[21] R. Tindell, *A counterexample on relative regular neighborhoods*, Bull. Amer. Math. Soc. 72 (1966), pp. 892–893.

[22] J. H. C. Whitehead, *Simplicial spaces, nuclei and m-groups*, Proc. London Math. Soc. 45 (1939), pp. 243–327.

[23] E. C. Zeeman, *Seminar on combinatorial topology*, Pub. Institut des Hautes Etudes Scient., 1963.

THE INSTITUTE FOR ADVANCED STUDY
Princeton, New Jersey

Reçu par la Rédaction le 20. III. 1968

Fiber homotopy type of associated loop spaces

by

F. H. Croom (Lexington, Ky.)

1. Introduction. Let E and B be topological spaces with base points and $\pi\colon E\to B$ a continuous map. This paper gives necessary and sufficient conditions that the fiber structures $(\Omega E, p, \Omega B)$ and $(\Omega B\times\Omega F, q, \Omega B)$ be fiber homotopy equivalent where F is the basic fiber of (E, π, B), ΩE is the space of based loops in E, $p\colon \Omega E\to\Omega B$ is the natural map induced by π and q is the projection on the first factor. From this result it is observed that if (E, π, B) is a Hurewicz fibration with cross section, $(\Omega E, p, \Omega B)$ and $(\Omega B\times\Omega F, q, \Omega B)$ are fiber homotopy equivalent. It follows that the higher loop space $\Omega^n E$ is H-isomorphic to $\Omega^n B\times\Omega^n F$ for $n\geqslant 2$. This naturally implies the known result ([3] p. 152):

$$\pi_n(E)\simeq\pi_n(B)+\pi_n(F)\quad\text{ for }\quad n\geqslant 2\,.$$

2. Preliminaries.

Definition. A fiber structure (E, π, B) is a *weak Hurewicz fibration* if there is a *weak lifting function*

$$\lambda\colon\ \Delta=\{(e, a)\in E\times B^I\colon\ p(e)=a(0)\}\to E^I$$

such that λ is continuous,

$$\pi\lambda(e, a)(t)=a(t)\qquad(t\in I)$$

and the map $(e, a)\to\lambda(e, a)(0)$ is fiberwise homotopic to the projection on the first factor.

The following analogue of the Curtis–Hurewicz theorem ([1], [4]) is easily proved:

Theorem 1. *The fiber structure (E, π, B) is a weak Hurewicz fibration if and only if for each space X, continuous $f\colon X\to E$ and homotopy $\varphi\colon X\times I\to B$ of πf there exists a homotopy $\Phi\colon X\times I\to E$ covering φ such that Φ_0 is fiberwise homotopic to f.*

Theorem 2. *In order that $(\Omega E, p, \Omega B)$ be fiber homotopy equivalent to $(\Omega B\times\Omega F, q, \Omega B)$, it is necessary and sufficient that $(\Omega E, p, \Omega B)$ be a weak Hurewicz fibration with cross section.*

The maps involved are understood to be base point preserving. A weak lifting function $\lambda\colon \Delta \to E^I$ preserves base points provided that $\lambda(e_0, C_B) = C_E$ where e_0 is the base point of E and C_E is the constant path $C_E(I) = e_0$.

3. Proof of Theorem 2.

Necessity. If $f = (f_1, f_2)\colon \Omega E \to \Omega B \times \Omega F$ and $g\colon \Omega B \times \Omega F \to \Omega E$ are a fiber homotopy equivalence pair, a weak lifting function

$$\lambda\colon \{(\alpha, \bar{\beta}) \in \Omega E \times \Omega B^I\colon p(\alpha) = \bar{\beta}(0)\} \to \Omega E^I$$

and cross section $\chi\colon \Omega B \to \Omega E$ are defined by

$$\lambda(\alpha, \bar{\beta})(t) = g\big(\bar{\beta}(t), f_2(\alpha)\big), \qquad \chi(\beta) = g(\beta, C_E)\,.$$

Sufficiency. Let λ be a weak lifting function and χ a cross section for $(\Omega E, p, \Omega B)$. Define a homotopy $R\colon \Omega B \times I \to \Omega B$ by

$$R(\beta, t)(s) = \begin{cases} \beta\big(1 - 2s(1-t)\big), & 0 \leqslant s \leqslant \frac{1}{2}, \\ \beta\big(1 - (1-t)(2-2s)\big), & \frac{1}{2} \leqslant s \leqslant 1, \ (\beta, t) \in \Omega B \times I, \end{cases}$$

and let $S\colon \Omega E \times I \to \Omega E$ denote the corresponding homotopy for ΩE. Define $\varphi\colon \Omega E \times I \to \Omega E$ by

$$\varphi(\alpha, t) = \lambda\big(\chi p(\alpha)' * \alpha, R(p(\alpha), \cdot)\big)(t)\,, \qquad (\alpha, t) \in \Omega E \times I\,,$$

where $\chi p(\alpha)'(s) = \chi p(\alpha)(1-s)$ and $*$ denotes the usual path multiplication. Observe that $p\varphi(\alpha, 1) = R\big(p(\alpha), 1\big) = C_B$ so that

$$\varphi_1(\Omega E) \subset p^{-1}(C_B) = \Omega F\,.$$

Define $f = (f_1, f_2)\colon \Omega E \to \Omega B \times \Omega F$ and $g\colon \Omega B \times \Omega F \to \Omega E$ by

$$f(\alpha) = \big(p(\alpha), \varphi_1(\alpha)\big), \qquad g(\beta, \sigma) = \chi(\beta) * \sigma\,.$$

Note that g is an H-homomorphism if ΩE is homotopy abelian and χ is an H-homomorphism. For $(\beta, \sigma) \in \Omega B \times \Omega F$,

$$f_1 g(\beta, \sigma) = p\big(\chi(\beta) * \sigma\big) = \beta * C_B$$

and

$$f_2 g(\beta, \sigma) = \varphi_1\big(\chi(\beta) * \sigma\big) = \lambda\big(\chi(\beta * C_B)' * \{\chi(\beta) * \sigma\},\ R(\beta * C_B, \cdot)\big)(1)\,.$$

Define homotopies $\mu\colon \Omega B \times I \to \Omega B$ and $\eta\colon \Omega B \times \Omega F \times I \to \Omega E$ by

$$\mu(\beta, t)(s) = \begin{cases} \beta(2s/1+t), & 0 \leqslant s \leqslant \frac{1}{2}(1+t)\,, \\ b_0, & \frac{1}{2}(1+t) \leqslant s \leqslant 1\,, \end{cases}$$

$$\eta(\beta, \sigma, t)(s) = \begin{cases} \chi\mu(\beta, t)'(4s/2-t)\,, & 0 \leqslant s \leqslant \frac{1}{4}(2-t)\,, \\ \chi(\beta)\left(\dfrac{8s-4+2t}{(1+t)(2-t)}\right), & \frac{1}{4}(2-t) \leqslant s \leqslant \frac{1}{8}(3+t)(2-t)\,, \\ \sigma\left(\dfrac{8s-(3+t)(2-t)}{8-(3+t)(2-t)}\right), & \frac{1}{8}(3+t)(2-t) \leqslant s \leqslant 1\,, \end{cases}$$

where b_0 denotes the base point of B. Then the homotopy $R^*\colon \Omega B \times \times I \times I \to \Omega B$ defined by

$$R^*(\beta, s, t) = \mu\big(R\big(\mu(\beta, t), s\big), 1-t\big)$$

has the following properties:

(1) $R^*(\beta, s, 0) = R(\beta * C_B, s)$,

(2) $R^*(\beta, s, 1) = R(\beta, s) * C_B$,

(3) $R^*(\beta, 1, t) = C_B$,

(4) $p\eta(\beta, \sigma, t) = R^*(\beta, 0, t)$, $\beta \in \Omega B$, $\sigma \in \Omega F$ and $s, t \in I$.

The lifting homotopy $M\colon \Omega B \times \Omega F \times I \to \Omega F$ defined by

$$M(\beta, \sigma, t) = \lambda\big(\eta(\beta, \sigma, t), R^*(\beta, \cdot, t)\big)(1)$$

satisfies

$$M(\beta, \sigma, 0) = \lambda\big(\chi(\beta * C_B)' * \{\chi(\beta) * \sigma\}, R(\beta * C_B, \cdot)\big)(1) = f_2 g(\beta, \sigma)$$

and

$$M(\beta, \sigma, 1) = \lambda\big(\{\chi(\beta)' * \chi(\beta)\} * \sigma, R(\beta, \cdot) * C_B\big)(1)\,.$$

Define a homotopy $L\colon \Omega B \times \Omega F \times I \to \Omega F$ by

$$L(\beta, \sigma, t) = \lambda\Big(S\big(\chi(\beta), t\big) * \sigma, R\big(\beta, t + \cdot(1-t)\big) * C_B\Big)(1)\,.$$

Observe that $L_0 = M_1$ and

$$L(\beta, \sigma, 1) = \lambda(C_E * \sigma, C_B^2)(1)$$

where C_B^2 is the constant loop $C_B^2(I) = C_B$. It thus follows that L_1 is homotopic to the projection of $\Omega B \times \Omega F$ on the second factor. Hence $f_2 g\colon \Omega B \times \Omega F \to \Omega F$ is homotopic to the projection on the second factor and fg is homotopic to the identity map on $\Omega B \times \Omega F$.

For $\alpha \in \Omega E$,

$$gf(\alpha) = \chi p(\alpha) * \varphi_1(\alpha)\,.$$

Since φ_1 is homotopic to φ_0, it follows that gf is homotopic to the identity map on ΩE.

The homotopy equivalence $f\colon \Omega E \to \Omega B \times \Omega F$ is a fiber map but the homotopy inverse g is not. Define $N\colon \Omega B \times I \to \Omega B$ and $Q\colon \Omega B \times I \to \Omega B$ by

$$N(\beta, s)(x) = \begin{cases} \beta(2x - xs), & 0 \leqslant x \leqslant \frac{1}{2}, \\ \beta(1 - s + xs), & \frac{1}{2} \leqslant x \leqslant 1; \end{cases}$$

$$Q(\beta, s) = \begin{cases} \beta * C_B, & 0 \leqslant s \leqslant \frac{1}{2}, \\ N(\beta, 2s - 1), & \frac{1}{2} \leqslant s \leqslant 1. \end{cases}$$

Define $\psi\colon \Omega B \times \Omega F \times I \to \Omega E$ by

$$\psi(\beta, \sigma, t) = \lambda\big(g(\beta, \sigma), Q(\beta, \cdot)\big)(t).$$

Then $\psi_1 \sim \psi_0 \sim g$ so that ψ_1 is also a homotopy inverse for f. If $(\beta, \sigma) \in \Omega B \times \Omega F$,

$$p\psi_1(\beta, \sigma) = Q(\beta, 1) = \beta = q(\beta, \sigma)$$

so ψ_1 is a fiber map. Note also that

$$qf\psi_1(\beta, \sigma) = q\big(p\psi_1(\beta, \sigma), \varphi_1\psi_1(\beta, \sigma)\big) = q\big(\beta, \varphi_1\psi_1(\beta, \sigma)\big) = \beta,$$

so that $f\psi_1$ is fiber homotopic to the identity map on $\Omega B \times \Omega F$. A straightforward computation shows that $\psi_1 f$ is fiber homotopic to the identity map on ΩE.

Now consider the fiber structures $(\Omega^n E, p^n, \Omega^n B)$ and $(\Omega^n B \times \Omega^n F, q^n, \Omega^n B)$ where p^n is the natural map induced by π and q^n is the projection on the first factor.

COROLLARY. *If (E, π, B) is a weak Hurewicz fibration with cross section, then $(\Omega^n E, p^n, \Omega^n B)$ and $(\Omega^n B \times \Omega^n F, q^n, \Omega^n B)$ are fiber homotopy equivalent for $n \geqslant 1$ and H-isomorphic for $n \geqslant 2$.*

Proof. Since (E, π, B) is a weak Hurewicz fibration, $(\Omega^n E, p^n, \Omega^n B)$ is also. Since the homotopy equivalence ψ_1 of the preceding theorem is an H-homomorphism if ΩE is homotopy abelian, it follows that the given fiber structures are H-isomorphic for $n \geqslant 2$.

References

[1] M. L. Curtis, *The covering homotopy theorem*, Proc. Amer. Math. Soc. 7 (1956), pp. 682–684.
[2] E. Fadell, *On fiber spaces*, Trans. Amer. Math. Soc. 90 (1961), pp. 1–14.
[3] S. T. Hu, *Homotopy Theory*, New York, 1959.
[4] W. Hurewicz, *On the concept of fiber space*, Proc. Nat. Acad. Sci. U.S.A. 41 (1955), pp. 956–961.

UNIVERSITY OF KENTUCKY

Reçu par la Rédaction le 20. 5. 1968

On nil semirings with ascending chain conditions

by

Erol Barbut (Moscow, Id.)

1. A set R, together with two operations $+$ and $\cdot$ is said to be a *semiring* if $(R, +)$ and $(R, \cdot)$ are semigroups, $(R, +)$ being a commutative semigroup with 0, with the distributive laws holding between addition and multiplication. Furthermore, we require that $x \cdot 0 = 0 \cdot x = 0$ for each x in R. If R is a semiring and $I \subseteq R$, then I is a *right ideal of* R if I is closed under addition, and for every $a \in R$, $b \in I$ we have $ba \in I$. Left and two-sided ideals are defined similarly, analogous to ring theory. If R is a semiring and S is a non-empty subset of R, then $S_r = \{x \in R \mid Sx = 0\}$. If I is a right ideal of R and $I = S_r$ for some $S \subseteq R$, then I is called a right annihilator ideal. Similarly $S_l = \{x \in R \mid xS = 0\}$ and we define left annihilator ideals. Finally, a left (right) ideal of R is called a left (right) k-ideal [1] if $x + y \in I$ and $y \in I$ implies that $x \in I$ for each x and y in R.

In this paper after defining the Levitzki radical $\mathfrak{L}(R)$ of a semiring R, we show that every nil subsemiring of a semiring with the ascending chain condition on left and right annihilator ideals is nilpotent, provided that $\mathfrak{L}(R)$ is a k-ideal.

2. If I is a two-sided ideal of a semiring R, then it is well known that R/I also becomes a semiring if we define a congruence relation $\equiv$ as follows:

$$a \equiv b \quad \text{iff} \quad a + i_1 = b + i_2 \quad \text{for} \quad i_1, i_2 \in I.$$

LEMMA 1. *If I is a k-ideal, then $x \equiv 0 \bmod I$ if $x \in I$.*

Proof. If $x \equiv 0 \bmod I$, then $x + y \in I$ for some $y \in I$. But then $x \in I$ since I is a k-ideal. Conversely if $x \in I$, then clearly $x \equiv 0 \bmod I$.

DEFINITION. A function φ from a semiring R to a semiring S is a *homomorphism* if

$$\varphi(x+y) = \varphi(x) + \varphi(y), \quad \varphi(xy) = \varphi(x)\varphi(y) \quad \text{and} \quad \varphi(0) = 0.$$

φ is a *semi-isomorphism* if φ is onto and $\operatorname{Ker}\varphi = 0$.