The homotopy equivalence $f\colon \Omega E\to\Omega B\times\Omega F$ is a fiber map but the homotopy inverse $g$ is not. Define $N\colon \Omega B\times I\to\Omega B$ and $Q\colon \Omega B\times I\to\Omega B$ by

$$N(\beta, s)(x)=\begin{cases}\beta(2x-xs), & 0\leqslant x\leqslant\frac{1}{2},\\ \beta(1-s+xs), & \frac{1}{2}\leqslant x\leqslant 1;\end{cases}$$

$$Q(\beta, s)=\begin{cases}\beta * C_B, & 0\leqslant s\leqslant\frac{1}{2},\\ N(\beta, 2s-1), & \frac{1}{2}\leqslant s\leqslant 1.\end{cases}$$

Define $\psi\colon \Omega B\times\Omega F\times I\to\Omega E$ by

$$\psi(\beta, \sigma, t)=\lambda\big(g(\beta, \sigma), Q(\beta, \cdot)\big)(t).$$

Then $\psi_1\sim\psi_0\sim g$ so that $\psi_1$ is also a homotopy inverse for $f$. If $(\beta, \sigma)\in\Omega B\times\Omega F$,

$$p\psi_1(\beta, \sigma)=Q(\beta, 1)=\beta=q(\beta, \sigma)$$

so $\psi_1$ is a fiber map. Note also that

$$qf\psi_1(\beta, \sigma)=q\big(p\psi_1(\beta, \sigma), \varphi_1\psi_1(\beta, \sigma)\big)=q\big(\beta, \varphi_1\psi_1(\beta, \sigma)\big)=\beta,$$

so that $f\psi_1$ is fiber homotopic to the identity map on $\Omega B\times\Omega F$. A straightforward computation shows that $\psi_1 f$ is fiber homotopic to the identity map on $\Omega E$.

Now consider the fiber structures $(\Omega^n E, p^n, \Omega^n B)$ and $(\Omega^n B\times\Omega^n F, q^n, \Omega^n B)$ where $p^n$ is the natural map induced by $\pi$ and $q^n$ is the projection on the first factor.

COROLLARY. *If $(E, \pi, B)$ is a weak Hurewicz fibration with cross section, then $(\Omega^n E, p^n, \Omega^n B)$ and $(\Omega^n B\times\Omega^n F, q^n, \Omega^n B)$ are fiber homotopy equivalent for $n\geqslant 1$ and $H$-isomorphic for $n\geqslant 2$.*

Proof. Since $(E, \pi, B)$ is a weak Hurewicz fibration, $(\Omega^n E, p^n, \Omega^n B)$ is also. Since the homotopy equivalence $\psi_1$ of the preceding theorem is an $H$-homomorphism if $\Omega E$ is homotopy abelian, it follows that the given fiber structures are $H$-isomorphic for $n\geqslant 2$.

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# On nil semirings with ascending chain conditions

by

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**1.** A set $R$, together with two operations $+$ and $\cdot$ is said to be a *semiring* if $(R, +)$ and $(R, \cdot)$ are semigroups, $(R, +)$ being a commutative semigroup with 0, with the distributive laws holding between addition and multiplication. Furthermore, we require that $x\cdot 0=0\cdot x=0$ for each $x$ in $R$. If $R$ is a semiring and $I\subseteq R$, then $I$ is a *right ideal of* $R$ if $I$ is closed under addition, and for every $a\in R$, $b\in I$ we have $ba\in I$. Left and two-sided ideals are defined similarly, analogous to ring theory. If $R$ is a semiring and $S$ is a non-empty subset of $R$, then $S_r=\{x\in R|\ Sx=0\}$. If $I$ is a right ideal of $R$ and $I=S_r$ for some $S\subseteq R$, then $I$ is called a right annihilator ideal. Similarly $S_l=\{x\in R|\ xS=0\}$ and we define left annihilator ideals. Finally, a left (right) ideal of $R$ is called a left (right) $k$-ideal [1] if $x+y\in I$ and $y\in I$ implies that $x\in I$ for each $x$ and $y$ in $R$.

In this paper after defining the Levitzki radical $\mathfrak{L}(R)$ of a semiring $R$, we show that every nil subsemiring of a semiring with the ascending chain condition on left and right annihilator ideals is nilpotent, provided that $\mathfrak{L}(R)$ is a $k$-ideal.

**2.** If $I$ is a two-sided ideal of a semiring $R$, then it is well known that $R/I$ also becomes a semiring if we define a congruence relation $\equiv$ as follows:

$$a\equiv b \quad\text{iff}\quad a+i_1=b+i_2 \quad\text{for}\quad i_1, i_2\in I.$$

LEMMA 1. *If $I$ is a $k$-ideal, then $x\equiv 0 \bmod I$ if $x\in I$.*

Proof. If $x\equiv 0 \bmod I$, then $x+y\in I$ for some $y\in I$. But then $x\in I$ since $I$ is a $k$-ideal. Conversely if $x\in I$, then clearly $x\equiv 0 \bmod I$.

DEFINITION. A function $\varphi$ from a semiring $R$ to a semiring $S$ is a *homomorphism* if

$$\varphi(x+y)=\varphi(x)+\varphi(y), \qquad \varphi(xy)=\varphi(x)\varphi(y) \qquad\text{and}\qquad \varphi(0)=0.$$

$\varphi$ is a *semi-isomorphism* if $\varphi$ is onto and $\mathrm{Ker}\varphi=0$.

LEMMA 2. *If $R$ and $S$ are semirings and $\varphi$ is a homomorphism from $R$ onto $S$, then $R/\mathrm{Ker}\varphi$ is semi-isomorphic to $S$.*

Proof. We map $\bar{r} \in R/\mathrm{Ker}\varphi$ to $\varphi(r) \in S$. Then this map $\psi$ is a homomorphism onto $A$. Let $\bar{r} \in \mathrm{Ker}\psi$. Then $\varphi(r) = 0$, that is, $r \in \mathrm{Ker}\varphi$. Hence $\bar{r} = 0$ and $\psi$ is a semi-isomorphism.

DEFINITION. A semiring $R$ is *locally nilpotent* if every finite subset $F$ of $R$ generates a nilpotent subsemiring. This is equivalent to the following condition: There exists an integer $N(F)$ such that every product $x_{i_1} \ldots x_{i_N} = 0$ for $x_{i_j}$ in $F$.

LEMMA 3. *If $A$ and $B$ are locally nilpotent ideals, then $A+B$ is a locally nilpotent ideal.*

Proof. Let $F = \{x_1, \ldots, x_n\}$ be a finite subset of $A+B$. Then $x_i = a_i + b_i$ where $a_i \in A$, $b_i \in B$, for $i = 1, \ldots, n$.

Let $G = \{a_1, \ldots, a_n\}$, $H = \{b_1, \ldots, b_n\}$, $K = \{a_{i_1} \ldots a_{i_k} b_j|\ j = 1, \ldots, n\}$, where $a_{i_1} \ldots a_{i_k}$ are all products from $G$ with $k \leqslant N_1$ where $G^{N_1} = 0$. $L = \{b_{i_1} \ldots b_{i_l} a_j|\ j = 1, \ldots, m\}$ where $b_{i_1} \ldots b_{i_l}$ are all products from $H$ with $l \leqslant N_2$ where $H^{N_2} = 0$.

Now $K \cup L \subseteq A \cap B$ since $A$ and $B$ are assumed to be two-sided ideals. Suppose that $N_3$ has been determined so that $(G \cup L)^{N_3} = 0$. Also let $N_4$ be determined so that $(H \cup K)^{N_4} = 0$. Now let $N = 2\max(N_3, N_4)$. In any monomial occurring in the product, $(a_{i_1} + b_{i_1}) \ldots (a_{i_N} + b_{i_N})$ the number of $a_{i_j}$'s plus the number of $b_{i_k}$'s must equal $N$.

Hence, $|a| + |b| = N$ where $|a|$ denotes the number of $a_{i_j}$'s occurring in a monomial and likewise for $|b|$.

Thus $|a| \geqslant \frac{1}{2}N = \max(N_3, N_4)$, or $|b| \geqslant \frac{1}{2}N$.

If $|a| \geqslant \max(N_3, N_4) \geqslant N_3$, then the monomial is zero by the choice of $N_3$. Similarly if $|b| \geqslant \max(N_3, N_4) \geqslant N_4$, then the monomial is again zero and thus $F^N = 0$. Hence $A+B$ is locally nilpotent.

The proofs for the following lemmas are similar to those in ring theory and are omitted.

LEMMA 4 ([3], p. 26). *The sum $\mathfrak{L}(R)$ of all locally nilpotent ideals of a semiring $R$ is a locally nilpotent ideal of $R$.*

LEMMA 5 ([2], p. 26). *If $A$ is a locally nilpotent $k$-ideal and $R/A$ is locally nilpotent, then $R$ is locally nilpotent.*

LEMMA 6 ([3], p. 27). *If $\mathfrak{L}(R)$ is a $k$-ideal, then $\mathfrak{L}\big(R/\mathfrak{L}(R)\big) = 0$.*

LEMMA 7 ([2], p. 81). *Let $R$ be a semiring satisfying the ascending chain condition on left annihilators. If $R$ is nil, then every non-zero homomorphic image of $R$ contains a non-zero nilpotent ideal.*

COROLLARY ([2], p. 83). *If $R$ is a nil semiring satisfying the ascending chain condition on left annihilators, then $R$ is locally nilpotent.*



LEMMA 8 ([2], p. 83). *Let $R$ be a nil semiring satisfying the ascending chain condition on left annihilators. Then there exists an element $x_0 \neq 0$ in $R$ such that $Rx_0 = 0$.*

LEMMA 9 ([2], p. 69). *If $R$ satisfies the ascending chain condition on left annihilators and if $A$ is a two-sided ideal which is a left annihilator in $R$, then $R/A$ satisfies the ascending chain condition on left annihilators.*

LEMMA 10 ([3], p. 27). *If $L$ locally nilpotent left (right) ideal, then $L \subseteq \mathfrak{L}(R)$.*

LEMMA 11. *Suppose that $R$ is a nil semiring such that $\mathfrak{L}(R)$ is a $k$-ideal. Let $T_n = \{x \in R|\ xR^n = 0\}$. Then $\mathfrak{L}(\bar{R})$ is also a $k$-ideal where $\bar{R} = R/T_n$.*

Proof. Let $x+y \in \mathfrak{L}(\bar{R})$ and $y \in \mathfrak{L}(\bar{R})$. We claim that $\overline{xR_1}$ is a locally nilpotent right ideal where $\overline{xR_1}$ is the right ideal generated by the set $\overline{xR} \cup \{n\bar{x}|\ n \in N\}$, where $N$ is the set of natural numbers. Suppose that $\{\bar{x}(\bar{r}_1 + n_1), \ldots, \bar{x}(\bar{r}_m + n_m)\}$ is a finite set in $\overline{xR_1}$, where $r_i \in R$ and $n_i \in N$.

Since $\bar{x} + \bar{y} \in \mathfrak{L}(\bar{R})$ and $\bar{y} \in \mathfrak{L}(\bar{R})$, $(\bar{x}+\bar{y})\bar{R}_1$ and $\overline{yR_1}$ are locally nilpotent right ideals. Hence there is some number $M$ such that $(\bar{x}+\bar{y})(\bar{r}_{i_1} + n_{i_1}) \ldots \ldots (\bar{x}+\bar{y})(\bar{r}_{i_M} + n_{i_M}) = 0$ and $K$ with $\bar{y}(\bar{r}_{i_1} + n_{i_1}) \ldots \bar{y}(\bar{r}_{i_K} + n_{i_K}) = 0$, for all $i_M$ and $i_K$.

Hence

$$(x+y)(r_{i_1} + n_{i_1}) \ldots (x+y)(r_{i_M} + n_{i_M}) R^n = 0$$

and

$$y(r_{i_1} + n_{i_1}) \ldots y(r_{i_K} + n_{i_K}) R^n = 0\,.$$

Thus

$$(x+y)(r_{i_1} + n_{i_1}) \ldots (x+y)(r_{i_{M+n}} + n_{i_{M+n}}) = 0$$

and

$$y(r_{i_1} + n_{i_1}) \ldots y(r_{i_{K+n}} + n_{i_{K+n}}) = 0$$

and hence $(x+y)R_1$ and $yR_1$ are locally nilpotent right ideals of $R$.

Therefore $x+y \in \mathfrak{L}(R)$ and $y \in \mathfrak{L}(R)$. Since $\mathfrak{L}(R)$ is assumed to be a $k$-ideal of $R$, $x \in \mathfrak{L}(R)$, and we have $x(r_{i_1} + n_{i_1}) \ldots x(r_{i_N} + n_{i_N}) = 0$ for all products of weight $N$, for some $N$. Hence $\bar{x}(\bar{r}_{i_1} + n_{i_1}) \ldots \bar{x}(\bar{r}_{i_N} + n_{i_N}) = 0$ in $\overline{xR_1}$ and we have shown that $\overline{xR_1}$ is a locally nilpotent right ideal of $\bar{R}$, so that $\bar{x} \in \mathfrak{L}(\bar{R})$. This shows that $\mathfrak{L}(\bar{R})$ is a $k$-ideal of $\bar{R}$.

THEOREM ([2], p. 84). *If $R$ is a semiring which satisfies the ascending chain condition on left and right annihilators and is such that $\mathfrak{L}(R)$ is a $k$-ideal, then any nil subsemiring of $R$ is nilpotent.*

Proof. Since the ascending chain conditions on left and right annihilators are inherited by subsemirings, we may assume that $R$ is nil.

Let $T_k = \{x \in R|\ xR^k = 0\}$. Since $T_1 \subseteq T_2 \subseteq \ldots$ is an ascending chain of left annihilators, there is an $n$ such that $T_n = T_{n+1} = \ldots$

If $T_n = R$ then $R^{n+1} = 0$ and the proof is completed. If $T_n \neq R$, then $\bar{R} = R/T_n \neq 0$ since $T_n$ is a $k$-ideal. By the previous lemma $\mathfrak{L}(\bar{R})$

is a $k$-ideal and by Lemma 9 $\bar{R}$ satisfies the ascending chain condition on right annihilators, and so by Lemma 8 there exists an $\bar{x} \neq 0$ in $\bar{R}$ such that $\overline{xR} = 0$.

Therefore $xR \subseteq R_n$ and $xRR^n = xR^{n+1} = 0$. By our choice of $n$, $x \in T_n$ so that $\bar{x} = 0$. This contradiction proves that $\bar{R} = 0$ and $R = T_n$. Hence $R^{n+1} = 0$.

COROLLARY. *If $R$ is a semiring satisfying the ascending chain condition on left and right $k$-ideals and such that $\mathfrak{L}(R)$ is a $k$-ideal, then any nil subsemiring of $R$ is nilpotent.*

Proof. Since every right or left annihilator ideal is a right or left $k$-ideal, the corollary follows from the theorem.

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# A proof of deRham's theorem

by

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It is the purpose of this note to give a short proof of deRham's theorem using a modification of Dugundji's cohomology comparison theorem [1] and a simple convexity lemma. We include a proof of this well-known lemma since we have been unable to find it in the literature.

LEMMA 1. *Let $f\colon U \to V$ be a homeomorphism, where $U$ and $V$ are open sets in $\mathbf{R}^n$. Assume* (1) *that $f$ is $C^1$ and that $g = f^{-1}$ is $C^2$. Then for each $x \in U$ there exists an $r(x) > 0$ such that the image $f\big(B(x, r)\big)$ of every ball $B(x, r)$ of radius $r \leqslant r(x)$ about $x$ is convex.*

Proof. We can assume $x = 0$ and that $U, V$ are small enough so that there exist real numbers $K > 0$, $M > 0$ satisfying

(1) If $\gamma$ is a curve obtained by restricting $f$ to any line segment in $U$, then

$$\|\gamma'(t)\| \leqslant K$$

(where $t$ is arc length on the segment and prime denotes differentiation).

(2) If $\varrho$ is a curve obtained by restricting $g$ to any line segment in $V$, then

$$\|\varrho''(t)\| \leqslant M .$$

Note that we also have $\|\varrho'(t)\| \geqslant 1/K$. Pick $\lambda > 0$ so small that

(3) $2M\lambda \leqslant 1/K^2$

and choose $s > 0$ so that

(4) $gB\big(f(0), s\big) \subset B(0, \lambda)$.

We are now going to show that

(5) For each ball $B(0, r) \subset gB\big(f(0), s\big)$, the image $fB(0, r)$ is convex.

In fact, given $y_0, y_1 \in fB(0, r)$, let $d = \|y_0 - y_1\|$, let $J$ be the closed interval $[0, d]$, and let $\sigma\colon J \to V$ be the line segment joining $y_0$ to $y_1$. We have $\sigma(J) \subset B\big(f(0), s\big)$, since the latter is a convex set containing $y_0$

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(1) Although the given hypotheses imply that $f$ itself is also $C^2$, we make no use of additional fact.