Nondegenerately continuous decompositions of 3-manifolds

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The principal result of this paper states essentially that if a countable number of 0-dimensional upper semicontinuous decompositions of a 3-manifold, \( \mathcal{M} \), "fit together" properly, and if each decomposition yields \( \mathcal{M} \), then the sum of the decompositions will also yield \( \mathcal{M} \). We apply this result to certain decompositions of \( \mathbb{E}^3 \) where the nondegenerate elements of the decompositions lie in various planes.

Let \( \mathcal{M} \) be a metric space, and suppose \( \mathcal{K} \) is a collection of mutually disjoint subsets of \( \mathcal{M} \). If \( g \in \mathcal{K} \), then \( \mathcal{K} \) is said to be continuous at \( g \) in case for each \( \epsilon > 0 \), there exists an open set, \( V \), in \( \mathcal{M} \) such that (1) \( g \subset V \) and (2) if \( g' \in \mathcal{K} \) and \( g' \cap V \neq \emptyset \), then \( g \subset S(g', \epsilon) \) and \( g' \subset S(g, \epsilon) \). A decomposition, \( \mathcal{G} \), of a metric space is said to be a nondegenerately continuous decomposition in case \( \mathcal{G}_0 \) (the collection of nondegenerate elements) is continuous at \( g \) for each \( g \in \mathcal{G}_0 \). In general terms this means that if a sequence of nondegenerate elements converges to a nondegenerate element, \( g_0 \), then the "size" of the elements of the sequence approaches the "size" of \( g \). Although placing obvious limitations on the nature of the decomposition space, the continuity restriction is not so severe as to eliminate such interesting spaces as Bing's dogbone space, which is, in fact a nondegenerately continuous decomposition.

Bing has shown the existence of a point-like decomposition of \( \mathbb{E}^3 \) with only countably many nondegenerate elements, such that \( \mathbb{E}^3/\mathcal{G} \) is not homeomorphic to \( \mathbb{E}^3 \) [4]. In addition, these nondegenerate elements may be made to lie in the union of two perpendicular planes. The following theorems and corollaries indicate among other things that similar situations can not exist if the decompositions in question are nondegenerately continuous.

Notation and Definitions. Let \( \mathcal{G} \) be an upper semicontinuous decomposition (henceforth, referred to simply as a decomposition) of

* These results form a part of the author's doctoral dissertation at the University of Iowa, prepared under the supervision of Professor B. Armentrout.
Let $G$ be a decomposition of a metric space, $M$. Then $G$ is said to be weakly shrinkable in case for each positive number, $\varepsilon$, and each open set, $U$, containing $H^c$, there exists a homeomorphism, $h$, from $M$ onto $M$ such that

1. For each $g \in H^c$, $\text{diam } g < \varepsilon$
2. If $x \in M - U$, then $h(x) = x$.

A proof for Theorem 1 when $M$ is a 3-manifold may be found in [8] and when $M = R^3$ in [2].

**Theorem 1.** Let $G$ be a cellular decomposition of a 3-manifold, $M$, such that $P[H^c] = 0$-dimensional. Then $M/G$ is homeomorphic to $M$ if and only if $G$ is weakly shrinkable.

The proof of the first four conclusions of the following theorem may be found in [2]. Part (5) is established in [8].

**Theorem 2.** Suppose $G$ is a monotone decomposition of an $n$-manifold, $M$, such that $P[H^c] = 0$-dimensional and $\cup U$ is a cover of $M$ of $H^c$ by saturated open sets. Then there exists an open covering, $\cup U$, in $M$ of $H^c$ such that

1. The sets of $U$ are mutually disjoint;
2. Each set of $U$ lies in some set of $U$;
3. If $V \in U$, then $(BdV) \cap H^c = \emptyset$;
4. For each $V \in U$, $f_V$ is a homeomorphism from $M$ onto $M$ such that for $x \in M - V$, $f_V(x) = x$, then the function, $f_V$, defined as follows is a homeomorphism
   1. If $x \in V$, then $f_V(x) = f_V(x)$;
   2. If $x \not\in \cup U$, then $f_V(x) = x$;
5. If $h$ is a homeomorphism from $M$ onto $M$ with the property that if $V \in U$ and $x \in BdV$, then $h(x) = x$, then for each $W \in U$, $h(W) = W$.

We now consider the principal theorem of the paper. It represents a generalization of a result of Loman (7).

**Theorem 3.** Let $G_1, G_2, \ldots$ be cellular decompositions of a 3-manifold, $M$, such that for each $i$, $M/G_i$ is homeomorphic to $M$. If $G = \bigcup_{i=1}^{n} G_i$ is an $\sigma$-weakly continuous decomposition of $M$ such that $P[H^c] = 0$-dimensional, then $M/G$ is homeomorphic to $M$.

**Proof.** During the proof we shall always assume that all open sets are saturated with respect to $G$. We shall show that $G$ is weakly shrinkable. Suppose that $H^c$ is contained in an open set, $U$, and let $\varepsilon$ be a positive number. Henceforth, we assume that all open sets are contained in $U$.
Step 1. We consider two cases.

A. Suppose \( g \in H_{\alpha} \).
   (i) \( \text{diam } g > \varepsilon \). Choose an open set, \( U_2 \), such that:
   (1) \( g \subset U_2 \subset S(g, 1) \).
   (2) if \( g' \subset U_2 \) and \( g' \not\in H_{\alpha} \), then \( \text{diam } g' > \varepsilon/2 \) (this possible since \( G \) is \( \sigma \)-weakly continuous).
   (ii) \( \text{diam } g < \varepsilon \). Choose \( U_2 \) such that:
   (1) \( g \subset U_2 \subset S(g, 1) \).
   (2) if \( g' \subset U_2 \) and \( g' \not\in H_{\alpha} \), then \( \text{diam } g' < \varepsilon \).

B. Suppose \( g \in H_{\alpha} \), \( g \not\in H_{\alpha} \).
   Choose \( U_2 \) such that:
   (1) \( g \subset U_2 \subset S(g, 1) \).
   (2) \( U_2 \cap H_{9} = \emptyset \) (possible since \( G \) is \( \sigma \)-weakly continuous).

Let \( U_3 = (U_2)^{c} \setminus H_{9} \). Let \( U_3 \) be a refinement of \( U_3 \) satisfying all the conclusions of Theorem 2. Let \( U_3 = (V^3)^{c} \cup U_3 \); there exists an element \( g \in H_{\alpha} \) such that \( g \subset V^3 \).

C. Suppose that \( g \in H_{\alpha} \), \( g \not\in H_{\alpha} \).
   Choose \( U_2 \) such that:
   (1) \( g \subset U_2 \subset S(g, 1) \).
   (2) \( U_2 \cap (H_{9} \cup H_{9}) = \emptyset \).

Let \( U_4 = (U_2)^{c} \). Let \( U_4 \) be a refinement of \( U_4 \) satisfying all the conclusions of Theorem 2. Let \( U_4 = (V^4)^{c} \cup U_4 \); there exists an element \( g \in H_{\alpha} \) such that \( g \subset V^4 \).

Hence, \( U_2 \) is an open set containing \( H_{9} \). Therefore, by Theorem 1 and (2), Theorem 1 there exists a homeomorphism, \( \tilde{h}_k \), from \( M \) onto \( M \) such that:
   (1) \( g \in H_{\alpha} \), \( \text{diam } \tilde{h}_k(g) < \varepsilon \).
   (2) \( \tilde{h}_k(x) = h_k(x) \) if \( x \in M - U_2 \).

Let \( U_5 = (V^5)^{c} \cup U_5 \); if \( g \subset V^5 \), then \( \text{diam } h_k(g) < \varepsilon \). Define a homeomorphism, \( h_k \), from \( M \) onto \( M \) by:
   (1) \( h_k(x) = h_k(x) \) if \( x \in M - U_5 \).
   (2) \( h_k(x) = h_k(x) \), otherwise.

We assume that \( h_k, U_1, U_2, U_4, \) and \( V_5 \) have been defined for \( k = 1, 2, \ldots, n-1 \).

Step 2. At this stage all elements of \( H_{\alpha} \), \( H_{\alpha} \), \( \ldots, H_{\alpha} \), have been shrunk to a diameter of \( \varepsilon/2 \) by \( h_{\alpha-1} \), and now we proceed to shrink the elements of \( H_{\alpha} \) without disturbing the nondegenerate sets of the first \( n-1 \) decompositions.

A. Suppose \( g \in H_{\alpha} \).
   Then \( \text{diam } h_{\alpha-1}(g) < \varepsilon \). Choose an open set, \( U_2 \), such that:
   (1) \( g \subset U_2 \subset S(g, 1/\varepsilon) \cup V_\alpha^{\alpha-1} \), where \( V_\alpha^{\alpha-1} \) is the unique open set in \( U_\alpha^{\alpha-1} \) containing \( g \).
   (2) \( g \subset U_2 \), then \( \text{diam } h_{\alpha}(g') < \varepsilon \).

B. Suppose \( g \in H_{\alpha} \), \( g \not\in H_{\alpha} \).
   (i) \( \text{diam } h_{\alpha}(g) > \varepsilon \). Choose \( U_2 \) such that:
   (1) \( g \subset U_2 \subset S(g, 1/\varepsilon) \cup \overline{V}_\alpha^{\alpha-1} \), where \( \overline{V}_\alpha^{\alpha-1} \) is the unique open set in \( U_\alpha^{\alpha-1} \) containing \( g \).
   (2) \( U_2 \cap H_{9} = \emptyset \).
   (3) if \( g' \subset U_2 \) and \( g' \not\in H_{\alpha} \), then \( \text{diam } h_{\alpha}(g') > \varepsilon/2 \).
   (ii) \( \text{diam } h_{\alpha}(g) < \varepsilon \). Choose \( U_2 \) such that:
   (1) same as B. (i) (1).
   (2) same as B. (i) (2).
   (3) if \( g' \subset U_2 \), then \( \text{diam } h_{\alpha}(g') < \varepsilon \).
Nondegenerately continuous decompositions of 3-manifolds

W. Vossman

Let $\mathcal{U}_x = (U^*_x; g \in \mathcal{H}_x)$. Let $\mathcal{U}_x$ be a refinement of $\mathcal{U}_x$ satisfying all the conclusions of Theorem 2. Let $\mathcal{U}_x = (V^*_x; g \in \mathcal{V}_x)$; there exists an element $g \in \mathcal{H}_x$ such that $g \in V^*_x$.

Hence, $V^*_x$ is an open set containing $H_x$. Therefore, as before, there exists a homeomorphism, $h_x$, from $M$ onto $M$ such that:
1. $h_x(x) = h_x(x_0)$, if $x \neq x_0$ in $U^*_x$;
2. $g \in H_x$, then $h_x(g) = g$.

Let $h_x = (V^*_x; g \in h_x)$ if $g \in H_x$, then $h_x(g) = g$. Define a homeomorphism, $h_x$, from $M$ onto $M$ by:
1. $h_x(x) = h_x(x_0)$, if $x \in U^*_x$;
2. $h_x(x) = h_x(x_0)$, otherwise.

Let $h = \text{Limit } h_x$. We shall now show that $h$ is a homeomorphism from $M$ onto itself.

Claim: For each $x \in M$, there exists an open neighborhood, $U_x$, of $x$ and a positive integer, $N_x$, such that if $n > N_x$, then $h_n(x) = h(x)$.

Proof. Let $x \in M$, and first suppose that $x \in H_x$. Then $x \in U^*_x$, and $x \neq g_x$. Now $\text{diam } h_x(g_x) < \varepsilon$. Suppose $g_x$ is contained in $V^*_x$. By Step 1, we have that for each $g \in V^*_x$, $\text{diam } h_x(g) < \varepsilon$. Thus, $h_x = h(x)$.

Now suppose that $x \in U_x$. Since $G$ is a decomposition of $M$, $G \cap \{g : g \in H_x \text{ and } \text{diam } h_x(g) < \varepsilon\}$ is a closed set. Let $U_x$ be an open neighborhood of $x$ such that $U_x \cap \{g : g \in H_x \text{ and } \text{diam } h_x(g) < \varepsilon\} = \emptyset$. Choose $N_x$ large enough such that if $n > N_x$, then whenever a set $V^*_x \in U_x$ contains an element $g \in H_x$ and $\text{diam } h_x(g) > \varepsilon/2$, $V^*_x$ will not intersect $U_x$. Suppose that a set $V^*_x \in U_x$ such that $V^*_x \cap U_x = \emptyset$. Hence, if $g \in H_x$ and $\text{diam } h_x(g) > \varepsilon/2$, then $h(x) = h(x)$.

In order to do this we need only show that for each $g \in H_x$ such that $g \in V^*_x$, $\text{diam } h_x(g) < \varepsilon$. This would imply that $h_x(V^*_x \in U_x) \subseteq V^*_x \cap V^*_x$, then $\text{diam } h_x(g) < \varepsilon$, and, hence, $h_x(V^*_x \in U_x) = h_x(V^*_x \in U_x)$ or in general, if $V^*_x \cap V^*_x$, then $h_x(V^*_x \in U_x) = h_x(V^*_x \in U_x)$.

Let $g \in V^*_x$. Let $x$ be the first positive integer such that $g \in H_x$. If $s = 1$, then $h_x(g) < \varepsilon$. Since $g \in H_x$ will not be moved by succeeding homeomorphisms, $\text{diam } h_x(g) < \varepsilon$. Suppose $s \neq 1$. If $g \in V^*_x \in U_x$, then since $\text{diam } g < \varepsilon$, $V^*_x \in U_x$. Therefore, $h_x(g) = g$. If $g \in U_x = \emptyset$, then, of course, we have that $h_x(g) = g$.

Thus, $g$ is not affected by $h_x$. A similar argument may be used to show that for $i = 1, 2, \ldots, h_x(g) = g$. But in Step 1, $\text{diam } h_x(g) < \varepsilon$, and further,
a circle perpendicular to \(Q\) and piercing \(Q\) in precisely the two points, \(x\) and \(y\). Let \(c_1\) and \(c_2\) be points of \(C\) which lie in different components of \(C\) -- \(Q\). Since \(D\) is cellular in \(E^3\), we may define a mapping \(f\), from the standard disk, \(D\), into \(E^3\) such that \(f_{|D} = f_{|\partial D}\) is a homeomorphism onto \(C\), and \(f_{|D} \cap A = \emptyset\). Then \(f^{-1}[Q \cap f_{|D}]\) separates \(f^{-1}[c_1]\) and \(f^{-1}[c_2]\). If \(H\) is connected, lies in \(Q\), and contains \(x\) and \(y\), it contradicts our assumption that \(A\) separated \(x\) and \(y\). The claim is then established.

For \(i = 1, 2, \ldots\), let \(G_i = (g \in G : g \in Q_i)\). From a result of Hamstrom-Dyer [5], \(E^3/G\) is homeomorphic to \(E^3\). By the collection \(\{G_i : i = 1, 2, \ldots\}\) satisfies the hypotheses of Theorem 3, the proof of Corollary 3.2 is complete.

Before proceeding to Theorem 4, we shall state a number of general propositions concerning the relationship between decompositions and certain continuous functions. It is rather difficult to find actual proofs of these propositions in the literature, but these results are regarded as being well known and they have found a secure place in the theory of decomposition spaces.

If \(X\) and \(Y\) are topological spaces and \(f\) is a function from \(X\) onto \(Y\), then \(f\) is closed if and only if for each closed set, \(M\), in \(X\), \(f^{-1}[M]\) is closed in \(Y\). If \(f\) is said to be compact in case for each compact set, \(K\), in \(Y\), \(f^{-1}[K]\) is compact in \(X\).

We shall, henceforth, assume that \(X\) and \(Y\) are metric spaces.

**Proposition 1.** Suppose \(f\) is a closed mapping from \(X\) onto \(Y\). Then \(\bar{f} = (f^{-1}[y] : y \in Y)\) is an upper semi-continuous decomposition of \(X\). It is referred to as the decomposition of \(X\) induced by \(f\).

Suppose as in Proposition 1, \(G\) is the decomposition induced by a closed mapping \(f\) from \(X\) onto \(Y\). Let \(P\) be the projection map from \(X\) onto \(X/G\). There exists a third function which we shall now describe.

\[
\begin{array}{ccc}
X & \rightarrow & Y \\
\downarrow & & \downarrow \\
X/G & \rightarrow & Y/G
\end{array}
\]

Let \(g\) be the function from \(X/G\) into \(Y\) defined as follows: If \(x \in X/G\), then \(y(x)\) is that element \(y\) of \(Y\) such that \(y = fP^{-1}[x]\).

**Proposition 2.** The function, \(y\), is a homeomorphism from \(X/G\) onto \(Y\).

It should be noted that upper semi-continuous decompositions may be induced in some cases by mappings which are not closed. The function, \(y\), will then be well defined, onto, one to one, and continuous.
Nondegenerately continuous decompositions of 3-manifolds

(1) \( H_a(x) = x \) for each \( x \in E^0 \).
(2) If \( g \in G \), then \( H_a(g) \) is a point in \( E^0 \).
(3) If \( g, g' \in G \) and \( g \neq g' \), then \( H_a(g) \neq H_a(g') \).

Furthermore, we may assume that \( H \) leaves the planes, \( Q_1 \) and \( Q_2 \), and the line, \( L \), set-wise fixed.

Proof. We shall use the following notation. If \( F \) is a bounded line segment lying in \( L_n \) then \( m(F) \) will denote the length of \( F \), \( r(F) \) and \( l(F) \) will denote the right and left endpoints, respectively, of \( F \). We may assume that \( Q_1 = \{(x, y, z) : z = 0\} \) and \( Q_2 = \{(x, y, z) : y = 0\} \). Therefore, \( L \) is the \( z \)-axis, and, hence, when the context is clear, we shall identify \( r(F) \) and \( l(F) \) with the \( x \)-coordinates of \( r(F) \) and \( l(F) \), respectively.

During the course of the proof we shall construct a number of boxes (cubes) containing selected nondegenerate elements of the decomposition.

It will always be assumed that the ends of each box are perpendicular to \( L \), and the sides are parallel either to \( Q_1 \) or \( Q_2 \). \( H \) is such a box, then \( r(L) \) will denote the right end of \( L \) and \( l(L) \) the left end.

We let \( g_1, g_2, \ldots \) be the elements of \( H_0 \). We shall construct the pseudomapping \( H : E^0 \times [0, 1] / E^0 \) "piece by piece", i.e., first by defining it on \([0, 1/2] \), then on \([0, 3/4] \), then on \([0, 7/8] \), etc.

Step 1. In this step we shrink all nondegenerate elements of the decomposition to a length less than or equal to \( 1/2 \). Let

\[ A_1 = \{ (g \in H_0) : m(g) > 1/2 \} \cup \{ g_1 \} \]

We cover the sets of \( A_1 \) with a sequence \( \{B'_i\} \) of mutually disjoint boxes with the following properties
(1) For each \( i \), \( B'_i \) contains precisely one element, \( g_i \), of \( A_1 \), and \( g_0 = g_1 \).
(2) \( g_i \cap l(B'_i) = \emptyset \).
(3) \( r(B'_i) \cap B'_i = \emptyset \).
(4) \( B'_i \subseteq S(g_i, 1/2) \).

For each \( i \), let \( p_i \) be the point lying in \( g_i \) such that \( d(p_i, l(B'_i)) = 1/2 \). Using the techniques of Step 1, we may obtain an isometry \( H : E^0 \times (1/2, 3/4) / E^0 \) where for each \( i \), \( H_{g_i}(p_i) = \frac{1}{2}(g_i, p_i) \) and the isometry satisfies the following properties
(1) \( \frac{1}{2} < \frac{1}{2} \), then \( H_{g_i}(x) = H_{g_i}(x) \) for \( 1/2 < t < 3/4 \),
(2) \( \frac{1}{4} < t < \frac{1}{2} \), then \( H_{g_i}(x) > H_{g_i}(x) \),
(3) \( \frac{1}{4} < t < \frac{1}{2} \), then \( H_{g_i}(x) < H_{g_i}(x) \),
(4) \( \frac{1}{4} < t < \frac{1}{2} \), then \( H_{g_i}(x) = H_{g_i}(x) \),
(5) \( \frac{1}{4} < t < \frac{1}{2} \), then \( H_{g_i}(x) = H_{g_i}(x) \).

Step 2. Now assume that \( H : E^0 \times (0, 2^{m-1}) / E^0 \) has been defined. We wish to "extend" \( H \) to \( E^0 \times (0, 2^{m-1} / 2^n) \). In order to simplify the notation we denote \( H_{2^n-1} \) by \( H' \). Let

\[ A_2 = \{ H_{g_i}(g) : g \in G_0 \} \]

We cover the sets of \( A_2 \) with a sequence \( \{B''_i\} \) of mutually disjoint boxes with the following properties
(1) \( a_{n-1}, a_{n-1} + 1 < 1/2 \), for each integer \( n \),
(2) if \( g \in H_0 \), then \( g \in H_0 \), and \( g \neq g' \), then \( g \neq g' \).
(3) Limit \( a_n = l(B''_i) \) (a a positive integer),
(4) Limit \( a_n = l(B''_i) \) (a a positive integer).
(1) For each \( i \), \( B_i^3 \) contains precisely one element, \( H_i[g_i^1] \), of \( A_n \), where \( g_i = g_i^1, g_i = g_i^2, \ldots, g_i = g_i^{a_i} \),

(2) \( H_i^0[g_i] \cap \{ B_i^3 \} \neq \emptyset \),

(3) \( \{ B_i^3 \} \cap H_i^0 = \emptyset \),

(4) \( B_i^2 \subset S(H_i[g_i^1], 1/2^k) \).

(5) \( H_i^0 \cap B_i^3 \subset S(g_i^1, \alpha_i) \),

(6) \( \{ B_i^3 \} \cap \{ B_j^3 \} = \emptyset \) for \( i \neq j \).

For each \( i \), let \( p_i^1 \) be the point lying in \( H_i[g_i^1] \) such that \( d(p_i^1, \{ H_i[g_i] \}) = 1/2^k \). If for \( i = 1, 2, \ldots, n \), or for \( m(H_i[g_i]) \subset 1/2^k \), then the isotopy which we shall now define will not move points in \( B_i^3 \), and, hence, \( p_i^1 \) is not defined. Using the techniques of Step 1, we may obtain an isotopy \( H : E^3 \times (2^{k+1} - 1)/2^k \rightarrow E^3 \) where for each \( i \), \( H_{\alpha = \text{largest} \{ \alpha_i \} = \{ \alpha_i \}}[g_i^1] \) and \( p_i^1 \) is the isotopy satisfies the following properties

(1) \( p \in \{ B_i^3 \} \), then \( H_i(x) = H_i(x), f_0(x) = f_0(x) \), for \( f_0(x) < 1/2^k \),

(2) \( f(x) \in E^3 \), and \( d(p_i^1, H_i^0) < t_i \), \( i = 1, 2, \ldots, n \), then \( H_i(x) \in H_i(x) \).

(3) \( g \in H_i^0 \), then \( H_i(x) = H_i(x) \), for \( d(p_i^1, H_i^0) < 1/2^k \).

(4) \( f \in H_i^0 \), \( f(x) < t_i \), \( i = 1, 2, \ldots, n \), then \( H_i(x) = H_i(x) \).

(5) \( x \in E^3 \), then \( d(p_i^1, H_i^0) < 1/2^k \).

Let \( H \rightarrow H_{\alpha = \text{largest} \{ \alpha_i \} = \{ \alpha_i \}} \subset H_{\alpha = \text{largest} \{ \alpha_i \} = \{ \alpha_i \}} \subset \ldots \subset H_{\alpha = 2^{k+1} \text{mod} \{ \alpha_i \}} \), for each positive integer \( k \).

Let \( H_k = \text{Limit} \). Since \( H \) is a sequence of functions which converges uniformly to \( H_k \), it follows that \( H_k \) is well defined, continuous, and onto. It remains to show that if \( g \) and \( g' \) are distinct elements of \( G \), then \( H_k(g) \neq H_k(g') \). This can be seen by noting that there exists a positive integer \( N \), such that \( H_{\alpha = N} - H_{\alpha = N+1} \subset \emptyset \) and \( H_{\alpha = g} \) and \( H_{\alpha = g'} \) will either lie in distinct boxes \( B_i^3 \) and \( B_j^3 \) or perhaps will not lie in any box (recall (5) of Step 6).

In either case it is clear from the construction of subsequence boxes that \( H_k(g) \neq H_k(g') \).

Since \( H \) leaves the planes, \( Q_1 \) and \( Q_2 \), and the line, \( L \) set-wise fixed, and for \( g \in G \), \( H_k(g) \) must be a point, the proof of Lemma 4.3 is complete.

Theorem 4. Let \( G \) be a cellular nondegenerately continuous decomposition of \( E^3 \). Let \( Q_1, Q_2, \ldots, Q_n \) be a finite sequence of planes such that for each \( g \in H, g \in E^3 \) is contained in at least one of these planes.

Then \( E^3 \) is homeomorphic to \( E^3 \).

Proof. We shall prove the theorem for \( m = 2 \), although a similar argument holds for any finite number of planes. We first consider the special case where no elements of \( H \) lie in \( Q_1 \). For this case we shall show that \( G \) is shrinkable, and, hence, by Lemma 4.2, \( E^3 \) is homeomorphic to \( E^3 \). Let \( H \) be a cover of \( H \) by saturated open sets. Let \( h \) be an arbitrary homeomorphism from \( E^3 \) onto itself, and let \( \epsilon \) be a positive number. We denote by \( G \) the nondegenerately continuous decomposition of \( E^3 \) whose nondegenerate elements consist of those nondegenerate elements of \( G \) which lie in \( Q_1 \), for \( i = 1, 2 \).

Our first step consists in separating \( H \), and \( H \) by suitably chosen open sets. If \( g \in H \), and \( g \in H \), let \( \delta_g = d(g, H \cup H), and for \( g \in H \), let \( \delta_g = d(g, H \cup H) \). Since \( G \) is a nondegenerately continuous decomposition of \( E^3 \), \( \alpha_g \), and \( \alpha_g \) will be positive numbers. For each \( g \in H \), let \( U_g \) be a saturated open set containing \( g \) such that

(1) \( U_g \subset S(g, \alpha_g^0) \),

(2) \( U_g \) is contained in some \( U \in \mathcal{U} \).

Suppose \( g \in H_1 \) and \( g \in H_2 \). We claim that \( U_{g_1} \cap U_{g_2} = \emptyset \). We may assume that \( \alpha_0 < \alpha_1 \). Suppose that \( \epsilon \in U_{g_1} \cap U_{g_2} \). Then \( \alpha_0 = \min \{ g_1, g_2, \alpha_1 \} < \min \{ g_1, g_2, \alpha_1 \} + \min \{ g_1, g_2, \alpha_1 \} < \min \{ g_1, g_2, \alpha_1 \} + \alpha_0 + \alpha_1 < \alpha_0 + \alpha_1 < 0 \), a contradiction.

For \( i = 1, 2 \), \( G_i \) is a decomposition of \( E^3 \) whose nondegenerate elements lie in a plane, and, hence, \( E^3 \) is homeomorphic to \( E^3 \). Therefore, by Lemma 4.2, \( H \) is shrinkable, and since \( \{ U \in \mathcal{U} \mid g \in H \} \) is an open cover of \( H \), there exists a homeomorphism, \( h \), from \( E^3 \) onto \( E^3 \) such that

(1) \( f(x) = h(x) \), \( x \in E^3 \), \( f(x) = h(x) \), \( x \in E^3 \), \( f(x) = h(x) \), \( x \in E^3 \), \( f(x) = h(x) \), otherwise.

Then it may be readily verified that \( f \) is the required "shrinking" homeomorphism for \( G \), and the proof of the special case is concluded.

Now we consider the more general situation, where elements of \( H \) may lie in \( Q_1 \) or \( Q_2 \). Let \( H_1 \) be the function obtained in Lemma 4.3 which shrinks the nondegenerate elements lying in \( Q_1 \) to \( Q_1 \) to points and is one to one otherwise. \( H_1 \) is clearly a compact map, and if we let \( G' \) be the nondegenerately continuous decomposition of \( E^3 \) whose elements are of the form, \( H_i[g_i^1] \) for \( g_i < G_i \), then by Lemma 4.1, \( E^3 \) is homeomorphic to \( E^3 \). But \( G' \) has no nondegenerate elements on \( Q_1 \), and, hence, by our above work, \( E^3 \) is homeomorphic to \( E^3 \), which completes the proof.

Theorem 5. Suppose that \( G \) is a cellular decomposition of \( E^3 \) such that the elements of \( H \) lie in either of two planes, \( Q_1 \) and \( Q_2 \). Furthermore, assume that if \( g \in H \), and \( g \cap (Q_1 \cap Q_2) \neq \emptyset \), then \( g \subset (Q_1 \cap Q_2) \).

Then \( E^3 \) is homeomorphic to \( E^3 \).

Proof. Let \( H_1 \) be the function obtained in Lemma 4.3 which shrinks the nondegenerate elements lying in \( Q_1 \) to \( Q_1 \) to points and is one to one
otherwise. Let \( G' = \{ H_i(g); g \in G \} \). As mentioned previously, \( H_i \) is compact and, hence, by Lemma 4.1, \( G' \) is a decomposition of \( E^3 \) and \( E^3/G' \) is homeomorphic to \( E^3/G \). With the aid of the techniques used in the proof of Theorem 4, it may be shown that \( E^3/G' \) is homeomorphic to \( E^3 \), and, thus, Theorem 5 is established.

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