N_i} = p_i^1$, and $a_0 = r(g_i^1)$,

(2) the bisequence is monotone increasing,

(3) Limit $a_n = L \cap r(B_i^1)$ (n a positive integer),

(4) Limit $a_{-n} = l(B_i^1) \cap L$,

(5) $d(a_k, a_{k+1}) \leqslant 1/2$, for each integer k,

(6) if $g \in H_G$, $g \subset B_i^1$, and $g \neq g_i^1$, then there exists a positive integer, k, such that $g \subset [a_k, a_{k+1}]$.

Let $H: E^3 \times [0, 1/2] \to E^3$ be the isotopy which slides points in each box to the left such that for each integer, k, $H_{1/2}(a_k) = a_{k-N_i}$. $H: E^3 \times [0, 1/2] \to E^3$ is constructed then to satisfy the following

(1) if $x \notin \bigcup B_i^1$, then $H_t(x) = x$, for $0 \leqslant t \leqslant 1/2$,

(2) if $x \in L$, and $0 \leqslant t_1 < t_2 \leqslant 1/2$, then $H_{t_1}(x) \geqslant H_{t_2}(x)$,

(3) if $g \in H_G$, then $m(H_{1/2}[g]) \leqslant 1/2$,

(4) for $0 \leqslant t \leqslant 1/2$, $H_t[L] = L$, and for $i = 1, 2$ $H_t[Q_i] = Q_i$.

Step 2. Nondegenerate elements will now be shrunk to a diameter less than or equal to 1/4.

Let

$$A_2 = \{H_{1/2}[g]: g \in H_G \text{ and } m(H_{1/2}[g]) \geqslant 1/4\} \cup \{H_{1/2}[g_1], H_{1/2}[g_2]\}.$$

We cover the sets of A_2 with a sequence $\{B_i^2\}$ of mutually disjoint boxes with the following properties

(1) for each i, B_i^2 contains precisely one element, $H_{1/2}[g_i^2]$, of A_2, where $g_1 = g_1^2$ and $g_2 = g_2^2$,

(2) $H_{1/2}[g_i^2] \cap l(B_i^2) = l(H_{1/2}[g_i^2])$,

(3) $r(B_i^2) \cap H_{1/2}[H_G^*] = \emptyset$,

(4) $B_i^2 \subset S(H_{1/2}[g_i^2], 1/8)$,

(5) $H_{1/2}{}^{-1}[B_i^2] \subset S(g_i^2, 1/2)$,

(6) if $B_i^2 \cap B_j^1 \neq \emptyset$, then $B_i^2 \subset B_j^1$.

For each i, let p_i^2 be the point lying in $H_{1/2}[g_i^2]$ such that $d(p_i^2, l(H_{1/2}[g_i^2])) = 1/4$. Using the techniques of Step 1, we may obtain an isotopy $H: E^3 \times [1/2, 3/4] \to E^3$ where for each i, $H_{3/4}[g_i^2] = [l(g_i^2), p_i^2]$ and the isotopy satisfies the following properties

(1) if $x \notin \bigcup B_i^2$, then $H_t(x) = H_{1/2}(x)$, for $1/2 \leqslant t \leqslant 3/4$,

(2) if $x \in L$, and $1/2 \leqslant t_1 < t_2 \leqslant 3/4$, then $H_{t_1}(x) \geqslant H_{t_2}(x)$,

(3) if $g \in H_G$, then $m(H_{3/4}[g]) \leqslant 1/4$,

(4) for $1/2 \leqslant t \leqslant 3/4$, $H_t[L] = L$, and $H_t[Q_i] = Q_i$,

(5) if $x \in E^3$, then $d(H_{1/2}(x), H_{3/4}(x)) \leqslant 1/4 + 1/8$.

Step n. We now assume that $H: E^3 \times [0, (2^{n-1}-1)/2^{n-1}] \to E^3$ has been defined. We wish to "extend" H to $E^3 \times [0, (2^n-1)/2^n]$. In order to simplify the notation we denote $H_{(2^{n-1}-1)/2^{n-1}}$ by H'. Let

$$A_n = \{H'[g]: g \in H_G \text{ and } m(H'[g]) \geqslant 1/2^n\} \cup \{H'[g_1], H'[g_2], \ldots, H'[g_n]\}.$$

We cover the sets of A_n with a sequence $\{B_i^n\}$ of mutually disjoint boxes with the following properties

(1) for each i, B_i^n contains precisely one element, $H'[g_i^n]$, of A_n, where $g_1 = g_1^n, g_2 = g_2^n, \ldots, g_n = g_n^n$,

(2) $H'[g_i^n] \cap l(B_i^n) = l(H'[g_i^n])$,

(3) $r(B_i^n) \cap H'[H_G^*] = \emptyset$,

(4) $B_i^n \subset S(H'[g_i^n], 1/2^{n+1})$,

(5) $H'^{-1}[B_i^n] \subset S(g_i^n, 1/n)$,

(6) if $B_i^n \cap B_j^{n-1} \neq \emptyset$, then $B_i^n \subset B_j^{n-1}$.

For each i, let p_i^n be the point lying in $H'[g_i^n]$ such that $d(p_i^n, l(H'[g_i^n])) = 1/2^n$. If for $i = 1, 2, \ldots$, or n $m(H'[g_i^n]) \leqslant 1/2^n$, then the isotopy which we shall now define will not move points in B_i^n, and, hence, p_i^n is not defined. Using the techniques of Step 1, we may obtain an isotopy $H: E^3 \times [(2^{n-1}-1)/2^{n-1}, (2^n-1)/2^n] \to E^3$ where for each i, $H_{(2^n-1)/2^n}[g_i^n] = [l(g_i^n), p_i^n]$ and the isotopy satisfies the following properties

(1) if $x \notin \bigcup B_i^n$, then $H_t(x) = H'(x)$, for $(2^{n-1}-1)/2^{n-1} \leqslant t \leqslant (2^n-1)/2^n$,

(2) if $x \in L$, and $(2^{n-1}-1)/2^{n-1} \leqslant t_1 < t_2 \leqslant (2^n-1)/2^n$, then $H_{t_1}(x) \geqslant H_{t_2}(x)$,

(3) if $g \in H_G$, then $m(H_{(2^n-1)/2^n}[g]) \leqslant 1/2^n$,

(4) for $(2^{n-1}-1)/2^{n-1} \leqslant t \leqslant (2^n-1)/2^n$, $H_t[L] = L$, and $H_t[Q_i] = Q_i$,

(5) if $x \in E^3$, then $d(H'(x), H_{(2^n-1)/2^n}(x)) \leqslant 1/2^n + 1/2^{n+1}$.

Let $\hat{H}_k = H_{(2^k-1)/2^k} \circ H_{(2^{k-1}-1)/2^{k-1}} \circ \ldots \circ H_{1/2}$, for each positive integer k. Let $H_1 = \text{Limit } \hat{H}_k$. Since $\{\hat{H}_k\}$ is a sequence of functions which converges uniformly to H_1, it follows that H_1 is well defined, continuous, and onto. It remains to show that if g and g' are distinct elements of G, then $H_1[g] \neq H_1[g']$. This may be seen by noting that there exists a positive integer, N, such that $H_{N-1}[g]$ and $H_{N-1}[g']$ will either lie in distinct boxes B_i^N and B_j^N or perhaps will not lie in any box (recall (5) of Step n). In either case it is clear from the construction of subsequent boxes that $H_1[g] \neq H_1[g']$.

Since H leaves the planes, Q_1 and Q_2, and the line, L set-wise fixed, and for $g \in G$, $H_1[g]$ must be a point, the proof of Lemma 4.3 is complete.

THEOREM 4. *Let G be a cellular nondegenerately continuous decomposition of E^3. Let $Q_1, Q_2, \ldots, Q_m$ be a finite sequence of planes such that for each $g \in H_G$, g is contained in at least one of these planes.*

Then E^3/G is homeomorphic to E^3.

Proof. We shall prove the theorem for $m = 2$, although a similar argument holds for any finite number of planes. We first consider the special case where no elements of H_G lie in $Q_1 \cap Q_2$. For this case we shall show that G is shrinkable, and, hence, by Lemma 4.2, E^3/G is homeomorphic to E^3. Let $\mathfrak{U}$ be a cover of H_G^* by saturated open sets. Let h be an arbitrary homeomorphism from E^3 onto itself, and let ε be a positive number. We denote by G_i the nondegenerately continuous decomposition

of E^3 whose nondegenerate elements consist of those nondegenerate elements of G which lie in Q_i, for $i = 1, 2$.

Our first step consists in separating $H_{G_1}^*$ and $H_{G_2}^*$ by suitably chosen open sets. If $g_1 \in H_{G_1}$, let $a_{g_1} = d(g_1, \text{Cl}H_{G_2}^*)$, and for $g_2 \in H_{G_2}$, let $a_{g_2} = d(g_2, \text{Cl}H_{G_1}^*)$. Since G is a nondegenerately continuous decomposition of E^3, a_{g_1} and a_{g_2} will be positive numbers. For each $g \in H_G$, let U_g be a saturated open set containing g such that:

(1) $U_g \subset S(g, a_g/3)$,

(2) U_g is contained in some $U \in \mathfrak{U}$.

Suppose $g_1 \in H_{G_1}$ and $g_2 \in H_{G_2}$. We claim that $U_{g_1} \cap U_{g_2} = \emptyset$. We may assume that $a_{g_1} \leqslant a_{g_2}$. Suppose that $z \in U_{g_1} \cap U_{g_2}$. Then

$$a_{g_2} = d(g_2, \text{Cl}H_{G_1}^*) \leqslant d(g_2, z) + d(z, \text{Cl}H_{G_1}^*) \leqslant d(g_2, z) + d(z, g_1)$$
$$\leqslant a_{g_2}/3 + a_{g_1}/3 \leqslant 2a_{g_2}/3,$$

a contradiction.

For $i = 1, 2$, G_i is a decomposition of E^3 whose nondegenerate elements lie in a plane, and, hence, E^3/G_i is homeomorphic to E^3[6]. Therefore, by Lemma 4.2, G_i is shrinkable, and since $\{U_g: g \in H_{G_i}\}$ is an open cover of $H_{G_i}^*$, there exists a homeomorphism, h_i, from E^3 onto E^3 such that:

(1) for each $g \in H_{G_i}$, $\operatorname{diam} h_i[g] < \varepsilon$,

(2) $h_i[g] \cup h[g] \subset h[D]$, where $D = U_{g'}$ for some $g' \in H_{G_i}$,

(3) if $x \in E^3 - \bigcup \{U_g: g \in H_{G_i}\}$, $h_i(x) = h(x)$.

Let f be a homeomorphism from E^3 onto itself such that

(1) $f(x) = h_1(x)$, if $x \in E^3 - \bigcup \{U_g: g \in H_{G_1}\}$,

(2) $f(x) = h_2(x)$, otherwise.

Then it may be readily verified that f is the required "shrinking" homeomorphism for G, and the proof of the special case is concluded.

Now we consider the more general situation, where elements of H_G may lie in $Q_1 \cap Q_2$. Let H_1 be the function obtained in Lemma 4.3 which shrinks the nondegenerate elements lying in $Q_1 \cap Q_2$ to points and is one to one otherwise. H_1 is clearly a compact map, and if we let G' be the nondegenerately continuous decomposition of E^3 whose elements are of the form, $H_1[g]$, for $g \in G$, then by Lemma 4.1, E^3/G' is homeomorphic to E^3/G. But G' has no nondegenerate elements on $Q_1 \cap Q_2$, and, hence, by our above work, E^3/G' is homeomorphic to E^3, which completes the proof.

THEOREM 5. *Suppose that G is a cellular decomposition of E^3 such that the elements of H_G lie in either of two planes, Q_1 and Q_2. Furthermore, assume that if $g \in H_G$, and $g \cap (Q_1 \cap Q_2) \neq \emptyset$, then $g \subset Q_1 \cap Q_2$.*

Then E^3/G is homeomorphic to E^3.

Proof. Let H_1 be the function obtained in Lemma 4.3 which shrinks the nondegenerate elements lying in $Q_1 \cap Q_2$ to points and is one to one otherwise. Let $G' = \{H_1[g]: g \in G\}$. As mentioned previously, H_1 is compact and, hence, by Lemma 4.1, G' is a decomposition of E^3 and E^3/G' is homeomorphic to E^3/G. With the aid of the techniques used in the proof of Theorem 4, it may be shown that E^3/G' is homeomorphic to E^3, and, thus, Theorem 5 is established.

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