A proof of deRham's theorem

by

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It is the purpose of this note to give a short proof of deRham's theorem using a modification of Dugundji's cohomology comparison theorem [1] and a simple convexity lemma. We include a proof of this well-known lemma since we have been unable to find it in the literature.

LEMMA 1. Let $f : U \to V$ be a homeomorphism, where $U$ and $V$ are open sets in $\mathbb{R}^n$. Assume (1) that $f$ is $C^0$ and that $g = f^{-1}$ is $C^0$. Then for each $x \in U$ there exists an $r(x) > 0$ such that the image $f(B(x, r))$ of every ball $B(x, r)$ of radius $r < r(x)$ about $x$ is convex.

Proof. We can assume $x = 0$ and that $U, V$ are small enough so that there exist real numbers $K > 0$, $M > 0$ satisfying

(1) If $y$ is a curve obtained by restricting $f$ to any line segment in $U$, then

$$|y'(t)| < K$$

(where $t$ is arc length on the segment and prime denotes differentiation).

(2) If $g$ is a curve obtained by restricting $g$ to any line segment in $V$, then

$$|g'(t)| < M$$

Note that we also have $|g'(0)| > 1/K$. Pick $\lambda > 0$ so small that

(3) $2\lambda < 1/K$

and choose $s > 0$ so that

(4) $gB(f(0), s) \subset B(0, \lambda)$.

We are now going to show that

(5) For each ball $B(0, r) \subset gB(f(0), s)$, the image $fB(0, r)$ is convex.

In fact, given $y_0, y_1 \in fB(0, r)$, let $d = |y_0 - y_1|$, let $J$ be the closed interval $[0, d]$, and let $\sigma : J \to V$ be the line segment joining $y_0$ to $y_1$. We have $\sigma(J) \subset B(f(0), s)$, since the latter is a convex set containing $y_0$.

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(1) Although the given hypotheses imply that $f$ itself is also $C^0$, we make no use of additional facts.
and \( y_t \) so, because of (4), we are sure that \( g(\bar{y}) \subseteq B(0, \bar{r}) \). Let \( g = g_0 \) and define \( H : J \rightarrow \mathbf{R} \) by

\[
H(t) = |g(t)|^p; \quad t \in J.
\]

Then

\[
H'(t) = 2|g(t)|^{p-2}<g'(t), g(t)>
\]

where \(<\cdot, \cdot>\) is the usual scalar product in \( \mathbf{R}^n \). Since

\[
|g'(t)|^2 \leq |g(t)|^p \leq |g(\bar{t})|^p \leq M2^p
\]

where \( |g(\bar{t})|^p \geq 1/|\mathbf{R}^n| \), we find from (3) that \( H''(t) > 0 \) on \( J \), so that \( H \) is a convex function on \( J \) and therefore

\[
H(t) \leq \max(H(0), H(\bar{t})) = \max(|g(0)|^p, |g(\bar{t})|^p) < r^p
\]

for all \( t \in J \). Thus, \( |g(t)| < r \) for all \( t \in J \), so \( g(\bar{J}) \subseteq B(0, r) \) and consequently \( \sigma(\bar{J}) = f_{\bar{t}}(J) \subseteq f_{\bar{t}}B(0, r) \). This completes the proof of both (5) and the Lemma.

Now let \( M \) be a paracompact \( C^\infty \)-manifold, and put a Riemannian metric on \( M \). Using the exponential map

\[
\exp_x : U_x \rightarrow M,
\]

where \( U_x \) is an open neighborhood of \( x \) in the tangent space \( T_x \) of \( M \) at \( x \in M \), a proof essentially the same (\(^1\)) as that for the Lemma 1 shows that the image of every sufficiently small ball \( B(0, r) \subseteq U_x \subseteq T_x \) is a geodesically convex neighborhood of \( x \) in \( M \). Being paracompact and locally compact, \( M \) is the free union of subspaces each having the form

\[
\bigcup U_x, \quad \text{where each } U_x \text{ is open, each } U_1 \text{ is compact, and } U_2 \subseteq U_1 \text{ for each } i \text{ covering } U_1 \text{ (resp. each } U_{i+1} \subseteq U_i \text{) by finitely many geodesically convex open sets, each contained in } U_i \text{ (resp. } U_{i+2} \subseteq U_{i+1} \text{), it follows that}
\]

\(^2\) A paracompact \( C^\infty \)-manifold \( M \) has a star-finite open covering by geodesically convex sets.

It follows that all intersections of these sets are geodesically convex and hence all sets in the covering and all intersections are open \( n \)-balls.

Definition. A structure \( S \) on a topological space \( X \) is a lattice of subsets (meet is intersection, join is union) such that the empty set \( \emptyset \) and the space \( X \) belong to \( S \). Given a structure \( S \) on \( X \), the structure category \( (X, S) \) has as objects the elements of \( S \), with the set \( \text{Hom}(A, B) \) of homomorphisms being the inclusion whenever \( A \subseteq B \) and empty otherwise. A cohomology theory on \( (X, S) \) is a sequence \( \{h^i\} \) of functors from \( (X, S) \) to the category of abelian groups and homomorphisms (any abelian category could be used), along with natural transformations

\[
\delta : h^i(A \cap B) \rightarrow h^{i+1}(A \cup B)
\]

for each \( g \in G \), \( A \subseteq B \) such that the following sequence is exact:

\[
\cdots \rightarrow h^{i-1}(A \cap B) \rightarrow h^i(A \cup B) \rightarrow h^i(A) \oplus h^i(B) \rightarrow h^i(A \cap B) \rightarrow \cdots
\]

Here

\[
i^*: (i_1^*, i_2^*) ; \quad i^* (t, \xi) = i_1^* \xi - i_2^* \xi \]

where \( i_A : A \rightarrow A \cup B \), \( i_A : A \cap B \rightarrow A \), etc.

If for any space \( X \) we take \( S \) to be the lattice of all open sets, then singular cohomology is a cohomology theory in the sense just defined. (See, for example, [2], page 239.) Given any covering \( \mathfrak{U} \) of \( X \) by open sets we get a structure \( S \) by taking the lattice generated by the members of \( \mathfrak{U} \). Singular cohomology is then a cohomology theory for \( (X, S) \).

If \( X \) is now a \( C^\infty \)-manifold, we also have the deRham groups (vector spaces) defined on open sets of \( X \). To see that this is also a cohomology theory (i.e., that we have a Mayer–Vietoris sequence) (\( \delta \)-form) it suffices to notice that a \( p \)-form \( \omega^p \) on an open set \( W \) is uniquely determined by its integrals on all differentiable \( p \)-simplexes in \( W \); the proof is then like that for singular cohomology. Just as for singular cohomology, the deRham cohomology is a cohomology theory for \( (X, S) \) where \( S \) is the lattice generated by any covering of \( X \) by open sets. Our objective, of course, is to show these cohomology theories are isomorphic for a paracompact \( C^\infty \) manifold.

Lemma 3. Let \( h, h \) be cohomology theories on \( (X, S) \) and let \( t : h \rightarrow h \) be a natural transformation. Assume that for some \( A, B \subseteq X \), both \( t(A) : h(A) \rightarrow h(A) \) and \( t(B) : h(B) \rightarrow h(B) \) are isomorphisms. Then \( t(A \cup B) \) is an isomorphism if and only if \( t(A) \cap B \) is an isomorphism.

Proof. Use the 5-lemma.

Theorem 1. Let \( \mathfrak{U} = \{U_T \} \) be a star-finite covering of \( X \) and let \( S \) be the structure generated by \( \mathfrak{U} \). Let \( h, h \) be cohomology theories on \( (X, S) \) and let \( t : h \rightarrow h \) be a natural transformation. Assume that for each finite intersection \( U_T \cap \cdots \cap U_N \subseteq \{U_T \cap \cdots \cap U_N \} \) is an isomorphism. Then \( t(X) : h(X) \rightarrow h(X) \) is an isomorphism.

Proof. First we establish

\( (*) \) If \( I_1, \ldots, I_n \) are finite intersections of elements of \( \mathfrak{U} \), then \( t(I_1 \cup \cdots \cup I_n) \) is an isomorphism.

We prove \( (*) \) by induction, noting that for \( n = 1 \) it is true by hypoth-
Some remarks on the consequence operation in sentential logics

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1. Preliminary notations. Let $S$ be the set of formulas formed by means of sentential variables $p_i (i \in \mathbb{S})$ (the set of indices $\mathbb{S}$ being at least denumerably infinite) and a finite number of connectives $F_1 , \ldots , F_n$. As known, $S = \langle S , F_1 , \ldots , F_n \rangle$ is an absolutely free algebra, and $\{p_i\}_{i \in \mathbb{S}}$ is the set of free generators of it. By a consequence in $S$ we understand (cf. [9]) an operation $Cn$ defined for every subset $X$ of $S$ and each that:

\begin{align}
X & \subseteq Cn(Cn(X)) \subseteq Cn(X) \subseteq S, \\
X & \subseteq Y \rightarrow Cn(X) \subseteq Cn(Y).
\end{align}

Given an algebra $S$, as described above, and a consequence $Cn$ in $S$, the couple $L = \langle S , Cn \rangle$ will be called a sentential logic; $S$ and $Cn$ will be called the language of $L$ and the consequence of $L$ respectively. Let $X \subseteq S$. $X$ is said to be consistent provided that $Cn(X) \neq S$. If $X = Cn(X)$, $X$ is said to be a $Cn$-system (or a system in $L$): The elements of the set $Cn(\emptyset)$, where $\emptyset$ denotes the empty set, are called the theorems of $L$.

A relation $R \subseteq 2^X \times S$ will be called a rule of inference in $S$. If $R(X , a)$, i.e. the relation $R$ holds for the arguments $X$ and $a$, we shall say that the set of premises $X$ entails the conclusion $a$ under the rule $R$. It is often convenient to assume that the first domain of $R$ consists of sets of a fixed cardinality, which is then called the cardinality of the rule $R$. A set $X$ is said to be closed under a rule $R$ provided that, for every $a \in S$ and every $Y \subseteq X$, if $R(Y , a)$, then $a \in X$. Given a set of rules of inference $\mathcal{R}$ and a consequence $Cn$, we say that $\mathcal{R}$ is a basis for $Cn$ if $Cn$ is based on $\mathcal{R}$ if those sets which are $Cn$-systems and only those are closed under all the rules in $\mathcal{R}$. Every consequence operation possesses a basis (see [3]).

The consequence based on $\mathcal{R}$ will be denoted by $Cn_{\mathcal{R}}$.

The cardinal number $\kappa$ is called the cardinality of a consequence $Cn$ in $S$ if it is the least cardinal number for which the following is valid:

\begin{align}
\forall Y \subseteq X \land Y \subseteq m \land a \in Cn(Y). 
\end{align}

References


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