is a k-ideal and by Lemma 9 $\bar{R}$ satisfies the ascending chain condition on right annihilators, and so by Lemma 8 there exists an $\bar{x} \neq 0$ in $\bar{R}$ such that $\overline{xR} = 0$.

Therefore $xR \subseteq R_n$ and $xRR^n = xR^{n+1} = 0$. By our choice of n, $x \in T_n$ so that $\bar{x} = 0$. This contradiction proves that $\bar{R} = 0$ and $R = T_n$. Hence $R^{n+1} = 0$.

COROLLARY. *If R is a semiring satisfying the ascending chain condition on left and right k-ideals and such that $\mathfrak{L}(R)$ is a k-ideal, then any nil subsemiring of R is nilpotent.*

Proof. Since every right or left annihilator ideal is a right or left k-ideal, the corollary follows from the theorem.

Note. This paper is part of the author's Ph. D. dissertation prepared under Professor Lawrence P. Belluce at the University of California, Riverside.

References

[1] D. R. LaTorre, *On h-ideals and k-ideals in semirings*, Depr. Hungary Publ. Math. 12 (1965), p. 219.

[2] I. N. Herstein, *Topics in ring theory*, Univ. Chicago Math. Lecture Notes (1965), pp. 75–102.

[3] — *Theory of rings*, Univ. Chicago Math. Lecture Notes (1961), pp. 25–28.

Reçu par la Rédaction le 25. 6. 1968

A proof of deRham's theorem

by

M. L. Curtis and **J. Dugundji** (Houston, Tex.)*

It is the purpose of this note to give a short proof of deRham's theorem using a modification of Dugundji's cohomology comparison theorem [1] and a simple convexity lemma. We include a proof of this well-known lemma since we have been unable to find it in the literature.

LEMMA 1. *Let $f\colon U \to V$ be a homeomorphism, where U and V are open sets in $\mathbf{R}^n$. Assume* (1) *that f is C^1 and that $g = f^{-1}$ is C^2. Then for each $x \in U$ there exists an $r(x) > 0$ such that the image $f\big(B(x, r)\big)$ of every ball $B(x, r)$ of radius $r \leqslant r(x)$ about x is convex.*

Proof. We can assume $x = 0$ and that U, V are small enough so that there exist real numbers $K > 0$, $M > 0$ satisfying

(1) If γ is a curve obtained by restricting f to any line segment in U, then

$$\|\gamma'(t)\| \leqslant K$$

(where t is arc length on the segment and prime denotes differentiation).

(2) If ϱ is a curve obtained by restricting g to any line segment in V, then

$$\|\varrho''(t)\| \leqslant M .$$

Note that we also have $\|\varrho'(t)\| \geqslant 1/K$. Pick $\lambda > 0$ so small that

(3) $2M\lambda \leqslant 1/K^2$

and choose $s > 0$ so that

(4) $gB\big(f(0), s\big) \subset B(0, \lambda)$.

We are now going to show that

(5) For each ball $B(0, r) \subset gB\big(f(0), s\big)$, the image $fB(0, r)$ is convex.

In fact, given $y_0, y_1 \in fB(0, r)$, let $d = \|y_0 - y_1\|$, let J be the closed interval $[0, d]$, and let $\sigma\colon J \to V$ be the line segment joining y_0 to y_1. We have $\sigma(J) \subset B\big(f(0), s\big)$, since the latter is a convex set containing y_0

* Partially supported by NSF Grants.

(1) Although the given hypotheses imply that f itself is also C^2, we make no use of additional fact.

and y_1 so, because of (4), we are sure that $g\sigma(J) \subset B(0, \lambda)$. Let $\varrho = g\sigma$ and define $H: J \to \boldsymbol{R}^1$ by

$$H(t) = \|\varrho(t)\|^2, \quad t \in J.$$

Then

$$H''(t) = 2\|\varrho'(t)\|^2 + 2\langle \varrho''(t), \varrho(t)\rangle$$

where $\langle \cdot, \cdot \rangle$ is the usual scalar product in $\boldsymbol{R}^n$. Since

$$|\langle \varrho''(t), \varrho(t)\rangle| \leqslant \|\varrho''(t)\| \cdot \|\varrho(t)\| \leqslant M\lambda$$

whereas $\|\varrho'(t)\|^2 \geqslant 1/K^2$, we find from (3) that $H''(t) > 0$ on J, so that H is a convex function on J and therefore

$$H(t) \leqslant \max\{H(0), H(d)\} = \max\{\|g(y_0)\|^2, \|g(y_1)\|^2\} < r^2$$

for all $t \in J$. Thus, $\|\varrho(t)\| < r$ for all $t \in J$, so $\varrho(J) \subset B(0, r)$ and consequently $\sigma(J) = f\varrho(J) \subset fB(0, r)$. This completes the proof of both (5) and the Lemma.

Now let M be a paracompact C^∞ n-manifold, and put a Riemannian metric on M. Using the exponential map

$$\exp_x: U_x \to M,$$

where U_x is an open neighborhood of 0 in the tangent space T_x of M at $x \in M$, a proof essentially the same [2] as that for the Lemma 1 shows that the image of every sufficiently small ball $B(0, r) \subset U_x \subset T_x$ is a geodesically convex neighborhood of x in M. Being paracompact and locally compact, M is the free union of subspaces each having the form $\bigcup_1^\infty U_i$, where each U_i is open, each $\overline{U}_i$ is compact, and $\overline{U}_i \subset U_{i+1}$ for each i; covering $\overline{U}_2$ (resp. each $\overline{U}_{i+1} - U_i$) by finitely many geodesically convex open sets, each contained in U_3 (resp. $U_{i+2} - \overline{U}_{i-1}$) it follows that

LEMMA 2. *A paracompact C^∞ manifold M^n has a star-finite open covering* [3] *by geodesically convex sets.*

It follows that all intersections of these sets are geodesically convex and hence all sets in the covering and all intersections are open n-balls.

DEFINITION. A *structure* $\mathcal{S}$ on a topological space X is a lattice of subsets (meet is intersection, join is union) such that the empty set $\emptyset$ and the space X belong to $\mathcal{S}$. Given a structure $\mathcal{S}$ on X, the *structure category* $(X, \mathcal{S})$ has as objects the elements of $\mathcal{S}$, with the set $\operatorname{Hom}(A, B)$ of

(2) The only formal difference is that the $B(\exp_x(0), s)$ in (5) must be replaced by a $B(x, s^*) \subset B(x, s)$ having the property that any geodesic with endpoints in $B(x, s^*)$ lies in $B(x, s)$.

(3) A covering of a space is star-finite if each set of the covering meets at most finitely many sets of the covering.

morphisms being the inclusion whenever $A \subset B$ and empty otherwise. A *cohomology theory* on $(X, \mathcal{S})$ is a sequence $\{h^q \mid q \in \boldsymbol{Z}\}$ of cofunctors from $(X, \mathcal{S})$ to the category of abelian groups and homomorphisms (any abelian category could be used), along with natural transformations

$$\delta: h^q(A \cap B) \to h^{q+1}(A \cup B)$$

for each (q, A, B) such that the following sequence is exact:

$$\ldots \longrightarrow h^{q-1}(A \cap B) \xrightarrow{\delta} h^q(A \cup B) \xrightarrow{i^*} h^q(A) \oplus h^q(B) \xrightarrow{j^*} h^q(A \cap B) \longrightarrow \ldots$$

Here

$$i^*(\xi) = (i_A^*\xi, i_B^*\xi), \quad j^*(\eta, \zeta) = j_A^*\eta - j_B^*\zeta$$

where $i_A: A \subset_\to A \cup B$, $j_A: A \cap B \subset_\to A$, etc.

If for any space X we take $\mathcal{S}$ to be the lattice of all open sets, then singular cohomology is a cohomology theory in the sense just defined. (See, for example, [2], page 239). Given any covering $\mathcal{U}$ of X by open sets we get a structure $\mathcal{S}$ by taking the lattice generated by the members of $\mathcal{U}$. Singular cohomology is then a cohomology theory for $(X, \mathcal{S})$.

If X is now a C^∞ manifold, we also have the deRham groups (vector spaces) defined on open sets of X. To see that this is also a cohomology theory (i.e., that we have a Mayer–Vietoris sequence) it suffices to notice that a p-form ω^p on an open set W is uniquely determined by its integrals on all differentiable p-simplexes in W; the proof is then like that for singular cohomology. Just as for singular cohomology, the deRham cohomology is a cohomology theory for $(X, \mathcal{S})$ where $\mathcal{S}$ is the lattice generated by any covering of X by open sets. Our objective, of course, is to show these cohomology theories are isomorphic for a paracompact C^∞ manifold.

LEMMA 3. *Let $h, \hat{h}$ be cohomology theories on $(X, \mathcal{S})$ and let $t: h \to \hat{h}$ be a natural transformation. Assume that for some $A, B \in \mathcal{S}$, both $t(A): h(A) \to \hat{h}(A)$ and $t(B): h(B) \to \hat{h}(B)$ are isomorphisms. Then $t(A \cup B)$ is an isomorphism if and only if $t(A \cap B)$ is an isomorphism.*

Proof. Use the 5-lemma.

THEOREM 1. *Let $\mathcal{U} = \{U_a \mid a \in \mathfrak{a}\}$ be a star-finite covering of X and let $\mathcal{S}$ be the structure generated by $\mathcal{U}$. Let $h, \hat{h}$ be cohomology theories on $(X, \mathcal{S})$ and let $t: h \to \hat{h}$ be a natural transformation. Assume that for each finite intersection $U_{a_1} \cap \ldots \cap U_{a_n}$, $t(U_{a_1} \cap \ldots \cap U_{a_n})$ is an isomorphism. Then $t(X): h(X) \to \hat{h}(X)$ is an isomorphism.*

Proof. First we establish

(∗) If $I_1, \ldots, I_n$ are finite intersections of elements of $\mathcal{U}$, then $t(I_1 \cup \ldots \cup I_n)$ is an isomorphism.

We prove (∗) by induction, noting that for $n = 1$ it is true by hypothesis. Assume it is true for all unions of n finite intersections and let $I_1, \ldots, I_{n+1}$ be $n+1$ finite intersections. Then

$$I_{n+1} \cap (I_1 \cup \ldots \cup I_n)$$

is a union of n finite intersections so $t\big(I_{n+1} \cap (I_1 \cup \ldots \cup I_n)\big)$ is an isomorphism and so is $t(I_1 \cup \ldots \cup I_n)$, both by the inductive hypotheses. By Lemma 3, $t(I_1 \cup \ldots \cup I_{n+1})$ is an isomorphism and (*) is established.

Now we well-order the index set α and proceed by transfinite induction.

Consider $t(\bigcup_{\beta<\alpha} U_\beta)$. Since $\mathfrak{U}$ is *star-finite*, $I = U_\alpha \cap \bigcup_{\beta<\alpha} U_\beta$ is a *finite* union of finite intersections, so by (*) $t(I)$ is an isomorphism. By hypothesis $t(U_\alpha)$ is an isomorphism, so by Lemma 3 $t(\bigcup_{\beta\leqslant\alpha} U_\beta)$ is an isomorphism. This proves $t(X)$ is an isomorphism.

We are now ready to prove deRham's theorem.

THEOREM 2. *If M^n is a paracompact C^∞ manifold, then the singular cohomology groups with real coefficients are isomorphic with the deRham groups.*

Proof. Let h denote singular cohomology and $\hat{h}$ denote deRham cohomology. As noted earlier, these are cohomology theories for the structure $\mathcal{S}$ generated by a covering of M by open sets. In particular we take a star-finite covering $\mathfrak{U} = \{U_\alpha\}$ as in Lemma 2′ by geodesically convex sets.

A p-form ω gives rise to a p-cochain by defining $\omega(\sigma_p) = \int_{\sigma_p} \omega$, for each singular p-simplex σ_p. By Stokes' theorem

$$\int_{\sigma_{p+1}} d\omega = \int_{\partial\sigma_{p+1}} \omega\,,$$

so that this induces a natural transformation

$$t\colon\ \hat{h} \to h\,.$$

By the Poincaré lemma $\hat{h}(U) = 0$, and by the cone construction $h(U) = 0$, where U is any finite intersection of the U_α's. Thus t satisfies the hypothesis of Theorem 1, so $t(M)$ is an isomorphism. q.e.d.

References

[1] J. Dugundji, *Comparison of homologies*, Ann. Scuola Norm. Sup. Pisa 20 (1966), pp. 745–751.

[2] E. Spanier, *Algebraic Topology*, 1966.

RICE UNIVERSITY
UNIVERSITY OF SOUTHERN CALIFORNIA

Reçu par la Rédaction le 10. 12. 1968

Some remarks on the consequence operation in sentential logics

by

R. Wójcicki* (Wrocław)

1. Preliminary notions. Let S be the set of formulas formed by means of sentential variables $p_\xi (\xi \in \Xi)$ (the set of indices Ξ being at least denumerably infinite) and a finite number of connectives $F_1, \ldots, F_n$. As known, $\mathcal{S} = \langle S, F_1, \ldots, F_n\rangle$ is an absolutely free algebra, and $\{p_\xi\}_{\xi\in\Xi}$ is the set of free generators of it. By a *consequence in* $\mathcal{S}$ we understand (cf. [5]) an operation Cn defined for every subset X of S and such that:

(1.1) $$X \subseteq \mathrm{Cn}\big(\mathrm{Cn}(X)\big) \subseteq \mathrm{Cn}(X) \subseteq S\,,$$

(1.2) $$X \subseteq Y \to \mathrm{Cn}(X) \subseteq \mathrm{Cn}(Y)\,.$$

Given an algebra $\mathcal{S}$, as described above, and a consequence Cn in $\mathcal{S}$, the couple $L = \langle \mathcal{S}, \mathrm{Cn}\rangle$ will be called a *sentential logic*; $\mathcal{S}$ and Cn will be called the *language* of L and the *consequence* of L respectively. Let $X \subseteq S$. X is said to be *consistent* provided that $\mathrm{Cn}(X) \neq S$. If $X = \mathrm{Cn}(X)$, X is said to be a Cn-*system* (or a *system in* L). The elements of the set $\mathrm{Cn}(\emptyset)$, where $\emptyset$ denotes the empty set, are called the *theorems of* L.

A relation $R \subseteq 2^S \times S$ will be called a *rule of inference* in $\mathcal{S}$. If $R(X, \alpha)$, i.e. the relation R holds for the arguments X and α, we shall say that *the set of premisses X entails the conclusion α under the rule R*. It is often convenient to assume that the first domain of R consists of sets of a fixed cardinality, which is then called the *cardinality of the rule* R. A set X is said to be *closed under a rule* R provided that, for every $\alpha \in S$ and every $Y \subseteq X$, if $R(Y, \alpha)$ then $\alpha \in X$. Given a set of rules of inference $\mathfrak{R}$ and a consequence Cn, we say that $\mathfrak{R}$ is *a basis for* Cn (Cn is *based on* $\mathfrak{R}$) if those sets which are Cn-systems and only those are closed under all the rules $R \in \mathfrak{R}$. Every consequence operation possesses a basis (see [2]). The consequence based on $\mathfrak{R}$ will be denoted by $\mathrm{Cn}_{\mathfrak{R}}$.

The cardinal number $\mathfrak{m}$ is called *the cardinality of a consequence* Cn in $\mathcal{S}$ if it is the least cardinal number for which the following is valid:

(1.3) $$\alpha \in \mathrm{Cn}(X) \equiv \bigvee_Y \big(Y \subseteq X \wedge \overline{\overline{Y}} < \mathfrak{m} \wedge \alpha \in \mathrm{Cn}(Y)\big)\,,$$

(*) The author is greatly indebted to Prof. J. Łoś for his valuable suggestions and comments on this paper.