Some remarks on the consequence operation in sentential logics

by

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1. Preliminary notions. Let \( S \) be the set of formulas formed by means of sentential variables \( p_\alpha (\alpha \in \Sigma) \) (the set of indices \( \Sigma \) being at least denumerably infinite) and a finite number of connectives \( F_1, \ldots, F_n \). As known, \( S = \langle S, F_1, \ldots, F_n \rangle \) is an absolutely free algebra, and \( \{p_\alpha \}_{\alpha \in \Sigma} \) is the set of free generators of it. By a consequence in \( S \) we understand (cf. [5]) an operation \( Cn \) defined for every subset \( X \) of \( S \) and such that:

\[
\begin{align*}
(1.1) & \quad X \subseteq Cn(Cn(X)) \subseteq Cn(X) \subseteq S, \\
(1.2) & \quad X \subseteq \exists Y : Cn(X) \subseteq Cn(Y).
\end{align*}
\]

Given an algebra \( S \), as described above, and a consequence \( Cn \) in \( S \), the couple \( L = \langle S, Cn \rangle \) will be called a sentential logic; \( S \) and \( Cn \) will be called the language of \( L \) and the consequence of \( L \) respectively. Let \( X \subseteq S \). \( X \) is said to be consistent provided that \( Cn(X) \neq S \). If \( X = Cn(X) \), \( X \) is said to be a \( Cn \)-system (or a system in \( L \)); the elements of the set \( Cn(\emptyset) \), where \( \emptyset \) denotes the empty set, are called the theorems of \( L \).

A relation \( R \subseteq 2^X \times S \) will be called a rule of inference in \( S \). If \( R(X, \alpha) \), i.e. the relation \( R \) holds for the arguments \( X \) and \( \alpha \), we shall say that the set of premises \( X \) entails the conclusion \( \alpha \) under the rule \( R \). It is often convenient to assume that the first domain of \( R \) consists of sets of a fixed cardinality, which is then called the cardinality of the rule \( R \). A set \( X \) is said to be closed under a rule \( R \) provided that, for every \( \alpha \in S \) and every \( Y \subseteq X \), if \( R(Y, \alpha) \) then \( \alpha \in X \). Given a set of rules of inference \( R \) and a consequence \( Cn \), we say that \( R \) is a basis for \( Cn \) (\( Cn \) is based on \( R \)) if those sets which are \( Cn \)-systems and only those are closed under all the rules \( R \in R \). Every consequence operation possesses a basis (see [2]).

The consequence based on \( R \) will be denoted by \( Cn_R \).

The cardinal number \( m \) is called the cardinality of a consequence \( Cn \) in \( S \) if it is the least cardinal number for which the following is valid:

\[
(1.3) \quad \alpha \in Cn(X) \iff \bigvee Y \subseteq X \land Y < m \land \alpha \in Cn(Y),
\]

[\( (*) \) The author is greatly indebted to Prof. J. Löb for his valuable suggestions and comments on this paper.]
for every $\alpha \in S$, $X \subseteq S$. Clearly, if $m$ is the cardinality of $\text{Cn}$, then $m < \aleph_0$. If the cardinality of $\text{Cn}$ is not greater than $n$, $\text{Cn}$ is said to be finite.

If for every endomorphism $e$ of $S$ and for every $X \subseteq S$,

\begin{equation}
\epsilon \text{Cn}(X) \subseteq \text{Cn}(\epsilon X),
\end{equation}

then $L = (S, \text{Cn})$ is said to be a structural logic and $\text{Cn}$ a structural consequence (cf. [2]). To the notion of a structural consequence corresponds that of a structural rule of inference $R$. $R$ is structural provided that, for every endomorphism $e$ of $S$, every $\alpha \in S$, and every $X \subseteq S$:

\begin{equation}
(R, X, \alpha) \rightarrow (e^\alpha X, e \alpha).
\end{equation}

A rule $R$ will sometimes be given in a schematic way as $X/\beta$. It will be understood then that $(R, X, \alpha)$ if and only if for an endomorphism $e$ of $S$, $X = eY$ and $\alpha = e^\beta$ (cf. the notion of a sequential rule [3]). All such rules are structural. A consequence $\text{Cn}$ is structural if and only if it possesses a structural basis $S$, i.e., a basis such that, for every $R \in R$, $R$ is structural (see [2]).

Let $V(X)$ denote the least set of variables which generates a subalgebra $S_k$ of $S$ such that $X \subseteq S_k$. Loosely speaking $V(X)$ is the set of variables occurring in the formulas which are elements of $S$. Assume that $X$ and $Y$ are subsets of $S$. We shall say that $X$ and $Y$ are mutually uniform in $L = (S, \text{Cn})$, in symbols $X \simeq Y$, if and only if for every $Z \subseteq S$, every $\alpha \in S$, and for every isomorphism $\nu_1, \nu_2$ of $S$ into $S$ such that $\nu_1^\vee(S) \subseteq V(X)$ (1 = 1, 2) and

\begin{equation}
V(\nu_1 X) \cap V(\nu_2 Y) = \nu_1(V(\nu_1 X)) \cap V(\nu_2 Y) \cap V(\nu_1 X) \cap V(\nu_2 Y) = \emptyset \text{ then } \alpha \in \text{Cn}(\nu_1, \nu_2 X) \subseteq \text{Cn}(\nu_1, \nu_2 Y).
\end{equation}

We shall need the following assertion.

**Assertion 1.1.** If $L = (S, \text{Cn})$ is a structural logic, then $\simeq L$ is an equivalence relation in the set of subsets of $S$.

Proof. It is immediately seen that $\simeq L$ is reflexive and symmetric. To prove that it is transitive, assume that for subsets $X, Y, Z, U$ of $S$, a formula $\alpha \in S$, and for isomorphisms $\nu_1, \nu_2$ of $S$ into $S$ such that $\nu_1^\vee(S) \subseteq V(X)$ (1 = 1, 2) and

\begin{equation}
V(\nu_1 X) \cap V(\nu_2 Y) = \emptyset \text{ then } \alpha \in \text{Cn}(\nu_1, \nu_2 X).
\end{equation}

Divide $V(S)$ into mutually disjoint subsets $V_1, V_2$ such that the cardinality of each of them equals the cardinality of $V(S)$. Let $\mu_1, \mu_2$ be isomorphisms of $S$ into $S$ such that $\mu_1^\vee(S) = V_1$ (1 = 1, 2). Since $L$ is assumed to be structural, we conclude from (4) that $\mu_1 \in \text{Cn}(\mu_1 Z \cup \mu_2 X)$. By (1) we obtain $\mu_1 \in \text{Cn}(\mu_1 Z \cup \mu_2 X)$, and by (2) and (3) we have $\mu_2 \in \text{Cn}(\mu_1 Z \cup \mu_2 X)$. As $\mu_1$ is an isomorphism of $S$ into $S$, this yields $\alpha \in \text{Cn}(\nu_1, \nu_2 X)$.

We shall discern between an isomorphism onto and an isomorphism onto.

Let $L = (S, \text{Cn})$ be a structural calculus. Denote by $A$ the set of all consistent systems in $L$. Let $m$ be a cardinal number. We shall say that $L$ is a $m$-uniform logic ($\text{Cn}$ is a $m$-uniform consequence) if and only if either $A$ is the empty set and $m = 1$ or $A$ is a non-empty set and $m$ is the cardinal of the quotient set $A/\simeq_L$ of $A$ with respect to $\simeq_L$. If $m$ is finite a $m$-uniform logic will be called finitely uniform. The $1$-uniform logics (consequences) will be called uniform. One may easily verify that a consequence $\text{Cn}$ in $S$ is uniform in the sense defined above if it is uniform in the sense defined by Los and Suszko [2], i.e., if it satisfies the condition: for every subsets $X, Y$ of $S$ and for every $\alpha \in S$,

\begin{equation}
\text{if } V(X) \cap V(Y) = \emptyset \text{ then } \alpha \in \text{Cn}(X) \cap \text{Cn}(Y).
\end{equation}

Again let $L$ be structural. If for every set $\{X_i\}_{\in R}$ of subsets of $S$ the conditions

\begin{equation}
X_i \simeq_X X_j, \text{ for every } r_i, r_j \in R
\end{equation}

\begin{equation}
V(X_i) \cap V(X_j) = \emptyset, \text{ for every } r_i, r_j \in R \text{ such that } r_i \neq r_j
\end{equation}

imply

\begin{equation}
X_{r_i} \simeq_{X_{r_j}} \simeq_{X_{r_k}} \text{ for every } r_i, r_j, r_k \in R,
\end{equation}

we shall say that $L$ is a regular logic ($\text{Cn}$ is a regular consequence). The consequence defined by the conditions $\text{Cn}(X_i) = X$ when there is an endomorphism $\epsilon$ of $S$ such that $X$ is finite and $\text{Cn}(X_i) = S$ otherwise is an example of a consequence which is structural, uniform but not regular.

**Assertion 1.2.** If $L = (S, \text{Cn})$ is a structural logic, $X_1 \simeq_{X_2} X_j \simeq_{X_k} X_3$ and $\epsilon$ is a $1$-uniform logic, $X_1, X_2, X_3$ are subsets of $S$ such that $V(X_1) \cap V(X_2) \cup V(X_3) \subseteq \emptyset$.

Proof. To prove this assertion, it is clearly enough to prove that if $L = (S, \text{Cn})$ is a structural logic, $X_1, X_2, X_3$ are subsets of $S$ such that $V(X_1) \cap V(X_2) \cup V(X_3) \subseteq \emptyset$ then for every $i (1 < i < \kappa) X_i \simeq_{X_1} X_2 \cup X_3 \cup X_i$. Since $L$ is assumed to be structural, we conclude from (4) that $\mu_1 \in \text{Cn}(\mu_1 Z \cup \mu_2 X)$. By (1) we obtain $\mu_1 \in \text{Cn}(\mu_1 Z \cup \mu_2 X)$ and by (2) and (3) we have $\mu_2 \in \text{Cn}(\mu_1 Z \cup \mu_2 X)$. As $\mu_1$ is an isomorphism of $S$ into $S$, this yields $\alpha \in \text{Cn}(\nu_1, \nu_2 X)$.
such that the cardinality of each of them equals the cardinality of $V(S)$. Let $\mu_1, \mu_2, \mu_3$ be isomorphisms of $S$ into $S$ such that $\mu_i V(S) = Y_i$ ($i = 1, 2, 3$). As $Cn$ is a structural consequence we obtain $\mu_2 a \in Cn(\mu_1 Z \cup \mu_2 Y_1 \cup \mu_3 Y_2)$. Taking into account that $X \subseteq Y_1$, we arrive at $\mu_1 a \in Cn(\mu_1 Z \cup \mu_2 X \cup \mu_3 Y_2)$. Now by $X \subseteq Y_2$, we obtain $\mu_1 a \in Cn(\mu_1 Z \cup \mu_2 X \cup \mu_3 Y_2)$. Clearly, the existentialness of $S$ such that $\varepsilon a = a, \varepsilon b = b, \varepsilon c = c, \varepsilon d = d$. By the structuralness of $Cn$ we have $\varepsilon a \in Cn(\mu_1 Z \cup \mu_2 X \cup \mu_3 Y_2)$ and this in turn yields $\alpha a \in Cn(\mu_1 Z \cup \mu_2 X \cup \mu_3 Y_2)$, completing the proof.

As an obvious consequence of Assertion 1.2 we have

**Assertion 1.3.** If $Cn$ is a finite structural consequence in $S$ then $L = \langle S, Cn \rangle$ is regular. (1)

Let $L = \langle S, Cn \rangle$ and $L_S = \langle S_0, Cn_0 \rangle$ be sentential logics. If $S_4$ is a subalgebra of $S$ such that for every $X \subseteq S_4$

$$Cn_0(X) = Cn(X) \cap S_4$$

(1.11)

then $L$ is said to be an extension of $L_S$, if, in addition,

$$Cn(X) = \bigcup_{\alpha a \in Cn_0(X)} Cn_0(\alpha a X)$$

(1.12)

where $\alpha a$ runs over all automorphisms of $S$ with $\alpha X \subseteq S_4$ and $X$ runs over all subsets of $X$ of the cardinality less that than of $Cn_0$, then we shall say that $L$ is a natural extension of $L_S$. Given $Cn_0$, condition (1.12) always defines a consequence operation; if $Cn_0$ is structural, it is structural also (cf. [2]).

2. Matrices adequate for sentential logics. Let $S$ be the language of a sentential logic $L$, and $A$ be an algebra similar to $S$. Let $\mathbb{B}_{w} = \mathbb{B}$ be a set of subsets of $A$. The pair $\mathbb{R} = \langle A, \mathbb{B}_{w} \rangle$ will be called a generalized matrix or shortly a matrix of $S$. A homomorphism $h : S \to A$ will be called a valuation of the formulas of $S$ in $\mathbb{R}$ or, if $S$ and $\mathbb{R}$ are fixed, a valuation. A formula $\alpha$ is said to be a tautology of $S$ and $\mathbb{R}$ and if only if, for every valuation $h$, $\alpha \in Cn_0(X)$. The set of valuations of $\mathbb{R}$ will be denoted by $E(\mathbb{R})$.

Let $\alpha \in S$, $X \subseteq S$ and let $h$ run over the set of valuations of formulas of $S$ in $\mathbb{R}$. As can easily be seen, the operation $Cn_\alpha$, defined as

$$a \in Cn_\alpha(X) = \bigwedge h \in E(\mathbb{R}) \cup (h X \subseteq S_4 \to a \in Cn_0(X))$$

(2.1)

is a consequence in $\mathbb{R}$. It will be called the matrix consequence determined by $\mathbb{R}$. One may easily verify that $Cn_\alpha(0) = E(\mathbb{R})$. The notions defined here are generalizations of well-known ones. Namely, if $\mathbb{B}_{w} = \mathbb{B}$ is a unitary set, they turn into the familiar notions of a matrix, a valuation, a tautology, (cf. e.g. [1]) and matrix consequence (the latter notion was introduced in [2]). The cardinality of the set $\mathbb{B}_{w}$ will be called the degree of $\mathbb{R}$. Thus the matrices in the usual sense are the matrices of degree 1.

We shall say that $\mathbb{R}$ is a matrix weakly adequate for $L = \langle S, Cn \rangle$ if and only if $\mathbb{R}$ is a matrix of $S$ and $Cn_\alpha(0) = Cn_\alpha(0)$, i.e. the set of tautologies of $\mathbb{R}$ coincides with that of theorems of $\mathbb{L}$. If for every $X \subseteq S$, $Cn_\alpha(X) = Cn_\alpha(X)$, then the matrix $\mathbb{R}$ will be called strongly adequate for $L$. The following theorem is known.

**Theorem 2.1. (Lindenbaum)** If $L$ is a structural logic then there is a matrix $\mathbb{R}$ of the degree 1 weakly adequate for $L$. (*)

The problem of existence of matrices strongly adequate for sentential logics was posed and investigated by Loi and Suszko [2]. They stated the following theorem: if $L$ is a structural and uniform logic, then there is a matrix $\mathbb{R}$ of degree 1 strongly adequate for $L$. However, a more close inspection of the proof they gave reveals that what they actually proved is:

**Theorem 2.2. (Loi and Suszko)** If $L$ is a structural, uniform and regular logic, then there is a matrix $\mathbb{R}$ of degree 1 strongly adequate for $L$.

The following argument shows that the requirement of regularity of $L$ cannot be omitted. Assume that $L = \langle S, Cn \rangle$ is structural and uniform but it is not regular (an example of such a logic has been given), and suppose that a matrix $\mathbb{R} = \langle A, B \rangle$ is strongly adequate for $L$. We shall show that this is impossible. Since $L$ is not regular, there is a set $X$ of subsets of $S$ such that (1.8) and (1.9) are satisfied but (1.10) is not. Since $L$ is 1-uniform, this means that the sets $X_\alpha(r \in R)$ are consistent but their union is not. Consider any formula $\alpha$ such that for a given set $X_\alpha$, $\alpha \in Cn_\alpha(X_\alpha)$. Then for every $X_\alpha$, such that $V(\alpha) \cap V(X_\alpha) = \emptyset$, $\alpha \in Cn_\alpha(X_\alpha)$ either, and hence there is at most a finite number of sets $X_\alpha_1, ..., X_\alpha_k$, such that $\alpha \in Cn_\alpha(X_\alpha_i)$ $(i = 1, ..., k)$. Put $X_\alpha = X - (X_\alpha_1, ..., X_\alpha_k)$. For every $r \in R$, there is a valuation $h_r$ for which $h_r X_\alpha \subseteq B$ and $h_r \alpha \in B$. The variables in the sets $X_\alpha$ are separated, and we may construct a valuation such that $h_r \cup X_\alpha \subseteq \emptyset$ but $h_r \alpha \notin B$. This proves that $\alpha \notin Cn_\alpha\left(\left\{X_\alpha\right\}\right)$.

The union of $X_\alpha(r \in R)$ is then consistent and therefore mutually uniform.

(*) This theorem is an obvious consequence of a well-known result of Lindenbaum (cf. [3]) which states that for every $X \subseteq S$ such that $\cup \in X \cup X_\alpha$, where $\alpha \in X$ runs over the set of endomorphisms of $S$, there is a matrix $\mathbb{R}$ of the degree 1 for which $E(\mathbb{R}) = X$. Clearly if $L$ is structural then $\cup \in X \cup Cn_\alpha(0) = Cn_\alpha(0)$.

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with each of the sets \( X_1, \ldots, X_n \). By Assertion 1.2 this proves that (1.10) is valid, which contradicts the assumptions and concludes the proof.

It is easily seen that if \( L \) possesses a matrix strongly adequate for \( s \), then \( L \) is structural; if, in addition, this matrix is a matrix of degree 1, \( L \) is uniform. Hence we have:

**Assertion 2.1.** If there is a matrix \( \mathfrak{M} \) of degree 1 strongly adequate for a logic \( L \), then \( L \) is structural, uniform, and regular.

Thus only logics of a particular, though important, kind may be interpreted in the strong sense by means of matrices of degree 1. Theorem 2.3. may be improved as follows.

**Theorem 2.4.** If \( L \) is a structural, \( m \)-uniform and regular logic, then there is a matrix \( \mathfrak{M} \) of degree \( m \) strongly adequate for \( L \).

**Proof.** The main idea of this proof is borrowed from L. de Loof and S. Maczko’s proof of Theorem 2.3. Assume that \( L = (S, C_n) \) is a structural, \( m \)-uniform, regular logic. Let \( A \) be the set of consistent systems of \( L \). Take any natural extension \( L' = (S', C_n') \) of \( L \) such that \( S' \supseteq S \). Let \( \{A_i\}_{i \in I} = A \cup \{\varepsilon\} \), where \( A \cap \varepsilon = \emptyset \) and let \( (X_i)_{i \in I} = A \). We put \( E = \bigcup_{i \in I} E_i \). Clearly \( (X_i)_{i \in E} \) is a hyperextension. Divide \( (S', E) \) into pairwise disjoint sets \( V_i (r \in E) \) such that \( V_r \supseteq V_i \), regularizes \( V_i \). Let \( \mu_r (r \in E) \) be an automorphism of \( S' \) such that \( \mu_r V_i \subseteq V_i \). We put

\[
B_r = C_n \cup \bigcup_{r \in E} \mu_r X_r,
\]

and we shall prove that \( \mathfrak{M} \) is strongly adequate for \( L \). Note that \( \mathfrak{M} \) is a matrix of degree \( m \). Assume that \( X \subseteq S \), \( a \in S \), \( a \in C_n(X) \). For a valuation \( h \), let \( h a \in B_r \). \( h \) may be extended to an endomorphism \( h^* \) of \( S' \) and we have \( h^* a \in C_n'(X) \). Hence \( h^* C_n(X) \cap C_n'(X) = C_n(X) \subseteq C_n'(B_r) \). This proves that \( C_n(X) \subseteq C_n(B_r) \). Assume now that \( a \in C_n(X), X \subseteq S \). Take as a valuation \( h \) the automorphism \( \mu_r \) restricted to \( S \). This gives \( h X \subseteq C_n(X) \subseteq B, S \). Suppose that

\[
\mu_r a \in B_r = C_n(a X_r \cup \bigcup_{r \in E} \mu_r X_r).
\]

Let \( m \) be the cardinality of \( C_n \) and hence of \( C_n' \). Then there is an \( R' \subseteq B_r \) such that \( \mu_r m < R' \) and \( \mu_r a \in C_n(a X_r \cup \mu_r X_r) \). \( m' < S \) and similarly \( X_r \subseteq S \), for every \( r \in E \). Therefore the cardinality of the union \( X_r \) is not greater than the cardinality of \( S \). This guarantees that there is an automorphism \( \nu \) of \( S' \) such that \( \nu h X_r \cup \bigcup_{r \in E} \mu_r X_r \subseteq S \) and also \( \nu a \in S \). We have \( \nu a \in C_n(a X_r \cup \bigcup_{r \in E} \mu_r X_r) \).

\[a \in C_n(X) \cap \bigcup_{Y \in E} (X \subseteq Y \rightarrow a \in Y)\]

is a consequence in \( S \). We clearly have \( C_n = C_n(S \cup B_r) \), and also

\[
C_n \subseteq C_n = \bigcup_{X \subseteq S} (X \subseteq C_n(X)).
\]

**3. The algebraic structure of the consequence operation.** To consider the result obtained from a more general point of view, we shall briefly examine the algebraic structure of the consequence operation. Throughout this section we shall assume that \( S \) is an arbitrarily chosen but fixed language of a sentential logic. It will then be understood that all the notions which will be employed in the sequel are related to \( S \).

By \( C(S) \) we shall denote the set of consequences in \( S \). Given a consequence \( C_n \subseteq C_n \) will denote the set of \( C_n \)-systems. Let \( \mathfrak{X} \) be a set of subsets of \( S \). The operation \( C_n \) defined as

\[
a \in C_n(X) \cap \bigcup_{Y \in E} (X \subseteq Y \rightarrow a \in Y)
\]

is a consequence in \( S \). We clearly have \( C_n = C_n(X) \), and also

\[
C_n \subseteq C_n = \bigcup_{X \subseteq S} (X \subseteq C_n(X)).
\]

Theorem 2.4. If \( L \) is a structural logic, then there is a matrix \( \mathfrak{M} \) strongly adequate for \( L \).

**Proof.** Consider the matrix \( \mathfrak{M} = (S, (X_e), (\tau)) \), where \( (X_e)_{e \in E} \) is the set of all systems of \( L \). The valuations in \( \mathfrak{M} \) are the endomorphisms \( \epsilon \) of \( S \) (we put \( L = (S, C_n) \)), and we have

\[
\alpha \in C_n(X) \cap \bigcup_{Y \in E} (X \subseteq Y \rightarrow a \in Y).
\]

Assume first that \( a \in C_n(X) \). \( L \) is structural and therefore \( a \epsilon C_n(X) \). This yields \( a \in C_n(X) \), for every \( Y \subseteq S \). Hence \( a \epsilon C_n(X) \). Assume in turn that \( a \in C_n(X) \). By (3.4) we obtain \( a \in C_n(X) \), for every \( Y \subseteq S \) and every endomorphism \( \epsilon \) of \( S \). Put \( Y = X \). Choosing the identity transformation as \( \epsilon \), we arrive at the formula: \( X \subseteq C_n(X) \rightarrow a \in C_n(X) \). This yields \( a \in C_n(X) \), concluding the proof.

We shall show in the next section that if \( L \) is not regular, then the degree of any matrix \( \mathfrak{M} \) which is strongly adequate for \( L \) must be infinite.
where $CN_{A} \subseteq CN_{B}$ stands for $\bigwedge_{T} X(CN_{A}(X) \subseteq CN_{B}(X))$. Let us now define two infinite operations $\bigcup$ and $\bigcap$ over the elements of $C(S)$. They will be called the sum and the product operation respectively. Given a set of consequences $(CN_{A})_{T}$ we pose

\begin{align*}
\bigcup_{T} CN_{A} &= CN_{A} \bigcup CN_{A} \bigcap CN_{A} \\
\bigcap_{T} CN_{A} &= CN_{A} \bigcap CN_{A}
\end{align*}

We shall write $CN_{A} \cup CN_{B} \cup \ldots \cup CN_{A}$ instead of $\bigcup_{T} CN_{A}$ and $CN_{A} \cap CN_{B} \cap \ldots \cap CN_{A}$, is to be understood in an analogous way. Using (3.2) one may prove by entirely obvious transformations that $\bigcap_{T} CN_{A}$ is the least upper bound and $\bigcap_{T} CN_{A}$ is the greatest lower bound of the set $(CN_{A})_{T}$ with respect to $\subseteq$. This proves that

**Assertion 3.1.** $C(S) = \langle C(S), \bigcap, \bigcup \rangle$ is a complete lattice with the lattice ordering $\subseteq$.

By (3.3) we almost immediately have

\begin{align*}
\bigcap_{T} CN_{A}(X) &= \bigcap_{T} CN_{A}(X) \bigcap CN_{A}(X).
\end{align*}

The analogue of (3.4) for $\bigcap$ is not, in general, valid (1). Still we may additionally characterize the $\bigcup$-operation as follows. For every $T \in T$, let $CN_{A} = CN_{A}(S)$, where $S$ is a rule basis of $CN_{A}$. Then

\begin{align*}
\bigcup_{T} CN_{A} &= CN_{A} \bigcup CN_{A} \bigcap CN_{A}.
\end{align*}

The identity (3.5) follows immediately from the fact that for every $X$, $X$ is closed under $T$ if and only if $X$ is closed under each $S_{A}(X)$, i.e., for every $T \in T$, $X$ is a $CN_{A}$-system.

Denote by $C(S)$ the set of structural consequences in $S$. The following is

**Assertion 3.2.** $C(S) = \langle C(S), \bigcup, \bigcap \rangle$ is a complete sublattice of the lattice $C(S)$ (1).

**Proof.** Assume that $CN_{A}(T \in T)$ are structural. By (3.4) we have

\begin{align*}
\bigcup_{T} CN_{A}(X) &= \bigcap_{T} CN_{A}(X) \bigcup CN_{A}(X) \\
\bigcap_{T} CN_{A}(X) &= \bigcap_{T} CN_{A}(X) \bigcap CN_{A}(X),
\end{align*}

for every $T \in T$. This is why we preferred to use $\bigcup$ as the sum and $\bigcap$ as the product symbol, rather than $\bigcup$ and $\bigcap$ respectively.

\[(1)\] is worth while to state here a few properties of the two lattices $C(S)$ and $C(S)$. By producing a suitable example it can be proved that neither of them is distributive. In either lattice the complement and $\cap$-complement of a given $CN_{A}$ may be non-existent. Notice also that, as can immediately be seen, they possess zero and unit elements. These are $CN_{A}$ and $CN_{A}$, respectively.

Endomorphism $\alpha$ of $S$. Hence $\bigcap_{T} CN_{A}$ is structural. $\bigcup_{T} CN_{A}$ is structural also because it has a structural basis. Any union of structural bases of all $CN_{A}(T \in T)$ is, by (3.5), such a basis.

We shall use the symbols $C(S)$ and $C(S)$ to denote the set of structural, uniform and regular consequences, and the set of structural, finitely uniform, and regular consequences respectively. As a consequence of Theorem 2.1, we have

**Corollary 3.1.** For every $CN_{A}(S)$, $CN_{A} = \bigcap_{T} CN_{A}$, where $CN_{A}(T \in T)$ is the set of all consequences $CN_{A}' \in C(S)$ such that $CN_{A} \subseteq CN_{A}'$.

**Proof.** If $CN_{A}$ is structural, then by Theorem 2.1, there is a matrix $M = \langle A_{A}, B_{A}, T \in T \rangle$ strongly adequate for $CN_{A}$. $CN_{A} = \bigcup_{T} CN_{A}(S)$, where $CN_{A}(T \in T)$ is structural for every $T \in T$. In $L = \langle S, CN_{A} \cap CN_{A}' \rangle$ and $L_{1} = \langle S, CN_{A} \rangle$, $i = 1, \ldots, n$. The relation $X \Rightarrow Y = \bigwedge_{T} (X \equiv_{A} Y)$ is an equivalence relation. It is a matter of obvious transformations to verify that $X \Rightarrow Y = X \equiv_{A} Y$. But this proves that $CN_{A} \cap CN_{A}'$ is both finitely uniform and regular. Indeed, let $A$ be the set of all consistent $CN_{A} \cap CN_{A}'$ systems. The cardinality of $A \Rightarrow$ is not greater than $2^{n}$, and at the same time we have $A \Rightarrow \Rightarrow A_{L_{1}}$. $CN_{A} \cap CN_{A}'$ is then at most $2^{n}$-uniform. We shall show now that it is regular. Let $X_{i} \Rightarrow$ be any set of subsets of $S$ which satisfies (1.8) and (1.9) and let $X_{i} \Rightarrow$ be an equivalence class of the set $X_{i} \Rightarrow$ under the relation $\Rightarrow$. We have $X_{i} \Rightarrow X_{i}$ for every $i = 1, \ldots, n$. Each $CN_{A}$ is regular and therefore $X \Rightarrow Y = X \equiv_{A} Y \Rightarrow$. Hence also $X \Rightarrow Y = X \equiv_{A} Y \Rightarrow$. The same may be proved about every equivalence class belonging to $(X_{i} \Rightarrow)_{i \in I}$. As we know, there is only a finite number of such classes, say $k$. Denote their unions by $Y_{1}, \ldots, Y_{k}$ respectively. Since each $Y_{i}$ is mutually uniform in $L_{1}$ with a set $X_{i}$, and the sets $X_{i}$ are pairwise mutually uniform, by the transitivity of $\equiv_{A}$, we have $Y_{i} \equiv_{A} Y_{k}$. By applying Assertion 1.2, we may show that $Y_{i} \cup Y_{i}, \ldots, Y_{k} \equiv_{A} Y_{k},$ for every $Y_{i} (i = 1, \ldots, k).$ But $Y_{1} \cup Y_{2}, \ldots, Y_{k} = \bigcup_{i \in I} X_{i}$, and once again by the transitivity of $\equiv_{A},$
we conclude that the latter union is mutually uniform with every $L$. Hence $Cn' \cap Cn''$ is regular.

As an obvious consequence of Theorem 3.1., we have

**Corollary 3.2.** No matrix $M$ of a finite degree is strongly adequate for a non-regular logic $L$.

There is unfortunately an asymmetry in the properties which are displayed by the operations $\cap$ and $\cup$. By analogy to Corollary 3.1. one may expect that the sum of consequences $Cn'$ and $Cn''$ such that, for a given $Cn$, $Cn'$ and $Cn''$ are identical with $Cn$. To see that this is not true consider $Cn, Cn'$ based on a single rule $R$ given by the schema $F_1(p), F_2(q)\vdash F_3(p)$, where $p, q$ are variables and $F_1, F_2, F_3$ are unary connectives. If for a consequence $Cn, F_1(p) \vdash Cn$ (for $F_2(p), F_3(q)$) then either $Cn$ is not uniform or, if $F_1(p) \vdash Cn(F_2(p), Cn$ is stronger than $Cn$), i.e. $Cn > Cn$. The following argument shows that the counterpart of Theorem 3.1. for $\cup$ is not valid either. Consider the sequence of consequences $Cn, Cn', Cn''$, $Cn, Cn', Cn''$ ..., where the rules $R_1, R_2, R_3, ...$ are respectively given by schemas:

$F_1(p) \vdash F_2(q)$;
$F_1(p) \equiv F_3(q)$;
...

(this time $F_3$ is taken as a binary connective). Take now the corresponding sequence $Cn, Cn', Cn''$ ..., with the corresponding rules: $F_1(q) \equiv F_3(p)$; $F_1(q) \equiv F_3(p)$; $F_1(q) \equiv F_3(p)$; ... All these consequences, besides being structural, are uniform and regular. This follows from the fact that the conclusion of any rule cannot be, in the rules of the same sequence, used as a premis. For the same reason the sum of consequences of any of these two sequences will be uniform and regular. But the infinite sum of the consequences of both sequences is $\kappa$-uniform. No two of the formulas $g, F_1(p), F_1(q), F_3(p)$, ..., are mutually uniform with respect to such a sum consequence.

These negative results seem to show that the properties of the consequences which are constructed by means of the $\cup$-operation cannot be described in terms of matrices in as simple a way as it was possible in the case of consequences obtained by applying $\cap$.

**References**
