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Finitely generated semigroups of continuous functions on [0,1]

by

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1. Introduction.

DEFINITION 1.1. Let C denote the topological semigroup of continuous function of $[0,1]$ into $[0,1]$ employing the composition product and uniform topology. We will use the norm notation for the uniform metric $\|f-g\| = \sup_{0\leqslant x\leqslant 1} |f(x)-g(x)|$.

DEFINITION 1.2. Let C_0 denote the subsemigroup of C consisting of those elements of C which map $[0,1]$ onto $[0,1]$.

In [2], the authors show that there exist two elements of C which together generate a dense subsemigroup of C. One of the functions is $g(x) = \frac{1}{2}+\frac{1}{2}x$ and the other contains "copies" of elements of a countable dense subset of C. The main result of this paper is Theorem 3.6 which asserts that there are two fairly elementary elements of C_0 which together generate a dense subsemigroup of C_0. The techniques of proof in this paper are entirely different from those in [2].

The motivation for this work comes from the theory of inverse limit spaces. One would like to choose the minimum number of functions and the simplest possible functions as bonding maps in an inverse limit system. In this regard, the corollaries following Theorem 3.6 may be useful. [1] and [5] are applications of [2] to inverse limit spaces.

Without specific reference, all of the functions in this paper are assumed to be in C_0.

2. The prime functions.

DEFINITION 2.1. A function $f \in C_0$ is called *prime* if f is not a homeomorphism and $f = f_1 f_2$ for $f_1, f_2 \in C_0$ implies that either f_1 or f_2 is a homeomorphism.

DEFINITION 2.2. Let PM denote the subsemigroup of C_0 consisting of those functions which are made up of a finite number of strictly monotone pieces. That is, $f \in \mathrm{PM}$ if there exists a partition $0 = a_0 < a_1 < \ldots$

$< a_n = 1$ $n \geqslant 1$ such that f is increasing or decreasing on each interval $[a_{i-1}, a_i]$, $i = 1, 2, \ldots, n$. Note that PM contains the polynomials in C_0.

LEMMA 2.3. *If f is prime then $f \in$ PM.*

Proof. We note first that f must be light. Suppose, on the contrary, that f is constant on some interval $[a, b]$. Let f_2 be a function such that $f_2(x) = x$ for all $x \notin [a, b]$, $f_2([a, b]) = [a, b]$ and f_2 is not monotone on $[a, b]$. Then $f = ff_2$ and neither f nor f_2 is a homeomorphism.

Since f is not a homeomorphism, there exist numbers a and b such that $0 \leqslant a < b \leqslant 1$ and $f(a) = f(b)$. We can now apply Theorem 1 of [4] and obtain a factorization $f = f_1 f_2$ where f_1 is a polynomial and for all $x \in [0, 1]$, $|f_2(x) - x| < \frac{1}{2}(b-a)$. If we suppose that f_2 is not a homeomorphism, then f_1 must be a homeomorphism and $f_1 f_2(a) = f_1 f_2(b)$ implies that $f_2(a) = f_2(b)$.

$$f_2(a) < a + \tfrac{1}{2}(b-a) = \tfrac{1}{2}(a+b) \quad \text{and} \quad f_2(a) = f_2(b) > b - \tfrac{1}{2}(b-a) = \tfrac{1}{2}(a+b)$$

and so the assumption that f_2 is not a homeomorphism leads to a contradiction. It follows that $f = f_1 f_2 \in$ PM.

DEFINITION 2.4. If $f \in$ PM, then the standard partition for f is the partition $0 = a_0 < a_1 < a_2 \ldots < a_n = 1$ such that f is monotone on each of the intervals $[a_{i-1}, a_i]$, $i = 1, 2, \ldots, n$, and

$$[f(a_i) - f(a_{i-1})][f(a_{i+1}) - f(a_i)] < 0, \qquad i = 1, 2, \ldots, n-1.$$

f is said to have n pieces.

LEMMA 2.5. *If f is prime and f has more than two pieces, then $f^{-1}(\{0, 1\}) = \{0, 1\}$.*

Proof. If we suppose that the theorem is false, then we can suppose without loss of generality that there exists a number a between 0 and 1 such that $f(a) = 0$ and $f([a, 1]) = [0, 1]$. Let b be such that $[0, b] = f([0, a])$.

Let f_1 and f_2 be defined as follows:

$$f_1(x) = \begin{cases} m_1(x), & \text{a linear function of } [0, a] \text{ onto } [0, b], \\ m_2(x), & \text{a linear function of } [a, 1] \text{ onto } [0, 1]; \end{cases}$$

$$f_2(x) = \begin{cases} m_1^{-1} f(x) & \text{on} \quad [0, a], \\ m_2^{-1} f(x) & \text{on} \quad [a, 1]. \end{cases}$$

f_1 and f_2 are well defined and continuous and $f = f_1 f_2$. f_2 is not a homeomorphism since by hypothesis, either $m_1^{-1} f$ or $m_2^{-1} f$ must have at least two pieces. We need to show that $f_2([0, 1]) = [0, 1]$. Choose $x \in [0, a]$ such that $f(x) = b$. Then $f_2(x) = m_1^{-1} f(x) = m_1^{-1}(b) = 0$. Next choose $y \in [a, 1]$ such that $f(y) = 1$. Then $f_2(y) = m_2^{-1} f(y) = m_2^{-1}(1) = 1$. Thus the factorization $f = f_1 f_2$ is contrary to the hypothesis that f is prime and so the lemma is proved.

The technique of factorization which was used in the proof of Lemma 2.5 will be used again in later proofs but with less detail.

DEFINITION 2.6. Let I denote the identity function and $J = 1 - I$.

DEFINITION 2.7. The functions L, V and Z are defined as follows:

$$L(x) = \begin{cases} 1 - 2x & \text{for} \quad 0 \leqslant x \leqslant \frac{1}{2}, \\ x - \frac{1}{2} & \text{for} \quad \frac{1}{2} \leqslant x \leqslant 1; \end{cases}$$

$$Z(x) = \begin{cases} 2x & \text{for} \quad 0 \leqslant x \leqslant \frac{1}{3}, \\ 1 - x & \text{for} \quad \frac{1}{3} \leqslant x \leqslant \frac{2}{3}, \\ 2x - 1 & \text{for} \quad \frac{2}{3} \leqslant x \leqslant 1; \end{cases}$$

$$V(x) = \begin{cases} 2x & \text{for} \quad 0 \leqslant x \leqslant \frac{1}{2}, \\ 2 - 2x & \text{for} \quad \frac{1}{2} \leqslant x \leqslant 1. \end{cases}$$

DEFINITION 2.8. The functions $f, g \in C_0$ are said to be *topologically equivalent* if there exist homeomorphisms $h, k \in C_0$ such that $f = hgk$. This is denoted by $f \simeq g$.

LEMMA 2.9. *If f is prime, then f is topologically equivalent to one of the functions L, V or Z.*

Proof. Every member of PM which has only two pieces is topologically equivalent to either L or V and so it remains to show that if f is prime and has more than two pieces, then $f \simeq Z$.

Suppose f is prime and has more than two pieces. By Lemma 2.5, we have either Case 1. $f(0) = 0$ and $f(1) = 1$ or Case 2. $f(0) = 1$ and $f(1) = 0$. We need to consider only Case 1 since Jf is prime if and only if f is prime.

Let $0 = a_0 < a_1 < a_2 < \ldots < a_n = 1$ be the standard partition for f. Now f is increasing on $[0, a_1]$ and decreasing on $[a_1, a_2]$ with $0 < f(a_2) < f(a_1) < 1$. Let $b = \min\{(a_1, 1] \cap f^{-1}f(a_1)\}$. For all $a_1 < x < b$, $f(x) < f(a_1)$. We can now show that f is increasing on $[b, 1]$. Suppose not and define f_1 and f_2 as follows:

$$f_1(x) = \begin{cases} m(x), & \text{a linear function of } [0, b] \text{ onto } [0, f(b)], \\ f(x) & \text{for} \quad b \leqslant x \leqslant 1; \end{cases}$$

$$f_2(x) = \begin{cases} m^{-1} f(x) & \text{for} \quad 0 \leqslant x \leqslant b, \\ x & \text{for} \quad b \leqslant x \leqslant 1; \end{cases}$$

$f = f_1 f_2$ and neither factor is a homeomorphism.

We now choose C between a_1 and b such that $f(c) = \min\limits_{a_1 \leqslant x \leqslant b} f(x)$.

Define f_1 and f_2 as follows:

$$f_1(x) = \begin{cases} f(x) & \text{for} \quad 0 \leqslant x \leqslant a_1, \\ m_1(x), & \text{a linear function of } [a_1, c] \text{ onto } [f(c), f(a_1)], \\ m_2(x), & \text{a linear function of } [c, b] \text{ onto } [f(c), f(b)], \\ m_3(x), & \text{a linear function of } [b, 1] \text{ onto } [f(b), 1]; \end{cases}$$

$$f_2(x) = \begin{cases} x & \text{for} \quad 0 \leqslant x \leqslant a_1\,, \\ m_1^{-1}f(x) & \text{for} \quad a_1 \leqslant x \leqslant c\,, \\ m_2^{-1}f(x) & \text{for} \quad c \leqslant x \leqslant b\,, \\ m_3^{-1}f(x) & \text{for} \quad b \leqslant x \leqslant 1\,; \end{cases}$$

$f = f_1 f_2$ and since f is prime and f_1 is not a homeomorphism, then f_2 is a homeomorphism. Therefore $f \simeq Z$ since $f_1 \simeq Z$.

THEOREM 2.10. *f is prime if and only if f is topologically equivalent to one of the functions L, V or Z.*

Theorem 2.10 completes the characterization of the primes but we will need to prove two more lemmas first.

DEFINITION 2.11. If $f \in \mathrm{PM}$ and $0 = a_0 < a_1 < \ldots < a_n = 1$ is the standard partition for f, then $\{a_1, \ldots, a_{n-1}\}$ is called the set of vertices of f. $v(f) = n-1$, the number of vertices of f. Note that $v(f) = 0$ if and only if f is a homeomorphism.

LEMMA 2.12. *If $f_1, f_2 \in \mathrm{PM}$, then $v(f_1 f_2) \geqslant v(f_1) + v(f_2)$.*

Proof. First we observe that if $f = f_1 f_2$, then

(1) each vertex of f_2 is a vertex of f,

and

(2) if a is a vertex of f_1, then each number in $f_2^{-1}(a)$ is a vertex of f.

If a is a vertex of f_1, then there exists a number in $f_2^{-1}(a)$ w. not a vertex of f_2. Let $0 = a_0 < a_1 < \ldots < a_n = 1$ be the standard tition for f_2 and for each $i = 1, 2, \ldots, n$, let A_i denote the number vertices of f_1 in $f_2\big((a_{i-1}, a_i)\big)$. Then by application of (1) and (2),

$$v(f) = v(f_1 f_2) \geqslant \sum_{i=1}^{n} A_i + v(f_2) \geqslant v(f_1) + v(f_2)\,.$$

LEMMA 2.13. *If $f = f_1 f_2 \in \mathrm{PM}$, then $f_1, f_2 \in \mathrm{PM}$.*

Proof. First suppose that $f_2 \notin \mathrm{PM}$. Then every partition of $[0,1]$ contains an interval on which f_2 is not a homeomorphism. Thus every partition of $[0,1]$ contains an interval which contains a point x in its interior such that if O is an open interval containing x, then there exist numbers $a, b \in O$ such that $f_2(a) = f_2(b)$. It follows that there exists a sequence $O_1, O_2, O_3, \ldots$ of disjoint open intervals such that for each $i \geqslant 1$, there are numbers $a_i, b_i \in O_i$ such that $f_2(a_i) = f_2(b_i)$. But if $f_2(a_i) = f_2(b_i)$, then $f(a_i) = f(b_i)$. This is a contradiction because $f \in \mathrm{PM}$ cannot have such a property.

Now suppose that $f_1 \notin \mathrm{PM}$ and choose sequences $\{O_i\}$, $\{a_i\}$ and $\{b_i\}$ in the same manner for f_1. For each $i \geqslant 1$, there exists an open interval U_i (open relative to $[0,1]$) such that $f_2(U_i) = O_i$. Choose $c_i, d_i \in U_i$ such that $f_2(c_i) = a_i$ and $f_2(d_i) = b_i$. We are led to a contradiction as before since the intervals $\{U_i\}$ are disjoint and $f(c_i) = f(d_i)$.

Proof of Theorem 2.10. We now need only to show that each of L, V and Z is indeed prime. Lemmas 2.12 and 2.13 imply that if $f \in \mathrm{PM}$ and f is not prime, then $v(f) \geqslant 2$. Thus each of L and V is prime.

Now suppose that Z is not prime. If $Z = f_1 f_2$, then $f_1, f_2 \in \mathrm{PM}$ and $v(f_1) = v(f_2) = 1$ by application of Lemmas 2.12 and 2.13. It follows that each of f_1 and f_2 is topologically equivalent to either L or V. So there exists a homeomorphism h such that one of the following holds: Case 1. $Z \simeq VhV$; Case 2. $Z \simeq LhV$; Case 3. $Z \simeq VhL$; or Case 4. $Z \simeq LhL$.

Case 1 and Case 2 are impossible because $VhV(0) - VhV(1) = Vh(0) - Vh(0) = 0$ and $LhV(0) - LhV(1) = Lh(0) - Lh(0) = 0$. Case 3 is impossible because $VhL(0) - VhL(\frac{1}{2}) = Vh(1) - Vh(0) = 0$.

Case 4 is disposed of by considering that $[0, \frac{1}{2}] \cap L^{-1}h^{-1}([0, \frac{1}{2}]) = A$ is a nondegenerate closed interval and LhL takes A homeomorphically onto $[0, 1]$. A contradiction is reached in each case and so the theorem is proved.

THEOREM 2.14. *If $f \in \mathrm{PM}$ and f is not a homeomorphism, then f can be factored $f = f_1 f_2 \ldots f_n$ where for each $i = 1, 2, \ldots, n$, f_i is topologically equivalent to L, V or Z. That is, there exists a positive integer n and a factorization $f = h_1 g_1 h_2 g_2 \ldots g_n h_{n+1}$ where h_i is a homeomorphism, $i = 1, 2, \ldots, n+1$, and g_i is one of the functions L, V or Z, $i = 1, 2, \ldots, n$.*

Proof. If f is a prime, then the conclusion follows trivially. If f is not a prime, then f is the product of finitely many nonhomeomorphisms in PM. The number of terms in the product may not exceed $v(f)$ because of Lemma 2.12. So if $f = f_1 f_2 \ldots f_n$ where n is maximum, then each factor f_i, $i = 1, 2, \ldots, n$, is prime. This completes the proof.

As a consequence of Theorem 2.14, we have that if H denotes the group of all homeomorphisms in C_0, then PM, a dense subsemigroup of C_0, is generated by $\{L, V, Z\} \cup H$. Furthermore, by making use of [3] we can still obtain a dense subsemigroup of C_0 by choosing only two elements of H.

COROLLARY 2.15. *There exist two increasing homeomorphisms θ_1, θ_2 such that $\{L, V, Z, \theta_1, \theta_2\}$ generates a dense subsemigroup of C_0.*

Proof. In [3], Knichal proves that there are two increasing homeomorphisms θ_1 and θ_2 which together generate a dense subsemigroup of the group of all increasing homeomorphisms. (Knichal's Theorem is generalized by Theorem 3.3 of this paper.) It follows immediately that $\{\theta_1, \theta_2, J\}$ generates a dense subsemigroup of the group of all homeomorphisms. The corollary will be proved if we can show that J can be approximated by finite compositions of L, θ_1 and θ_2. For this purpose, suppose $1 > \varepsilon > 0$ and let h and k be the increasing homeomorphisms defined as follows: $h(\frac{1}{2}) = \varepsilon$ and h is linear on each of the intervals $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$. $k(1-\varepsilon) = \frac{1}{4}$ and k is linear on each of the intervals $[0, 1-\varepsilon]$ and $[1-\varepsilon, 1]$.

We will show that $\|J - hLk\| < \varepsilon$. First suppose $0 \leqslant x \leqslant 1-\varepsilon$. hLk is linear on $[0, 1-\varepsilon]$ and since $hLk(0) = 1$ and $hLk(1-\varepsilon) = \varepsilon$, it follows that $J(x) = hLk(x)$. Now suppose $1-\varepsilon \leqslant x \leqslant 1$. Then $\frac{1}{4} \leqslant k(x) \leqslant 1$ and $0 \leqslant Lk(x) \leqslant \frac{1}{2}$ and so $0 \leqslant hLk(x) \leqslant \varepsilon$. It follows that $\|J - hLk\| < \varepsilon$.

Since h and k can be approximated by finite compositions of θ_1 and θ_2, the proof is now complete.

3. A dense subsemigroup of C_0 generated by two elements. We begin this section by showing that there are two homeomorphisms which together generate a dense subsemigroup of the group of all homeomorphisms.

DEFINITION 3.1. An increasing homeomorphism h is said to be *above the identity* if for all $0 < x < 1$, $h(x) > x$ or *below the identity* if for all $0 < x < 1$, $h(x) < x$.

LEMMA 3.2. *There exists an increasing homeomorphism φ above the identity with the following property:*

Given sequences $0 < a_1 < a_2 < \ldots < a_r < 1$ and $0 < b_1 < b_2 < \ldots < b_r < 1$ and $\varepsilon > 0$, there exist sequences $0 < c_1 < c_2 < \ldots < c_r < 1$ and $0 < d_1 < d_2 < \ldots < d_r < 1$ and positive integers m and n such that $|c_i - a_i| < \varepsilon$, $|d_i - b_i| < \varepsilon$ and $\varphi^m J \varphi^n J(c_i) = d_i$, $i = 1, 2, \ldots, r$.

Proof. By the lemma of [3], there exist increasing homeomorphisms θ_1 above the identity and θ_2 below the identity with the following property:

(*) Given sequences $0 < a_1 < a_2 < \ldots < a_r < 1$ and $0 < b_1 < b_2 < \ldots < b_r < 1$ of rational numbers, there exist positive integers m and n such that $\theta_1^m \theta_2^n(a_i) = b_i$, $i = 1, 2, \ldots, r$.

We choose such a pair θ_1, θ_2 and choose a number a such that $0 < a < \frac{1}{2}$ and $\theta_1(a) < \frac{1}{2}$. Note that

$$J\theta_2 J(1-a) = J\theta_2(a) > J(a) > \tfrac{1}{2}.$$

Let φ be the homeomorphism defined by letting $\varphi(x) = \theta_1(x)$ for $0 \leqslant x \leqslant a$ $\varphi(x) = J\theta_2 J(x)$ for $1-a \leqslant x \leqslant 1$ and φ is linear on $[a, 1-a]$. It is not difficult to check that φ is above the identity and each of $J\varphi J$ and φ^{-1} is below the identity.

Let $0 < a_1 < a_2 < \ldots < a_r < 1$ and $0 < b_1 < b_2 < \ldots < b_r < 1$ be given. There exists a positive integer n_1 such that $J\varphi^{n_1} J(a_r) < a$ and there exists a positive integer m_1 such that $\varphi^{-m_1}(b_r) < a$.

Now we can choose sequences of rational numbers $0 < a_1' < a_2' < \ldots a_r' < a$ and $0 < b_1' < b_2' < \ldots b_r' < a$ such that

$$|J_\varphi^{-n_1} J(a_i') - a_i| < \varepsilon \quad \text{and} \quad |\varphi^{m_1}(b_i') - b_i| < \varepsilon, \quad i = 1, 2, \ldots, r.$$

Let $c_i = J\varphi^{-n_1} J(a_i')$ and $d_i = \varphi^{m_1}(b_i')$.

By (*), we can choose positive integers m_2 and n_2 such that $\theta_1^{m_2}\theta_2^{n_2}(a_i') = b_i'$, $i = 1, 2, \ldots, r$. Now consider $\varphi^m J \varphi^n J(c_i)$ where $m = m_1 + m_2$ and $n = n_1 + n_2$.

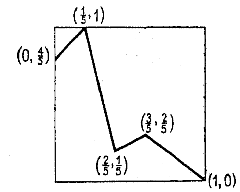
$$\varphi^{m_1+m_2} J \varphi^{n_1+n_2} J(c_i) = \varphi^{m_1+m_2} J \varphi^{n_1+n_2} J\big(J\varphi^{-n_1} J(a_i')\big) = \varphi^{m_1+m_2} J \varphi^{n_2} J(a_i')$$
$$= \varphi^{m_1+m_2} J J \theta_2^{n_2} J J(a_i') = \varphi^{m_1} \theta_1^{m_2} \theta_2^{n_2}(a_i') = \varphi^{m_1}(b_i') = d_i.$$

THEOREM 3.3. *There exists an increasing homeomorphism φ above the identity such that if h is a homeomorphism and $\varepsilon > 0$, there exist positive integers m and n such that $\|h - \varphi^m J \varphi^n J\| < \varepsilon$ in case h is increasing and $\|h - \varphi^m J \varphi^n\| < \varepsilon$ in case h is decreasing.*

Proof. Let φ be as in Lemma 3.2. In case h is increasing, the proof is essentially the same as the proof of Theorem 1 in [3]. If h is decreasing, then hJ is increasing and $\|hJ - \varphi^m J \varphi^n J\| = \|h - \varphi^m J \varphi^n\|$.

DEFINITION 3.4. The function f is defined as follows:

$$f(x) = \begin{cases} \frac{4}{5} + x & \text{for} \quad 0 \leqslant x \leqslant \frac{1}{5} \\ \frac{9}{5} - 4x & \text{for} \quad \frac{1}{5} \leqslant x \leqslant \frac{2}{5} \\ x - \frac{1}{5} & \text{for} \quad \frac{2}{5} \leqslant x \leqslant \frac{3}{5} \\ 1 - x & \text{for} \quad \frac{3}{5} \leqslant x \leqslant 1 \end{cases} =$$



LEMMA 3.5. *If φ is a homeomorphism as in Theorem 3.3 and $\varepsilon > 0$, then there exist positive integers m and n such that $\|\varphi^m f \varphi^n - J\| < \varepsilon$.*

Proof. If $1 > \varepsilon > 0$, there exist positive integers m and n such that $\|\varphi^m J \varphi^n - J\| < \varepsilon$ and large enough that $\varphi^n(\varepsilon) > \frac{4}{5}$ and $\varphi^m(\frac{1}{5}) > 1-\varepsilon$.

First suppose that $0 \leqslant x \leqslant \varphi^{-n}(\frac{4}{5})$ and note that $\varphi^{-n}(\frac{4}{5}) < \varepsilon$. Then $0 \leqslant \varphi^n(x) \leqslant \frac{4}{5}$ and so

$$\tfrac{1}{5} \leqslant f\varphi^n(x) \leqslant 1 \quad \text{and} \quad 1 - \varepsilon < \varphi^m(\tfrac{1}{5}) \leqslant \varphi^m f \varphi^n(x) \leqslant 1.$$

It follows that if $0 \leqslant x \leqslant \varphi^{-n}(\frac{4}{5})$, then

$$|\varphi^m f \varphi^n(x) - J(x)| < \varepsilon.$$

Now suppose $\varphi^{-n}(\frac{4}{5}) \leqslant x \leqslant 1$. Then $\frac{4}{5} \leqslant \varphi^n(x) \leqslant 1$ and so

$$f\varphi^n(x) = J\varphi^n(x) \quad \text{and} \quad \varphi^m f \varphi^n(x) = \varphi^m J \varphi^n(x).$$

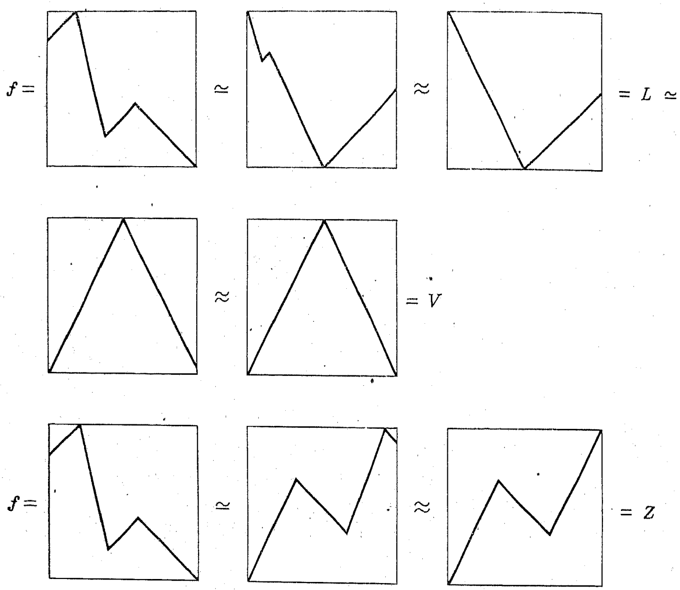
Thus if $0 \leqslant x \leqslant 1$, then

$$|\varphi^m f \varphi^n(x) - J(x)| < \varepsilon.$$

THEOREM 3.6. *If φ is a homeomorphism as in Theorem 3.3, then $\{\varphi, f\}$ generates a dense subsemigroup of C_0.*

Proof. Let Γ denote the closure of the semigroup generated by $\{\varphi, f\}$. Lemma 3.5 guarantees that $J \in \Gamma$ and thus by Theorem 3.3, every homeomorphism is in Γ. If we can show that L, V and Z are in Γ, then Corollary 2.15 implies that $\Gamma = C_0$.

Instead of presenting a detailed proof that $L, V, Z \in \Gamma$, we can outline the proof by the diagrams which follow. The symbols $\approx$ and $\simeq$ stand for "is topologically equivalent to" and "is approximately equal to" respectively.



Corollary 3.7. *$\{L, Z\} \cup H$ generates a dense subsemigroup of C_0 where H denotes the group of all homeomorphisms.*

Proof. By Theorem 2.14, f can be expressed as a finite product of functions each topologically equivalent to L, V or Z. In fact, there exist homeomorphisms h_1, h_2 and h_3 such that $f = h_1 Z h_2 L h_3$. Since by Theorem 3.6, $\{f\} \cup H$ generates a dense subsemigroup of C_0, then $\{L, Z\} \cup H$ does also.

Corollary 3.8 *$\{L, Z, \varphi, J\}$ generates a dense subsemigroup of C_0 where φ is as in Theorem 3.3.*

Proof. This is a consequence of Corollary 3.7 and Theorem 3.3.

Corollary 3.9. *There exists a function L' topologically equivalent to L such that $\{L', Z, \varphi\}$ generates a dense subsemigroup of C_0. Again φ is as in Theorem 3.3.*

Proof. We choose L' such that L' is topologically equivalent to L and $L'(x) = 1 - x = J(x)$ for all $\frac{4}{5} \leqslant x \leqslant 1$. In the statement of Lemma 3.5, the function f can be replaced by L. Once J is approximated by finite compositions of L' and φ, we can apply Theorem 3.3 and Corollary 3.8.

Corollary 3.10. *Same as Corollary 3.9 except $\{L, Z', \varphi\}$.*

We conclude with a theorem which indicates that Theorem 3.6 is "best possible."

Theorem 3.11. *If $\{f_1, f_2\}$ generates a dense subsemigroup of C_0, then one of the functions, say f_1, is monotone and the other function f_2 must have the following properties:*

(1) *f_2 does not map two non-overlapping intervals onto* [0, 1].

(2) *f_2 does not map an interval homeomorphically onto* [0, 1].

(3) *Either $f_2(0)$ or $f_2(1)$ is between 0 and 1.*

Outline of proof. If neither f_1 nor f_2 is monotone, then monotone functions cannot be approximated by finite compositions of f_1 and f_2. Consider the following lemma which is stated without proof.

Lemma 3.12. *If $g \in C_0$ and g is not monotone, then there exists a positive number δ such that for any $g' \in C_0$ and monotone h, $\|gg' - h\| \geqslant \delta$.*

Properties (1) and (2) are necessary in order to approximate Z by finite compositions of f_1 and f_2. Property (3) is necessary in order to approximate L.

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