esis. Assume it is true for all unions of n finite intersections and let $I_1, \ldots, I_{n+1}$ be $n+1$ finite intersections. Then

$$I_{n+1} \cap (I_1 \cup \ldots \cup I_n)$$

is a union of n finite intersections so $t\big(I_{n+1} \cap (I_1 \cup \ldots \cup I_n)\big)$ is an isomorphism and so is $t(I_1 \cup \ldots \cup I_n)$, both by the inductive hypotheses. By Lemma 3, $t(I_1 \cup \ldots \cup I_{n+1})$ is an isomorphism and $(*)$ is established.

Now we well-order the index set a and proceed by transfinite induction.

Consider $t(\bigcup_{\beta<a} U_\beta)$. Since $\mathfrak{U}$ is *star-finite*, $I = U_a \cap \bigcup_{\beta<a} U_\beta$ is a *finite* union of finite intersections, so by $(*)$ $t(I)$ is an isomorphism. By hypothesis $t(U_a)$ is an isomorphism, so by Lemma 3 $t(\bigcup_{\beta\leqslant a} U_\beta)$ is an isomorphism. This proves $t(X)$ is an isomorphism.

We are now ready to prove deRham's theorem.

THEOREM 2. *If M^n is a paracompact C^∞ manifold, then the singular cohomology groups with real coefficients are isomorphic with the deRham groups.*

Proof. Let h denote singular cohomology and $\hat{h}$ denote deRham cohomology. As noted earlier, these are cohomology theories for the structure $\mathcal{S}$ generated by a covering of M by open sets. In particular we take a star-finite covering $\mathfrak{U} = \{U_a\}$ as in Lemma 2′ by geodesically convex sets.

A p-form ω gives rise to a p-cochain by defining $\omega(\sigma_p) = \int_{\sigma_p} \omega$, for each singular p-simplex σ_p. By Stokes' theorem

$$\int_{\sigma_{p+1}} d\omega = \int_{\partial\sigma_{p+1}} \omega\,,$$

so that this induces a natural transformation

$$t\colon\ \hat{h} \to h\,.$$

By the Poincaré lemma $\hat{h}(U) = 0$, and by the cone construction $h(U) = 0$, where U is any finite intersection of the U_a's. Thus t satisfies the hypothesis of Theorem 1, so $t(M)$ is an isomorphism. q.e.d.

References

[1] J. Dugundji, *Comparison of homologies*, Ann. Scuola Norm. Sup. Pisa 20 (1966), pp. 745–751.

[2] E. Spanier, *Algebraic Topology*, 1966.

RICE UNIVERSITY
UNIVERSITY OF SOUTHERN CALIFORNIA

Reçu par la Rédaction le 10. 12. 1968

Some remarks on the consequence operation in sentential logics

by

R. Wójcicki* (Wrocław)

1. Preliminary notions. Let S be the set of formulas formed by means of sentential variables $p_\xi (\xi \in \Xi)$ (the set of indices Ξ being at least denumerably infinite) and a finite number of connectives $F_1, \ldots, F_n$. As known, $\mathcal{S} = \langle S, F_1, \ldots, F_n\rangle$ is an absolutely free algebra, and $\{p_\xi\}_{\xi\in\Xi}$ is the set of free generators of it. By a *consequence in* $\mathcal{S}$ we understand (cf. [5]) an operation Cn defined for every subset X of S and such that:

(1.1) $$X \subseteq \mathrm{Cn}\big(\mathrm{Cn}(X)\big) \subseteq \mathrm{Cn}(X) \subseteq S\,,$$

(1.2) $$X \subseteq Y \to \mathrm{Cn}(X) \subseteq \mathrm{Cn}(Y)\,.$$

Given an algebra $\mathcal{S}$, as described above, and a consequence Cn in $\mathcal{S}$, the couple $L = \langle \mathcal{S}, \mathrm{Cn}\rangle$ will be called a *sentential logic*; $\mathcal{S}$ and Cn will be called the *language* of L and the *consequence* of L respectively. Let $X \subseteq S$. X is said to be *consistent* provided that $\mathrm{Cn}(X) \neq S$. If $X = \mathrm{Cn}(X)$, X is said to be a Cn-*system* (or a *system in* L). The elements of the set $\mathrm{Cn}(\emptyset)$, where $\emptyset$ denotes the empty set, are called the *theorems of* L.

A relation $R \subseteq 2^S \times S$ will be called a *rule of inference* in $\mathcal{S}$. If $R(X, \alpha)$, i.e. the relation R holds for the arguments X and α, we shall say that *the set of premisses X entails the conclusion α under the rule R*. It is often convenient to assume that the first domain of R consists of sets of a fixed cardinality, which is then called the *cardinality of the rule* R. A set X is said to be *closed under a rule* R provided that, for every $\alpha \in S$ and every $Y \subseteq X$, if $R(Y, \alpha)$ then $\alpha \in X$. Given a set of rules of inference $\mathfrak{R}$ and a consequence Cn, we say that $\mathfrak{R}$ is *a basis for* Cn (Cn is *based on* $\mathfrak{R}$) if those sets which are Cn-systems and only those are closed under all the rules $R \in \mathfrak{R}$. Every consequence operation possesses a basis (see [2]). The consequence based on $\mathfrak{R}$ will be denoted by $\mathrm{Cn}_{\mathfrak{R}}$.

The cardinal number $\mathfrak{m}$ is called *the cardinality of a consequence* Cn in $\mathcal{S}$ if it is the least cardinal number for which the following is valid:

(1.3) $$\alpha \in \mathrm{Cn}(X) \equiv \bigvee_Y \big(Y \subseteq X \wedge \overline{\overline{Y}} < \mathfrak{m} \wedge \alpha \in \mathrm{Cn}(Y)\big)\,,$$

(*) The author is greatly indebted to Prof. J. Łoś for his valuable suggestions and comments on this paper.

for every $a \in S$, $X \subseteq S$. Clearly, if $\mathfrak{m}$ is the cardinality of Cn, then $\mathfrak{m} \leqslant \bar{\bar{S}}$. If the cardinality of Cn is not greater then $\aleph_0$, Cn is said to be *finite*.

If for every endomorphism ε of S and for every $X \subseteq S$,

$$(1.4) \qquad \varepsilon \mathrm{Cn}(X) \subseteq \mathrm{Cn}(\varepsilon X),$$

then $L = \langle S, \mathrm{Cn}\rangle$ is said to be *a structural logic* and Cn a *structural consequence* (cf. [2]). To the notion of a structural consequence corresponds that of *a structural rule of inference R*. R is *structural* provided that, for every endomorphism ε of S, every $a \in S$, and every $X \subseteq S$:

$$(1.5) \qquad R(X, a) \rightarrow R(\varepsilon X, \varepsilon a).$$

A rule R will sometimes be given in a schematic way as Y/β. It will be understood then that $R(X, a)$ if and only if for an endomorphism ε of S, $X = \varepsilon Y$ and $a = \varepsilon\beta$ (cf. the notion of a sequential rule [3]). All such rules are structural. A consequence Cn is structural if and only if it possesses a structural basis $\mathfrak{R}$, i.e. a basis such that, for every $R \in \mathfrak{R}$, R is structural (see [2]).

Let $V(X)$ denote the least set of variables which generates a subalgebra S_0 of S such that $X \subseteq S_0$. Loosely speaking $V(X)$ is the set of variables occurring in the formulas which are elements of S. Assume that X and Y are subsets of S. We shall say that X and Y are *mutually uniform* in $L = \langle S, \mathrm{Cn}\rangle$, in symbols $X \simeq_L Y$, if and only if for every $Z \subseteq S$, every $a \in S$, and for every isomorphism ν_1, ν_2 of S into S [1] such that $\nu_i(V(S)) \subseteq V(S)$ $(i = 1, 2)$ the following condition holds:

(1.6) if $V(\nu_1 X) \cap V(Z) = V(\nu_2 Y) \cap V(Z) = V(\nu_1 X) \cap V(a) = V(\nu_2 Y) \cap \cap V(a) = \emptyset$ then $a \in \mathrm{Cn}(Z \cup \nu_1 X) \equiv a \in \mathrm{Cn}(Z \cup \nu_2 Y)$.

We shall need the following assertion.

ASSERTION 1.1. *If $L = \langle S, \mathrm{Cn}\rangle$ is a structural logic, then $\simeq_L$ is an equivalence relation in the set of subsets of S.*

Proof. It is immediately seen that $\simeq_L$ is reflexive and symmetric. To prove that it is transitive, assume that for subsets X, Y, Z, U of S, a formula $a \in S$, and for isomorphisms ν_1, ν_2 of S into S such that $\nu_i V(S) \subseteq V(S)$ $(i = 1, 2)$: (1) $X \simeq_L U$, (2) $U \simeq_L Y$, (3) the antecedent of (1.6) holds, (4) $a \in \mathrm{Cn}(Z \cup \nu_1 X)$. Divide $V(S)$ into mutually disjoint subsets V_1, V_2 such that the cardinality of each of them equals the cardinality of $V(S)$. Let μ_1, μ_2 be isomorphisms of S into S such that $\mu_i(V(S)) = V_i$ $(i = 1, 2)$. Since L is assumed to be structural, we conclude from (4) that $\mu_1 a \in \mathrm{Cn}(\mu_1 Z \cup \mu_1 \nu_1 X)$. By (1) we obtain $\mu_1 a \in \mathrm{Cn}(\mu_1 Z \cup \mu_2 U)$, and by (2) and (3) we have $\mu_1 a \in \mathrm{Cn}(\mu_1 Z \cup \mu_1 \nu_2 Y)$. As μ_1 is a isomorphisms and Cn is structural, this yields $a \in \mathrm{Cn}(Z \cup \nu_2 Y)$. If we replace (4) by $a \in \mathrm{Cn}(Z \cup \nu_2 Y)$ we shall prove by an entirely analogous argument that $a \in \mathrm{Cn}(Z \cup \nu_1 X)$. This gives $X \simeq_L Y$, concluding the proof.

[1] We shall discern between an isomorphism *into* and an isomorphism *onto*.

Let $L = \langle S, \mathrm{Cn}\rangle$ be a structural calculus. Denote by $\mathcal{A}$ the set of all consistent systems in L. Let $\mathfrak{m}$ be a cardinal number. We shall say that L is a $\mathfrak{m}$*-uniform logic* (Cn is a $\mathfrak{m}$*-uniform consequence*) if and only if either $\mathcal{A}$ is the empty set and $\mathfrak{m} = 1$ or $\mathcal{A}$ is a non-empty set and $\mathfrak{m}$ is the cardinal of the quotient set $\mathcal{A}/\simeq_L$ of $\mathcal{A}$ with respect to $\simeq_L$. If $\mathfrak{m}$ is finite a $\mathfrak{m}$-uniform logic will be called *finitely uniform*. The 1-uniform logics (consequences) will be called *uniform*. One may easily verify that a consequence Cn in S is uniform in the sense defined above if it is uniform in the sense defined by Łoś and Suszko [2], i.e. if it satisfies the condition: for every subsets X, Y of S and for every $a \in S$,

(1.7) if $V(X) \cap V(Y) = V(a) \cap V(Y) = \emptyset$, $\mathrm{Cn}(Y) \neq S$, and $a \in \mathrm{Cn}(X \cup Y)$ then $a \in \mathrm{Cn}(X)$.

Again let L be structural. If for every set $\{X_r\}_{r\in R}$ of subsets of S the conditions

$$(1.8) \qquad X_{r_1} \simeq_L X_{r_2}, \text{ for every } r_1, r_2 \in R$$

$$(1.9) \qquad V(X_{r_1}) \cap V(X_{r_2}) = \emptyset, \text{ for every } r_1, r_2 \in R \text{ such that } r_1 \neq r_2$$

imply

$$(1.10) \qquad X_{r_0} \simeq_L \bigcup_{r\in R} X_r,$$

for every $r_0 \in R$, then we shall say that L is *a regular logic* (Cn is *a regular consequence*). The consequence defined by the conditions $\mathrm{Cn}(X) = X$ when there is an endomorphism ε of S such that εX is finite and $\mathrm{Cn}(X) = S$ otherwise is an example of a consequence which is structural, uniform but not regular.

ASSERTION 1.2. *If $L = \langle S, \mathrm{Cn}\rangle$ is a structural logic, $X_1 \simeq_L X_2 \simeq_L \ldots \simeq_L X_k$, and for every i, j $(1 \leqslant i, j \leqslant k)$ if $i \neq j$ then $V(X_i) \cap V(X_j) = \emptyset$, then for every i $(1 \leqslant i \leqslant k)$ $X_i \simeq_L X_1 \cup X_2 \cup \ldots \cup X_k$.*

Proof. To prove this assertion it is clearly enough to prove that if $L = \langle S, \mathrm{Cn}\rangle$ is a structural logic, X, Y_1, Y_2 are subsets of S such that $V(Y_1) \cap V(Y_2) \neq \emptyset$, $X \simeq_L Y_1$, $X \simeq_L Y_2$, then $X \simeq_L (Y_1 \cup Y_2)$. Assume that for a formula $a \in S$, a set $Z \subseteq S$ and isomorphisms $\nu_1, \nu_2 \colon S \rightarrow S$ such that $\nu_i V(S) \subseteq V(S)$ $(i = 1, 2)$ the following condition holds: $V(\nu_2(Y_1 \cup Y_2)) \cap \cap V(Z) = V(\nu_1 X) \cap V(Z) = V(\nu_2(Y_1 \cup Y_2)) \cap V(a) = V(\nu_1 X) \cap V(a) = \emptyset$. We have to prove that $a \in \mathrm{Cn}(Z \cup \nu_1 X) \equiv a \in \mathrm{Cn}(Z \cup \nu_2(Y_1 \cup Y_2))$. The implication from left to right is obvious. It is yielded by the assumption $X \simeq_L Y_1$. To prove the converse, assume that $a \in \mathrm{Cn}(Z \cup \nu_2(Y_1 \cup Y_2)) = \mathrm{Cn}(Z \cup \nu_2 Y_1 \cup \nu_2 Y_2)$. Divide $V(S)$ into pairwise disjoint sets V_1, V_2, V_3

such that the cardinality of each of them equals the cardinality of $V(S)$. Let μ_1, μ_2, μ_3 be isomorphisms of S into S such that $\mu_i V(S) = V_i$ $(i = 1, 2, 3)$. As Cn is a structural consequence we obtain $\mu_1 \alpha \in \mathrm{Cn}(\mu_1 Z \cup \mu_1\nu_2 Y_1 \cup \mu_1\nu_2 Y_2)$. Taking into account that $X \simeq_L Y$, we arrive at $\mu_1 \alpha \in \mathrm{Cn}(\mu_1 Z \cup \mu_2 X \cup \mu_1\nu_2 Y_2)$. Now by $X \simeq_L Y_2$ we obtain $\mu_1 \alpha \in \mathrm{Cn}(\mu_1 Z \cup \mu_2 X \cup \mu_3 X)$. Clearly there is an endomorphism ε of S such that: $\varepsilon\mu_1\alpha = \alpha$, $\varepsilon\mu_1 Z = Z$, $\varepsilon\mu_2 X = \nu_1 X$, $\varepsilon\mu_3 X = \nu_1 X$. By the structurality of Cn we have $\varepsilon\mu_1\alpha \in \mathrm{Cn}(\varepsilon\mu_1 Z \cup \varepsilon\mu_2 X \cup \varepsilon\mu_3 X)$ and this in turn yields $\alpha \in \mathrm{Cn}(Z \cup \nu_1 X)$, completing the proof.

As an obvious consequence of Assertion 1.2 we have

ASSERTION 1.3. *If* Cn *is a finite structural consequence in* S *then* $L = \langle S, \mathrm{Cn}\rangle$ *is regular.* (²)

Let $L = \langle S, \mathrm{Cn}\rangle$ and $L_0 = \langle S_0, \mathrm{Cn}_0\rangle$ be sentential logics. If S_0 is a subalgebra of S such that for every $X \subseteq S_0$,

$$\mathrm{Cn}_0(X) = \mathrm{Cn}(X) \cap S_0\,, \tag{1.11}$$

then L is said to be *an extension of* L_0. If, in addition,

$$\mathrm{Cn}(X) = \bigcup_{Y} \bigcup_{\nu} \nu^{-1}\mathrm{Cn}_0(\nu Y)\,, \tag{1.12}$$

where ν runs over all automorphisms of S with $\nu Y \subseteq S_0$ and Y runs over all subsets of X of the cardinality less than that of Cn_0, then we shall say that L is *a natural extension of* L_0. Given Cn_0, condition (1.12) always defines a consequence operation; if Cn_0 is structural, it is structural also (cf. [2]).

2. Matrices adequate for sentential logics. Let S be the language of a sentential logic L, and let A be an algebra similar to S. Let $\{B_u\}_{u \in U}$ be a set of subsets of A. The pair $\mathfrak{M} = \langle A, \{B_u\}_{u\in U}\rangle$ will be called a *generalized matrix* or shortly a *matrix of* S. A homomorphism $h\colon S \to A$ will be called a *valuation of the formulas of* S *in* $\mathfrak{M}$ or, if S and $\mathfrak{M}$ are fixed, a *valuation*. A formula α is said to be *a tautology of* $\mathfrak{M}$ if and only if, for every valuation h, $h\alpha \in \bigcup_{u \in U} B_u$. The set of tautologies of $\mathfrak{M}$ will be denoted by $E(\mathfrak{M})$. Let $\alpha \in S$, $X \subseteq S$ and let h run over the set of valuations of formulas of S in $\mathfrak{M}$. As can easily be seen, the operation $\mathrm{Cn}_{\mathfrak{M}}$, defined as

$$\alpha \in \mathrm{Cn}_{\mathfrak{M}}(X) \equiv \bigwedge h \bigwedge u \in U (hX \subseteq B_u \to h\alpha \in B_u) \tag{2.1}$$

is a consequence in S. It will be called the *matrix consequence determined by* $\mathfrak{M}$. One may easily verify that $\mathrm{Cn}_{\mathfrak{M}}(\emptyset) = E(\mathfrak{M})$. The notions defined here are generalizations of well-known ones. Namely, if $\{B_u\}_{u\in U}$ is a unitary set, they turn into the familiar notions of a matrix, a valuation, a tautology, (cf. e.g. [1]) and matrix consequence (the latter notion was introduced in [2]). The cardinality of the set $\{B_u\}_{u\in U}$ will be called the degree of $\mathfrak{M}$. Thus the matrices in the usual sense are the matrices of degree 1.

(²) Let $R(X, \alpha) \equiv [\sim \bigvee \varepsilon(\varepsilon X < \aleph_0) \wedge V(\alpha) \subseteq V(X) \wedge \sim \bigvee Y \neq \emptyset\ (Y \not\subseteq X \wedge V(Y) \cap \cap V(X - Y) = \emptyset]$, where ε runs over endomorphisms of S. The consequence $\mathrm{Cn}_{(R)}$ may serve as an example of a consequence which being structural, uniform and regular is not finite.

We shall say that $\mathfrak{M}$ is a matrix *weakly adequate for* $L = \langle S, \mathrm{Cn}\rangle$ if and only if $\mathfrak{M}$ is a matrix of S and $\mathrm{Cn}(\emptyset) = \mathrm{Cn}_{\mathfrak{M}}(\emptyset)$, i.e. the set of tautologies of $\mathfrak{M}$ coincides with that of theorems of L. If for every $X \subseteq S$, $\mathrm{Cn}(X) = \mathrm{Cn}_{\mathfrak{M}}(X)$, then the matrix $\mathfrak{M}$ will be called *strongly adequate for* L. The following theorem is known.

THEOREM 2.1. (Lindenbaum) *If* L *is a structural logic then there is a matrix* $\mathfrak{M}$ *of the degree* 1 *weakly adequate for* L. (³)

The problem of existence of matrices strongly adequate for sentential logics was posed and investigated by Łoś and Suszko [2]. They stated the following theorem: if L is a structural and uniform logic, then there is a matrix $\mathfrak{M}$ of degree 1 strongly adequate for L. However, a more close inspection of the proof they gave reveals that what they actually proved is:

THEOREM 2.2. (Łoś and Suszko) *If* L *is a structural, uniform and regular logic, then there is a matrix* $\mathfrak{M}$ *of degree* 1 *strongly adequate for* L.

The following argument shows that the requirement of regularity of L cannot be omitted. Assume that $L = \langle S, \mathrm{Cn}\rangle$ is structural and uniform but it is not regular (an example of such a logic has been given), and suppose that a matrix $\mathfrak{M} = \langle A, B\rangle$ is strongly adequate for L. We shall show that this is impossible. Since L is not regular, there is a set $\{X_r\}_{r\in R}$ of subsets of S such that (1.8) and (1.9) are satisfied but (1.10) is not. Since L is 1-uniform, this means that the sets $X_r (r \in R)$ are consistent but their union is not. Consider any formula α such that for a given set X_{r_0}, $\alpha \mathrel{\bar{\in}} \mathrm{Cn}(X_{r_0})$. Then for every X_r such that $V(\alpha) \cap V(X_r) = \emptyset$, $\alpha \mathrel{\bar{\in}} \mathrm{Cn}(X_r)$ either, and hence there is at most a finite number of sets $X_{r_1}, \ldots, X_{r_k}$ such that $\alpha \in \mathrm{Cn}(X_{r_i})$ $(i = 1, \ldots, k)$. Put $R' = R - (r_1, \ldots, r_k)$. For every $r \in R'$ there is a valuation h_r for which $h_r X_r \subseteq B$ and $h_r\alpha \mathrel{\bar{\in}} B$. The variables in the sets X_r are separated, and we may construct a valuation h such that $h \bigcup_{r\in R'} X_r \subseteq B$ but $h\alpha \mathrel{\bar{\in}} B$. This proves that $\alpha \mathrel{\bar{\in}} \mathrm{Cn}(\bigcup_{r\in R'} X_r)$. The union of $X_r (r \in R')$ is then consistent and therefore mutually uniform

(³) This theorem is an obvious consequence of a well-known result of Lindenbaum (cf. [3]) which states that for every $X \subseteq S$ such that $\bigcup_{\varepsilon} \varepsilon X \subseteq X$, where ε runs over the set of endomorphisms of S, there is a matrix $\mathfrak{M}$ of the degree 1 for which $E(\mathfrak{M}) = X$. Clearly if L is structural then $\bigcup_{\varepsilon} \varepsilon \mathrm{Cn}(\emptyset) = \mathrm{Cn}(\emptyset)$.

with each of the sets $X_{r_1}, \ldots, X_{r_k}$. By Assertion 1.2 this proves that (1.10) is valid, which contradicts the assumptions and concludes the proof.

It is easily seen that if L possesses a matrix strongly adequate for it, then L is structural; if, in addition, this matrix is a matrix of degree 1, L is uniform. Hence we have:

ASSERTION 2.1. *If there is a matrix $\mathfrak{M}$ of degree 1 strongly adequate for a logic L, then L is structural, uniform, and regular.*

Thus only logics of a particular, though important, kind may be interpreted in the strong sense by means of matrices of degree 1. Theorem 2.2. may be improved as follows.

THEOREM 2.3. *If L is a structural, $\mathfrak{m}$-uniform and regular logic, then there is a matrix $\mathfrak{M}$ of degree $\mathfrak{m}$ strongly adequate for L.*

Proof. The main idea of this proof is borrowed from Łoś and Suszko's proof of Theorem 2.2. Assume that $L = \langle S, \mathrm{Cn}\rangle$ is a structural, $\mathfrak{m}$-uniform, regular logic. Let $\mathcal{A}$ be the set of consistent systems of L. Take any natural extension $L^* = \langle S^*, \mathrm{Cn}^*\rangle$ of L such that $\overline{\overline{S^*}} \geqslant \overline{\overline{\mathcal{A}}}$. Let $\{\mathcal{A}_u\}_{u \in U} = \mathcal{A}/\simeq_L$, where $\mathcal{A}/\simeq_L$ is the quotient set of $\mathcal{A}$ with respect to $\simeq_L$, and let $\{X_r\}_{r \in R_u} = \mathcal{A}_u$. We put $R = \bigcup_{u \in U} R_u$. Clearly $\{X_r\}_{r \in R} = \mathcal{A}$. Divide $V(S^*)$ into pairwise disjoint sets $V_r (r \in R)$ such that $\overline{\overline{V_r}} \geqslant \overline{\overline{V(S)}}$, for every $r \in R$. Let $\mu_r (r \in R)$ be an automorphism of S^* such that $\mu_r V(S) \subseteq V_r$. We put

(2.2) $$B_u = \mathrm{Cn}^*\big(\bigcup_{r \in R_u} \mu_r X_r\big),$$

(2.3) $$\mathfrak{M} = \langle S^*, \{B_u\}_{u \in U}\rangle,$$

and we shall prove that $\mathfrak{M}$ is strongly adequate for L. Note that $\mathfrak{M}$ is a matrix of degree $\mathfrak{m}$. Assume that $X \subseteq S$, $\alpha \in S$, $\alpha \in \mathrm{Cn}(X)$. For a valuation h, let $h\alpha \in B_u$. h may be extended to an endomorphism h^* of S^* and we have $h^*\alpha \in h^*\mathrm{Cn}(X) \subseteq h^*\mathrm{Cn}^*(X) \subseteq \mathrm{Cn}^*(h^*X) = \mathrm{Cn}^*(hX) \subseteq \mathrm{Cn}^*(B_u) = B_u$. This proves that $\mathrm{Cn}(X) \subseteq \mathrm{Cn}_{\mathfrak{M}}(X)$. Assume now that $\alpha \,\overline{\in}\, \mathrm{Cn}(X)$, $\mathrm{Cn}(X) = X_{r_0}$, $r_0 \in R_{u_0}$, and take as a valuation h the automorphism μ_{r_0} restricted to S. This gives $hX \subseteq hX_{r_0} \subseteq B_{u_0}$. Suppose that

$$h\alpha \in B_{u_0} = \mathrm{Cn}^*\big(hX_{r_0} \cup \bigcup_{r_0 \neq r \in R_{u_0}} \mu_r X_r\big)$$

Let $\mathfrak{m}'$ be the cardinality of Cn and hence of Cn*. Then there is an $R' \subseteq R_{u_0}$ such that $\overline{\overline{R'}} \leqslant \mathfrak{m}'$ and $h\alpha \in Cn^*(hX_{r_0} \cup \bigcup_{r \in R'} \mu_r X_r)$. $\mathfrak{m}' \leqslant \overline{\overline{S}}$ and similarly $\overline{\overline{X_r}} \leqslant \overline{\overline{S}}$, for every $r \in R$. Therefore the cardinality of the union $hX_{r_0} \cup \bigcup_{r \in R'} \mu_r X_r$ must not be greater than the cardinality of S. This guarantees that there is an automorphism ν of S^* such that $\nu h X_{r_0} \cup \bigcup_{r \in R'} \nu\mu_r X_r \subseteq S$ and also $\nu h\alpha \in S$. We have $\nu h\alpha \in \mathrm{Cn}(\nu h X_{r_0} \cup \bigcup_{r \in R'} \nu\mu_r X_r)$.

As Cn is regular, this yields $\nu h\alpha \in \mathrm{Cn}(\nu h X_{r_0})$. In turn we have $\nu^{-1}\nu h\alpha = h\alpha \in \mathrm{Cn}(\nu^{-1}\nu h X_{r_0}) = \mathrm{Cn}(hX_{r_0}) \subseteq \mathrm{Cn}^*(hX_{r_0})$. Replace h by μ_{r_0}. This gives $\mu_{r_0}\alpha \in \mathrm{Cn}^*(\mu_{r_0}X_{r_0})$. μ_{r_0} is an automorphism, and therefore by employing $\mu_{r_0}^{-1}$ we obtain in an obvious way $\alpha \in \mathrm{Cn}^*(X_{r_0}) = \mathrm{Cn}^*\big(\mathrm{Cn}(X)\big)$. $\alpha \in S$, and this gives $\alpha \in \mathrm{Cn}(X)$, contradicting the assumption. Hence $\mathrm{Cn}_{\mathfrak{M}}(X) \subseteq \mathrm{Cn}(X)$. This, together with the inclusion proved before, yields $\mathrm{Cn}_{\mathfrak{M}}(X) = \mathrm{Cn}(X)$, concluding the proof.

Since for every structural logic L there is a cardinal $\mathfrak{m}$ such that L is $\mathfrak{m}$-uniform, all structural and regular logics possess matrices strongly adequate for them. It can be proved that non-regular logics may also be interpreted in the strong manner by means of generalized matrices.

THEOREM 2.4. *If L is a structural logic, then there is a matrix $\mathfrak{M}$ strongly adequate for L.*

Proof. Consider the matrix $\mathfrak{M} = \langle S, \{X_w\}_{w \in W}\rangle$, where $\{X_w\}_{w \in W}$ is the set of all systems of L. The valuations in $\mathfrak{M}$ are the endomorphisms ε of S (we put $L = \langle S, \mathrm{Cn}\rangle$), and we have

(2.4) $$\alpha \in \mathrm{Cn}_{\mathfrak{M}}(X) \equiv \bigwedge Y \subseteq S \bigwedge \varepsilon \big(\varepsilon X \subseteq \mathrm{Cn}(Y) \rightarrow \varepsilon\alpha \in \mathrm{Cn}(Y)\big).$$

Assume first that $\alpha \in \mathrm{Cn}(X)$. L is structural and therefore $\varepsilon\alpha \in \mathrm{Cn}(\varepsilon X)$, for every endomorphism ε. This yields $\varepsilon X \subseteq Cn(Y) \rightarrow \varepsilon\alpha \in \mathrm{Cn}(Y)$, for every $Y \subseteq S$. Hence $\alpha \in \mathrm{Cn}_{\mathfrak{M}}(X)$. Assume in turn that $\alpha \in \mathrm{Cn}_{\mathfrak{M}}(X)$. By (2.4) we obtain: $\varepsilon X \subseteq \mathrm{Cn}\, Y \rightarrow \varepsilon\alpha \in \mathrm{Cn}(Y)$, for every $Y \subseteq S$ and every endomorphism ε of S. Put $Y = X$. Choosing the identity transformation as ε, we arrive at the formula: $X \subseteq \mathrm{Cn}(X) \rightarrow \alpha \in \mathrm{Cn}(X)$. This yields $\alpha \in \mathrm{Cn}(X)$, concluding the proof.

We shall show in the next section that if L is not regular, then the degree of any matrix $\mathfrak{M}$ which is strongly adequate for L must be infinite.

3. The algebraic structure of the consequence operation. To consider the result obtained from a more general point of view, we shall briefly examine the algebraic structure of the consequence operation. Throughout this section we shall assume that S is an arbitrarily chosen but fixed language of a sentential logic. It will then be understood that all the notions which will be employed in the sequel are related to S.

By $\mathfrak{C}(S)$ we shall denote the set of consequences in S. Given a consequence Cn, $\mathfrak{S}(\mathrm{Cn})$ will denote the set of Cn-systems. Let $\mathfrak{X}$ be a set of subsets of S. The operation $\mathrm{Cn}_{\mathfrak{X}}$ defined as

(3.1) $$\alpha \in \mathrm{Cn}_{\mathfrak{X}}(X) \equiv \bigwedge Y \in \mathfrak{X} (X \subseteq Y \rightarrow \alpha \in Y)$$

is a consequence in S. We clearly have $\mathrm{Cn} = \mathrm{Cn}_{\mathfrak{S}(\mathrm{Cn})}$, and also

(3.2) $$\mathrm{Cn}_1 \leqslant \mathrm{Cn}_2 \equiv \mathfrak{S}(\mathrm{Cn}_2) \subseteq \mathfrak{S}(\mathrm{Cn}_1)$$

where $Cn_1 \subseteq Cn_2$ stands for $\bigwedge X(Cn_1(X) \subseteq Cn_2(X))$. Let us now define two infinite operations $\Cup$ and $\Cap$ over the elements of $\mathfrak{C}(S)$. They will be called *the sum* and *the product operation* respectively. Given a set of consequences $\{Cn_t\}_{t\in T}$ we pose

(3.3) $$\Cup_{t\in T} Cn_t = Cn_{\bigcup_{t\in T} \mathfrak{S}(Cn_t)},$$

(3.4) $$\Cap_{t\in T} Cn_t = Cn_{\bigcap_{t\in T} \mathfrak{S}(Cn_t)}.$$

We shall write $Cn_1 \Cup Cn_2 \Cup \ldots \Cup Cn_k$ instead of $\Cup_{i=1,\ldots,k} Cn_i$; $Cn_1 \Cap Cn_2 \Cap \ldots \Cap Cn_k$ is to be understood in an analogous way. Using (3.2) one may prove by entirely obvious transformations that $\Cup_{t\in T} Cn_t$ is the least upper bound and $\Cap_{t\in T} Cn_t$ is the greatest lower bound of the set $\{Cn_t\}_{t\in T}$ with respect to $\leqslant$. This proves that

ASSERTION 3.1. *$\mathfrak{C}(S) = \langle \mathfrak{C}(S), \Cap, \Cup \rangle$ is a complete lattice with the lattice ordering $\leqslant$.*

By (3.3) we almost immediately have

(3.4) $$[\Cap_{t\in T} Cn_t](X) = \bigcap_{t\in T} Cn_t(X).$$

The analogue of (3.4) for $\Cup$ is not, in general, valid (⁴). Still we may additionally characterize the $\Cup$-operation as follows. For every $t \in T$, let $Cn_t = Cn_{\mathfrak{R}_t}$, where $\mathfrak{R}_t$ is a rule basis for Cn_t. Then

(3.5) $$\Cup_{t\in T} Cn_t = Cn_{\bigcap_{t\in T} \mathfrak{R}_t}.$$

The identity (3.5) follows immediately from the fact that for every X, X is closed under $\bigcup_{t\in T} \mathfrak{R}_t$ if and only if X is closed under each $\mathfrak{R}_t (t \in T)$, i.e., for every $t \in T$, X is a Cn_t-system.

Denote by $\mathfrak{C}_s(S)$ the set of structural consequences in S. The following is valid:

ASSERTION 3.2. *$\mathfrak{C}_s(S) = \langle \mathfrak{C}_s(S), \Cup, \Cap \rangle$ is a complete sublattice of the lattice $\mathfrak{C}(S)$* (⁵).

Proof. Assume that $Cn_t (t \in T)$ are structural. By (3.4) we have $\varepsilon[\Cap_{t\in T} Cn_t](X) = \varepsilon \bigcap_{t\in T} Cn_t(X) \subseteq \bigcap_{t\in T} Cn_t(\varepsilon X) = [\Cap_{t\in T} Cn_t](\varepsilon X)$, for every endomorphism ε of S. Hence $\Cap_{t\in T} Cn_t$ is structural. $\Cup_{t\in T} Cn_t$ is structural also because it has a structural basis. Any union of structural bases of all $Cn_t (t \in T)$ is, by (3.5), such a basis.

We shall use the symbols $\mathfrak{C}_s^0(S)$ and $\mathfrak{C}_s^1(S)$ to denote the set of structural, uniform and regular consequences, and the set of structural, finitely uniform, and regular consequences respectively. As a consequence of Theorem 2.4. we have

COROLLARY 3.1. *For every $Cn \in \mathfrak{C}_s(S)$, $Cn = \Cap_{t\in T} Cn_t$, where $\{Cn_t\}_{t\in T}$ is the set of all consequences $Cn' \in \mathfrak{C}_s^0(S)$ such that $Cn \subseteq Cn'$.*

Proof. If Cn is structural, then by Theorem 2.4. there is a matrix $\mathfrak{M} = \langle A, \{B_u\}_{u\in U} \rangle$ strongly adequate for Cn. $Cn = \Cup_{t\in T} Cn_{\mathfrak{M}_u}$, where $\mathfrak{M}_u = \langle A, B_u \rangle (u \in U)$. We have also $Cn \leqslant Cn_{\mathfrak{M}_u}$ for every $u \in U$.

We shall now prove

THEOREM 3.1. *$\mathfrak{C}_s^1(S)$ is the least set containing $\mathfrak{C}_s^0(S)$ and closed under the operation $\Cap$.*

Proof. It follows from Theorem 2.3. by an argument similar to that used to prove Corollary 3.1. that every consequence $Cn \in \mathfrak{C}_s^1(S)$ is identical with a finite product of consequences which are elements of $\mathfrak{C}_s^0(S)$. Thus we need only to prove that if $Cn', Cn'' \in \mathfrak{C}_s^1(S)$ then also $Cn' \Cap Cn'' \in \mathfrak{C}_s^1(S)$. Let $Cn' = Cn_1 \Cap Cn_2 \Cap, \ldots, \Cap Cn_m$ and $Cn'' = Cn_{m+1} \Cap, \ldots, \Cap Cn_n$, where $Cn_i \in \mathfrak{C}_s^0(S)$ for every $i = 1, 2, \ldots, n$. Put $L = \langle S, Cn' \Cap Cn'' \rangle$ and $L_i = \langle S, Cn_i \rangle$ $(i = 1, \ldots, n)$. The relation $X \sim Y \equiv \bigwedge i(X \simeq_{L_i} Y)$ is an equivalence relation. It is a matter of obvious transformations to verify that $X \sim Y \to X \simeq_L Y$. But this proves that $Cn' \Cap Cn''$ is both finitely uniform and regular. Indeed, let $\mathcal{A}$ be the set of all consistent $Cn' \Cap Cn''$-systems. The cardinality of $\mathcal{A}/\sim$ is not greater than 2^n, and at the same time we have $\overline{\overline{\mathcal{A}/\sim}} \geqslant \overline{\overline{\mathcal{A}/\simeq_L}}$. $Cn' \Cap Cn''$ is then at most 2^n-uniform. We shall show now that it is regular. Let $\{X_r\}_{r\in R}$ be any set of subsets of S which satisfies (1.8) and (1.9) and let $\{X_r\}_{r\in R'}$ be an equivalence class of the set $\{X_r\}_{r\in R}$ under the relation $\sim$. We have $X_{r_1} \simeq_{L_i} X_{r_2}$ for every $r_1, r_2 \in R'$ and for every $i = 1, \ldots, n$. Each Cn_i is regular and therefore $\bigcup_{r\in R'} X_r \simeq_{L_i} X_{r_0}$ for every $r_0 \in R'$. Hence also $\bigcup_{r\in R'} X_r \simeq_L X_{r_0}$. The same may be proved about every equivalence class belonging to $\{X_r\}_{r\in R}/\sim$. As we know, there is only a finite number of such classes, say k. Denote their unions by $Y_1, \ldots, Y_k$ respectively. Since each Y_i is mutually uniform in L with a set X_r, and the sets X_r are pairwise mutually uniform, by the transitivity of $\simeq_L$ we have $Y_i \simeq_L Y_j$. By applying Assertion 1.2. we may show that $Y_1 \cup Y_2, \ldots, \cup Y_n \simeq_L Y_i$, for every Y_i $(i = 1, \ldots, n)$. But $Y_1 \cup, \ldots, \cup Y_n = \bigcup_{r\in R} X_r$, and once again by the transitivity of $\simeq_L$

(⁴) This is why we preferred to use $\Cup$ as the sum and $\Cap$ as the product symbol, rather then $\cup$ and $\cap$ respectively.

(⁵) Perhaps it is worth while to state here a few properties of the two lattices $\mathfrak{C}(S)$ and $\mathfrak{C}_s(S)$. By producing a suitable example it may be proved that neither of them is distributive. In either lattice the $\Cap$-complement and $\Cup$-complement of a given Cn may be non-existent. Notice also that, as can immediately be seen, they possess zero and unit elements. These are Cn_{2^S} and $Cn_{(S)}$ respectively.

we conclude that the latter union is mutually uniform with every X_r. Hence $\mathrm{Cn}' \dot{\cap} \mathrm{Cn}''$ is regular.

As an obvious consequence of Theorem 3.1., we have

COROLLARY 3.2. *No matrix $\mathfrak{M}$ of a finite degree is strongly adequate for a non-regular logic L.*

There is unfortunately an asymmetry in the properties which are displayed by the operations $\dot{\cap}$ and $\dot{\cup}$. By analogy to Corollary 3.1. one may expect that the sum of consequences $\mathrm{Cn}' \in \mathrm{C}_s^0(S)$ such that, for a given Cn, $\mathrm{Cn}' \subseteq \mathrm{Cn}$ is identical with *Cn*. To see that this is not true consider $\mathrm{Cn}_{(R)}$ based on a single rule R given by the schema $F_2 p, F_3 q / F_1 p$, where p, q are variables and F_1, F_2, F_3 are unary connectives. If for a consequence Cn, $F_1 p \in \mathrm{Cn}(F_2 p, F_3 q)$ then either Cn is not uniform or, if $F_1 p \in \mathrm{Cn}(F_2 p)$, Cn is stronger than $\mathrm{Cn}_{(R)}$, i.e. $\mathrm{Cn}_{(R)} \not\leqslant \mathrm{Cn}$. The following argument shows that the counterpart of Theorem 3.1. for $\dot{\cup}$ is not valid either. Consider the sequence of consequences $\mathrm{Cn}_{(R_1)}, \mathrm{Cn}_{(R_1,R_2)}, \mathrm{Cn}_{(R_1,R_2,R_3)} \ldots$ where the rules $R_1, R_2, R_3, \ldots$ are respectively given by schemas:

$$p, F_1 q / F_2(q, p);$$
$$p, F_1 F_1 q / F_2(F_2(q, p)p);$$
$$p, F_1 F_1 F_1 q / F_2(F_2(F_2(q, p)p)p);$$
$$\ldots$$

(this time F_2 is taken as a binary connective). Take now the corresponding sequence $\mathrm{Cn}_{(R_1')}, \mathrm{Cn}_{(R_1', R_2')}, \ldots$ with the corresponding rules: $F_2(q, p)/F_3 p$; $F_2(F_2(q, p)p)/F_3 F_3 p$; ... All these consequences, besides being structural, are uniform and regular. This follows from the fact that the conclusion of any rule cannot be, in the rules of the same sequence, used as a premiss. For the same reason the sum of consequences of any of these two sequences will be uniform and regular. But the infinite sum of the consequences of both sequences is $\aleph_0$-uniform. No two of the formulas $q, F_1 q, F_1 F_1 q, \ldots$ are mutually uniform with respect to such a sum consequence.

These negative results seem to show that the properties of the consequences which are constructed by means of the $\dot{\cup}$-operation cannot be described in terms of matrices in as simple a way as it was possible in the case of consequences obtained by applying $\dot{\cap}$.

References

[1] J. Łoś, *O matrycach logicznych*, Travaux de la Société des Sciences et des Lettres de Wrocław, ser. B, no 19 (1949).

[2] — and R. Suszko, *Remarks on Sentential Logics*, Indagationes Math., 20 (1958), pp. 177–183.

[3] J. Łukasiewicz and A. Tarski, *Untersuchungen über den Aussagenkalkül*, Comptes rendus de la Société des Sciences et des Lettres de Varsovie, 23 (1930), pp. 30–50.

[4] R. Suszko, *Formalna teoria wartości logicznych* I, Studia Logica IV (1957), pp. 145–236.

[5] A. Tarski, *Über einige fundamentalen Begriffe der Metamathematik*, Comptes Rendus des séances de la Société des Sciences et des Lettres de Varsovie, 23 (1930), pp. 22–29.

INSTITUTE OF PHILOSOPHY AND SOCIOLOGY
OF THE POLISH ACADEMY OF SCIENCES

Reçu par la Rédaction le 13. 1. 1969

FUNDAMENTA
MATHEMATICAE
LXVIII (1970)

ERRATA

Page, line	Au lieu de	Lire
276_{17}	$\mathrm{Cn} \bigcap_{t \in T} \mathfrak{R}_t$	$\mathrm{Cn} \bigcup_{t \in T} \mathfrak{R}_t$
276_3	$\dot{\cap}$-complement and $\dot{\cap}$-complement	$\dot{\cup}$-complement and $\dot{\cap}$-complement
278^9	identical with *Cn*	identical with Cn