Fiber homotopy type of associated loop spaces

by

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1. Introduction. Let $E$ and $B$ be topological spaces with base points and $\pi : E \to B$ a continuous map. This paper gives necessary and sufficient conditions that the fiber structures $(\Omega E, p, \Omega B)$ and $(\Omega B \times \Omega F, q, \Omega B)$ be fiber homotopy equivalent where $F$ is the basic fiber of $(E, \pi, B)$, $\Omega E$ is the space of based loops in $E$, $p : \Omega E \to \Omega B$ is the natural map induced by $\pi$ and $q$ is the projection on the first factor. From this result it is observed that if $(E, \pi, B)$ is a Hurewicz fibration with cross section, $(\Omega E, p, \Omega B)$ and $(\Omega B \times \Omega F, q, \Omega B)$ are fiber homotopy equivalent. It follows that the higher loop space $\Omega^2 E$ is $H$-isomorphic to $\Omega^2 B \times \Omega^2 F$ for $n \geq 2$. This naturally implies the known result ([3] p. 152):

$$\pi_n(B) \cong \pi_n(B) + \pi_n(F) \quad \text{for} \quad n \geq 2.$$ 

2. Preliminaries.

DEFINITION. A fiber structure $(E, \pi, B)$ is a weak Hurewicz fibration if there is a weak lifting function

$$\lambda : A = \{(s, a) : E \times B^I ; p(s) = a(0) \} \rightarrow E^I$$

such that $\lambda$ is continuous,

$$\pi \lambda(s, a)(t) = a(t) \quad (t \in I)$$

and the map $(s, a) \rightarrow \lambda(s, a)(0)$ is fiberwise homotopic to the projection on the first factor.

The following analogue of the Curtis–Hurewicz theorem ([1], [4]) is easily proved.

Theorem 1. The fiber structure $(E, \pi, B)$ is a weak Hurewicz fibration if and only if for each space $X$, continuous $f : X \rightarrow E$ and homotopy $\varphi : X \times I \rightarrow B$ of $f$ there exists a homotopy $\Phi : X \times I \rightarrow E$ covering $\varphi$ such that $\Phi_s$ is fiberwise homotopic to $f$.

Theorem 2. In order that $(\Omega E, p, \Omega B)$ be fiber homotopy equivalent to $(\Omega B \times \Omega F, q, \Omega B)$, it is necessary and sufficient that $(\Omega E, p, \Omega B)$ be a weak Hurewicz fibration with cross section.
The maps involved are understood to be base point preserving. A weak lifting function \( \lambda : \beta \to E \) preserves base points provided that \( \lambda(a_0, C_\beta) = C_\beta \) where \( a_0 \) is the base point of \( E \) and \( C_\beta \) is the constant path \( C_\beta(t) = a_0 \).

3. Proof of Theorem 2.

Necessity. If \( f = (f_1, f_2) : \Omega E \to \Omega B \times \Omega E \) and \( g : \Omega B \times \Omega E \to \Omega E \) are fiber homotopy equivalence pairs, a weak lifting function

\[ \lambda : (\alpha, \beta) \in \Omega E \times \Omega F : p(\alpha) = \tilde{p}(0) \to \Omega F \]

and cross section \( \chi : \Omega B \to \Omega E \) are defined by

\[ \lambda(\alpha, \beta)(t) = g(\tilde{\beta}(t), f_2(\alpha)_t), \quad \chi(\beta) = g(\beta, C_\beta). \]

Sufficiency. Let \( \lambda \) be a weak lifting function and \( \chi \) a cross section for \((\Omega E, p, \Omega B)\). Define a homotopy \( R : \Omega B \times I \to \Omega E \) by

\[ R(\beta, t)(s) = \begin{cases} \beta(2s(1-t)), & 0 \leq s \leq \frac{1}{2}, \\ \beta(1-t), & \frac{1}{2} \leq s \leq 1, \end{cases} \quad (\beta, t) \in \Omega B \times I, \]

and let \( S : \Omega E \times I \to \Omega E \) denote the corresponding homotopy for \( \Omega E \). Define a homotopy \( \varphi : \Omega B \times I \to \Omega E \) by

\[ \varphi(\alpha, t) = \lambda(\chi(\beta)(\alpha)_t), \quad (\alpha, t) \in \Omega E \times I, \]

where \( \chi(\beta)(\alpha)_t = \chi(\beta)(1-t) \) and \( \lambda \) denotes the usual path multiplication.

Observe that \( \varphi(\alpha, 1) = R(p(\alpha), 1) = C_\beta \) so that

\[ \varphi(\alpha) \subset p^{-1}(C_\beta) = \Omega E. \]

Define \( f = (f_1, f_2) : \Omega E \to \Omega B \times \Omega E \) and \( g : \Omega B \times \Omega E \to \Omega E \) by

\[ f_1(\alpha) = \varphi(\alpha), \quad g(\beta, \sigma) = \chi(\beta) \cdot \sigma. \]

Note that \( g \) is an \( H \)-homomorphism if \( \Omega E \) is homotopy abelian and \( \varphi \) is an \( H \)-homomorphism. For \( (\beta, \sigma) \in \Omega B \times \Omega E \),

\[ f_2 g(\beta, \sigma) = p(\chi(\beta) \cdot \sigma) = \beta \cdot C_\beta \]

and

\[ f_2 g(\beta, \sigma) = \varphi(\chi(\beta) \cdot \sigma) = \lambda(\chi(\beta \cdot C_\beta) \cdot (\chi(\beta) \cdot \sigma), R(\beta \cdot C_\beta, \cdot))(1). \]

Define homotopies \( \mu : \Omega B \times I \to \Omega B \) and \( \eta : \Omega B \times \Omega F \times I \to \Omega F \) by

\[ \mu(\beta, t)(s) = \begin{cases} \beta(2s(1-t)), & 0 \leq s \leq \frac{1}{2}(1-t), \\ \beta(1-t), & \frac{1}{2}(1-t) \leq s \leq 1, \end{cases} \]

\[ \eta(\beta, \sigma, t)(s) = \begin{cases} \chi(\beta)(\eta(\beta, \sigma, t)(s)), & 0 \leq s \leq \frac{1}{2}(2-t), \\ \beta \cdot \sigma, & \frac{1}{2}(2-t) \leq s \leq 1. \end{cases} \]

where \( \eta \) denotes the base point of \( B \). Then the homotopy \( R^* : \Omega B \times \Omega F \times I \to \Omega F \) defined by

\[ R^*(\beta, \sigma, t) = \mu[R(\beta(\sigma, t), 0), 0] \]

has the following properties:

1. \( R^*(\beta, \sigma, 0) = \mu[R(\beta \cdot C_\beta), 0] \)
2. \( R^*(\beta, 1, s) = \beta \cdot C_\beta \)
3. \( R^*(\beta, 1, s) = C_\beta \)
4. \( \varphi(\alpha, 1) = R(p(\alpha), 1) = C_\beta \)

The lifting homotopy \( M : \Omega B \times \Omega F \times I \to \Omega F \) defined by

\[ M(\beta, \sigma, t) = \lambda(\chi(\beta)(\sigma), \chi(\beta)(\sigma), R(\beta \cdot C_\beta, \cdot))(1) \]

satisfies

\[ M(\beta, \sigma, 0) = \lambda(\chi(\beta)(\sigma), \chi(\beta)(\sigma), R(\beta \cdot C_\beta, \cdot))(1) = f_2 g(\beta, \sigma) \]

and

\[ M(\beta, \sigma, 1) = \lambda(\chi(\beta)(\sigma), \chi(\beta)(\sigma), R(\beta \cdot C_\beta, \cdot))(1). \]

Define a homotopy \( L : \Omega B \times \Omega F \times I \to \Omega F \) by

\[ L(\beta, \sigma, t) = \lambda(\chi(\beta)(\sigma), \chi(\beta)(\sigma), R(\beta \cdot C_\beta, \cdot))(1) \]

Observe that \( L_0 = M_0 \) and

\[ L(\beta, \sigma, 1) = \lambda(\chi(\beta)(\sigma), \chi(\beta)(\sigma), R(\beta \cdot C_\beta, \cdot))(1). \]

where \( C_\beta \) is the constant path \( C_\beta(1) = C_\beta \). It thus follows that \( L_1 \) is homotopic to the projection of \( \Omega B \times \Omega F \) on the second factor. Hence \( f_2 g : \Omega B \times \Omega F \to \Omega F \) is homotopic to the projection on the second factor and \( f_2 g \) is homotopic to the identity map on \( \Omega B \times \Omega F \).

For \( \sigma \in \Omega F \),

\[ g(\sigma) = \chi(\beta)(\sigma) \cdot \varphi(\alpha). \]

Since \( \varphi_1 \) is homotopic to \( \varphi_1 \), it follows that \( g \) is homotopic to the identity map on \( \Omega E \).

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The homotopy equivalence \( f: \Omega E \to \Omega B \times \Omega F \) is a fiber map but the homotopy inverse \( g \) is not. Define \( N: \Omega B \times I \to \Omega B \) and \( Q: \Omega B \times I \to \Omega B \) by

\[
N(\beta, s)(e) = \begin{cases} 
(\beta(2s - 2e), & 0 < s < \frac{1}{2}, \\
(\beta(1 - s + 2e), & \frac{1}{2} < s < 1; 
\end{cases} \\
Q(\beta, s) = \begin{cases} 
\beta \times Q_B, & 0 < s < \frac{1}{2}, \\
N(\beta, 2s - 1), & \frac{1}{2} < s < 1.
\end{cases}
\]

Define \( \psi: \Omega B \times \Omega F \times I \to \Omega B \) by

\[
\psi(\beta, \sigma, t) = \lambda(g(\beta, \sigma), Q(\beta, \cdot))(t).
\]

Then \( \psi \circ \varphi \sim g \) so that \( \psi \) is also a homotopy inverse for \( f \). If \( (\beta, \sigma) \in \Omega B \times \Omega F \),

\[
p\psi(\beta, \sigma) = Q(\beta, 1) = \beta = g(\beta, \sigma)
\]

so \( \psi \) is a fiber map. Note also that

\[
g \circ p \circ (\beta, \sigma) = g(q \circ p(\beta, \sigma), \varphi(q \circ p(\beta, \sigma))) = g(\beta, \varphi(q \circ p(\beta, \sigma))) = \beta,
\]

so that \( f \circ p \) is fiber homotopic to the identity map on \( \Omega B \times \Omega F \). A straightforward computation shows that \( \psi \circ f \) is fiber homotopic to the identity map on \( \Omega B \).

Now consider the fiber structures \( (\Omega^m E, p^m, \Omega^m B) \) and \( (\Omega^m B \times \Omega^m F, q^m, \Omega^m B) \) where \( p^m \) is the natural map induced by \( \pi \) and \( q^m \) is the projection on the first factor.

**Corollary.** If \( (E, \pi, B) \) is a weak Hurewicz fiberation with cross section, then \( (\Omega^m E, p^m, \Omega^m B) \) and \( (\Omega^m B \times \Omega^m F, q^m, \Omega^m B) \) are fiber homotopy equivalent for \( m > 0 \) and \( H \)-isomorphic for \( m > 2 \).

**Proof.** Since \( (E, \pi, B) \) is a weak Hurewicz fiberation, \( (\Omega^m E, p^m, \Omega^m B) \) is also. Since the homotopy equivalence \( \varphi \) of the preceding theorem is an \( H \)-homomorphism if \( \Omega E \) is homotopy abelian, it follows that the given fiber structures are \( H \)-isomorphic for \( m > 2 \).

### References


### On nil semirings with ascending chain conditions

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1. A set \( E \), together with two operations \( + \) and \( \cdot \) is said to be a semiring if \( (E, +) \) and \( (E, \cdot) \) are semigroups, \( (E, +) \) being a commutative semigroup with 0, with the distributive laws holding between addition and multiplication. Furthermore, we require that \( x \cdot 0 = 0 = x \cdot 0 \) for each \( x \in E \). If \( E \) is a semiring and \( I \subseteq E \), then \( I \) is a right ideal of \( E \) if \( I \) is closed under addition, and for every \( x \in E \), \( x \cdot I \) we have \( x \cdot I \). Right and two-sided ideals are defined similarly, analogous to ring theory. If \( E \) is a semiring and \( S \) is a non-empty subset of \( E \), then \( S_0 = \{ x \in E : xS = 0 \} \) and \( I \) is called a right annihilator ideal. Similarly \( S_0 = \{ x \in E : xS = 0 \} \) and we define left annihilator ideals. Finally, a left (right) ideal of \( E \) is called a left (right) \( k \)-ideal if \( x \cdot y \in I \) and \( y \in I \) implies that \( x \cdot I \) for each \( x \) and \( y \) in \( E \).

In this paper after defining the Levitzki radical \( L(E) \) of a semiring \( R \), we show that every nil subsemiring of a semiring with the ascending chain condition on left and right annihilator ideals is nilpotent, provided that \( L(E) \) is a \( k \)-ideal.

2. If \( I \) is a two-sided ideal of a semiring \( E \), then it is well known that \( E/I \) also becomes a semiring if we define a congruence relation \( = \) as follows:

\[
a = b \iff a + i_1 = b + i_2 \text{ for } i_1, i_2 \in I.
\]

**Lemma 1.** If \( E \) is a \( k \)-ideal, then \( x = 0 \mod I \) if \( x \in I \).

**Proof.** If \( x \equiv 0 \mod I \), then \( x + y \in I \) for some \( y \in I \). But then \( x \cdot I \) since \( I \) is a \( k \)-ideal. Conversely if \( x \in I \), then clearly \( x = 0 \mod I \).

**Definition.** A function \( \varphi \) from a semiring \( E \) to a semiring \( S \) is a \( \varphi \)-homomorphism if

\[
\varphi(x + y) = \varphi(x) + \varphi(y), \quad \varphi(xy) = \varphi(x)\varphi(y) \quad \text{and} \quad \varphi(0) = 0.
\]

\( \varphi \) is a semi-isomorphism if \( \varphi \) is onto and \( \ker \varphi = 0 \).