morphism for every $i \in I$. Equivalently, this can be expressed in $\mathcal{A}$ as follows: there is an epimorphism γ^* mapping a into a free product $g^* = \sum_{i \in I} a_i(\pi_i^*)$ such that $\pi_i^* \gamma^* = a_i^*$ is a (normal) monomorphism for each $i \in I$. In this case the object a is called a *transfree image* of the objects a_i, $i \in I$. (This concept is introduced and discussed in [5].) The object a_i is said to be *transfreely irreducible* if the union of all its proper ideals is again a proper one. The notion of transfree irreducibility is dual to that of subdirect irreducibility.

In particular, if $C(\infty) \in \mathcal{A}$ is a transfree image of objects a_i, then every a_i can be regarded as a subgroup of $C(\infty)$, and so each a_i is isomorphic to $C(\infty)$. Since the union of all proper subgroups of $C(\infty)$ is $C(\infty)$ itself, the components a_i cannot be transfreely irreducible. Dualizing, we find that $C(\infty)$, as an object of $\mathcal{A}^*$, cannot be subdirectly embedded in a direct product of subdirectly irreducible objects, and the theorem is proved.

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Generalized connected functions

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1. Introduction. A function $f\colon S \to T$ is said to be *connected* if it maps every connected set in S onto a connected set in T. Every continuous function is connected and the question as to when a connected function is continuous has been studied by many authors; for example, [2]–[5]. In this article S will denote a regular topological T_1-space with a base $\mathfrak{B}$ for the open sets such that every $U \in \mathfrak{B}$ is connected. The generalized connected function studied here will be a function f taking S to a T_1-space T such that $f(\overline{U})$ is connected in T for every $U \in \mathfrak{B}$. Such functions will be called functions *connected with respect to* $\mathfrak{B}$ or, simply, *connected* $(\mathfrak{B})$ functions. These functions have been studied in [1] for a domain restricted to euclidean space and for a range which is separable metric.

In this article some theorems on conditions implying continuity of connected $(\mathfrak{B})$ functions are presented as well as a sufficient condition as to when a connected $(\mathfrak{B})$ function is a connected function. In Section 3 it is shown that Theorem 2.1 is a generalization of the well known result in functional analysis (a linear functional f is continuous if and only if the null space of f is closed). It is shown that a linear functional is continuous if and only if it is connected. Finally, in Theorem 4.1, a condition is given as to when a certain type of function is a homeomorphism.

It is clear that a connected function on S is a connected $(\mathfrak{B})$ function and if f is a connected $(\mathfrak{B})$ function on S, then it can be easily shown that f takes all connected, open sets onto connected sets. In particular, it follows that $f(U)$ is connected for each $U \in \mathfrak{B}$. An example of a function which is connected $(\mathfrak{B})$ with respect to a certain base $(\mathfrak{B})$, but which is not connected, is provided in [1]. Another interesting example is given in Section 3 below.

2. Continuity of connected $(\mathfrak{B})$ functions. The following theorem gives a necessary and sufficient condition under which a connected $(\mathfrak{B})$ function is continuous. This is a generalization of Theorem 3 of [1] and of Theorem C of [3]. In particular, if f is real valued, then f is continuous

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if and only if the inverse image of any point is closed. The boundary of a set N will be denoted by $\mathrm{bd}N$.

THEOREM 2.1. *If $f\colon S\to T$ is a connected ($\mathfrak{B}$) function, then f is continuous if and only if $f^{-1}(\mathrm{bd}N)$ is closed for each set N belonging to a base for the open sets in T.*

Proof. To show continuity of f at $x \in S$ consider $f(x)$ and neighborhood N of $f(x)$. By hypothesis, $f^{-1}(\mathrm{bd}N)$ is closed and with $x \notin f^{-1}(\mathrm{bd}N)$, there is a member $U \in \mathfrak{B}$ such that $x \in U \subset \sim f^{-1}(\mathrm{bd}N)$ ($\sim$ denotes complement). Now, recalling that, by a remark in the introduction $f(U)$ is connected. Since $f(U)$ contains $f(x) \in N$ and misses $\mathrm{bd}N$ it follows that $f(U) \subset N$. This shows continuity of f at x, and since x is arbitrary, f is continuous on S.

The converse is obvious.

Remark 2.1. In the above theorem if f is simply a function which takes connected, open sets to connected sets, the proof holds without the assumption of regularity on S.

DEFINITION 2.1. A function $f\colon S\to T$ has *at worst a removable discontinuity* at $x \in S$ if there is a $y \in T$ such that for each neighborhood V of y there is a neighborhood U of x such that $f(U-\{x\}) \subset V$.

Theorem 3 of [2] generalizes Theorem 3.6 of [4] and the following theorem extends the result of [2] to connected ($\mathfrak{B}$) functions. The proof is analogous to that in [4] and Remark 2.1 holds here also.

THEOREM 2.2. *Let S be as above and let T be a Hausdorff space. A connected ($\mathfrak{B}$) function $f\colon S\to T$ is continuous at $x \in S$ if and only if f has at worst a removable discontinuity at x.*

DEFINITION 2.2. Let $f\colon S\to T$ be any function and denote by $C(f;\, x)$ the set of all $y \in T$ such that for each neighborhood N of y and each neighborhood M of x the set $f^{-1}(N) \cap M$ is not empty.

It can be shown that $y \in C(f;\, x)$ if and only if there is some net $\{x_\alpha\}$ converging to x for which the net $\{f(x_\alpha)\}$ converges to y. Note that $f(x) \in C(f;\, x)$ for every $x \in S$.

LEMMA 2.1. *Let $\mathfrak{N}$ denote the neighborhood system of $x \in S$. Then $C(f;\, x) = \bigcap \overline{f(N)}$, $(N \in \mathfrak{N})$.*

Proof. For any $y \in C(f;\, x)$ there exists a net $\{x_\alpha\}$ converging to x such that the net $\{f(x_\alpha)\}$ converges to y. The net $\{x_\alpha\}$ is eventually in N for each $N \in \mathfrak{N}$ and, consequently, the net $\{f(x_\alpha)\}$ is eventually in each $f(N)$. Since $\{f(x_\alpha)\}$ converges to y it follows that y is in each $\overline{f(N)}$. Thus, $y \in \bigcap \overline{f(N)}$, $(N \in \mathfrak{N})$.

Conversely, pick $y \in \bigcap \overline{f(N)}$, $(N \in \mathfrak{N})$ and let $\mathfrak{M}$ denote the system of neighborhoods of $y \in T$. For each $N \in \mathfrak{N}$ and for each $M \in \mathfrak{M}$ choose a point $y(M, N) \in M \cap f(N)$ and let the point $x(M, N) \in N$ be such that its image is $y(M, N)$. This can be done since $y \in \overline{f(N)}$ for each $N \in \mathfrak{N}$. Thus, $\{y(M, N)\} = \{f(x(M, N))\}$ is a net which converges to y and $\{x(M, N)\}$ converges to x. From the definition $y \in C(f;\, x)$.

COROLLARY 2.1. *For any $f\colon S\to T$ and any $x \in S$, the set $C(f;\, x)$ is closed in T.*

LEMMA 2.2. *Let S be as above and let T be a compact Hausdorff space. If $f\colon S\to T$ is a connected ($\mathfrak{B}$) function, then $C(f;\, x)$ is connected in T for each $x \in S$.*

Proof. With only minor modifications the proof is analogous to that of Theorem 3.7 of [4].

THEOREM 2.3. *With S and T as in Lemma 2 a connected ($\mathfrak{B}$) function $f\colon S\to T$ is continuous at $x \in S$ if and only if $C(f;\, x)$ is finite or denumerable.*

The proof is analogous to that Theorem 3.8 of [4].

THEOREM 2.4. *Let S and T be as in Lemma 2. If $f\colon S\to T$ is a connected ($\mathfrak{B}$) function such that for each non-degenerate connected subset C of S $C(f;\, x) \subset f(C)$ for each $x \in C$, then f is a connected function.*

Proof. Suppose that for some connected subset C of S, $f(C)$ is not connected and that $f(C) = A \cup B$ is a separation. If $A_1 = \{x \in C | f(x) \in A\}$ and $B_1 = \{x \in C | f(x) \in B\}$, then $C = A_1 \cup B_1$, $A_1 \cap B_1 = \varphi$ and $A_1 \neq \varphi$, $B_1 \neq \varphi$. Since C is connected we may, without loss of generality, suppose that $\bar{A}_1 \cap B_1 \neq \varphi$. If $x \in \bar{A}_1 \cap B_1$, then $f(x) \in B$ and there is a net $\{x_\alpha\} \subset A_1$ which converges to x. Since $\{f(x_\alpha)\} \subset A \subset \bar{A}$ and since $\bar{A}$ is compact there is a subnet $\{f(x_\beta)\}$ of $\{f(x_\alpha)\}$ which converges to some point $y \in \bar{A}$ and the subnet $\{x_\beta\}$ of $\{x_\alpha\}$ will still converge to x. Thus, $y \in C(f;\, x)$ and since $C(f;\, x) \subset f(C)$ it follows that $y \in f(C)$; in particular, $y \in A$ since $\bar{A} \cap B = \varphi$. By hypothesis. By Lemma 2, $C(f;\, x)$ is a connected subset of $f(C)$ and, thus, cannot intersect both A and B. However, $y \in A$ and $f(x) \in B$ so this is a contradiction.

EXAMPLE 2.1. The following well known function g satisfies all the conditions of Theorem 4, but is not a continuous function; in fact, it is not a connectivity function. Let $I = [0, 1]$ and define $f\colon I\to I$ by

$$f(x) = \limsup_{n=1,2,\ldots} \frac{a_1 + a_2 + \ldots + a_n}{n},$$

for $0 \leqslant x \leqslant 1$, where $x = (0 \cdot a_1 a_2 \ldots)$ is the dyadic development of x. The function takes on each value in I on each interval and is thus a connected function. Now consider the function $g\colon I\to I$ defined by $g(x) = 0$ when $x = f(x)$ and $g(x) = f(x)$, otherwise. The function g still takes on each value in I on each interval but the graph of g does not meet the diagonal $y = x$ in $I\times I$ and so it is not connected. However, $C(g;\, x) \subset g(C)$ for each interval $C \subset I$ and for each $x \in C$.

3. Linear functionals. Denote by L a separated topological linear space with real or complex scalar field Φ and let $f\colon L\to\Phi$ be a linear functional. It is known [6] that if f is not continuous, then $f^{-1}(0)$, the *null space of* f, is dense in L. A continuous function is a connected function and the following theorem shows that the converse is also true for linear functionals.

THEOREM 3.1. *If $f\colon L\to\Phi$ is a non-continuous linear functional, then f is not connected.*

Proof. Every connected function g must satisfy the property $g(\overline{C})\subset\overline{g(C)}$ for every connected set C in the domain if the range is an R_0-space (Sanderson, [5]). Since f is not continuous $f^{-1}(0)$ is dense in L and is also connected since it is a linear subspace. If $K=f^{-1}(0)$, it is easy to see that $f(\overline{K})\not\subset\overline{f(K)}$.

THEOREM 3.2. *If $f\colon L\to\Phi$ is a non-continuous linear functional and C is a subset of L with a non-empty interior, then $f(C)=\Phi$.*

Proof. If $f(C)\neq\Phi$, pick $t\in\Phi$ such that $t\notin f(C)$ and consider the dense set $f^{-1}(t)$ in L. Since C has a non-empty interior, $C\cap f^{-1}(t)\neq\varphi$. Therefore $t\in f(C)$ and this is a contradiction.

COROLLARY 3.1. *Every linear functional $f\colon L\to\Phi$ is a connected* $(\mathfrak{B})$ *function.*

Proof. It need only be remarked that every topological linear space is locally connected, and members of $\mathfrak{B}$ have non-empty interiors.

Since a linear functional $f\colon L\to\Phi$ is continuous if and only if its real part is continuous, Corollary 3.1 shows that the result from functional analysis (a linear functional f is continuous if and only if $f^{-1}(0)$ is closed) is a special case of Theorem 2.1. It is not a special case of the existing theorems in the literature since a linear functional need not be connected.

4. Homeomorphic spaces. A topological space is called *rim-compact* if it is Hausdorff and the topology has a base for the open sets such that the boundary of each member of the base is compact. It is known that a rim-compact space is regular.

THEOREM 4.1. *Let S and T be rim-compact spaces with bases $\mathfrak{B}$ and $\mathfrak{B}'$, respectively, for the open sets consisting of connected open sets. Consider a one-to-one connected $(\mathfrak{B})$ function $f\colon S\to T$ such that $f(\overline{U})$ is also closed for each $U\in\mathfrak{B}$. Suppose also that f^{-1} is connected $(\mathfrak{B}')$ and that $f^{-1}(\overline{U'})$ is closed for each $U'\in\mathfrak{B}'$. Then f is a homeomorphism.*

Proof. To show f is continuous, by Theorem 2.1 we need only show that $f^{-1}(\mathrm{bd}\,N)$ is closed for each neighborhood N of each $y\in T$. Without loss of generality, suppose N is a neighborhood of y with a compact boundary. For any $x\notin f^{-1}(\mathrm{bd}\,N)$, $f(x)\notin\mathrm{bd}\,N$ so by hypothesis on $\mathrm{bd}\,N$ and by Hausdorff property there is a finite cover of $\mathrm{bd}\,N$ by U'_i,

$i=1,2,\ldots,n$, with each $U'_i\in\mathfrak{B}'$, such that $f(x)\notin\bigcup_{i=1}^{n}\overline{U'_i}$. Now $x\notin\bigcup_{i=1}^{n}f^{-1}(\overline{U'_i})$ and since each term of the finite union is closed by hypothesis, x is in an open set which does not meet $f^{-1}(\mathrm{bd}\,N)$. Therefore $f^{-1}(\mathrm{bd}\,N)$ is closed.

A similar argument shows that f^{-1} is continuous.

COROLLARY 4.1. *If S and T are locally connected and locally compact Hausdorff spaces, then any biconnected function $f\colon S\to T$ is a homeomorphism* (see Theorem 3.10 of [4]).

Proof. Locally compact, Hausdorff spaces are rim-compact and, for spaces S and T as general as R_0-spaces, a biconnected function $f\colon S\to T$ is such that both f and f^{-1} take closed connected sets to closed connected sets.

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