Small ambient isotopies of a 3-manifold which transform one embedding of a polyhedron into another

by

Robert Craggs* (Urbana Ill.)

1. Introduction. This is the second of three papers in which we investigate the global relation between two tame embeddings of a polyhedron in a 3-manifold where both embeddings closely approximate a given topological embedding of the polyhedron. In Section 8 we show that if a polyhedron in the situation just mentioned has no local cut points then there is a small ambient isotopy of the manifold which pushes the first tame embedding into the second. Sanderson and Kister [15, 20] have results like this for more special classes of polyhedra where the given topological embedding of the polyhedron is tame. A local cut point in a polyhedron is a point which has a closed polyhedral neighborhood equivalent to a cone over a disconnected polyhedron. An example showing why it is necessary to exclude polyhedra with local cut points in our theorem is obtained by considering an unknotted polyhedral simple closed curve in E^3. Every point of a simple closed curve is a local cut point. By breaking open the curve at a point, tying an overhand knot near one end, and then gluing the curve back together again one can obtain an arbitrarily close polyhedral approximation to the curve which is differently embedded in E^3. The same construction can be used for any embedding of a polyhedron with a local cut point to show examples where our theorems fails.

Section 9 and 10 contain some applications of the isotopy theorem in Section 8. In Section 9 we show that two tame embeddings of a polyhedron with no local cut points in a 3-manifold are ambient isotopic if they are isotopic. In Section 10 we show that any closed embedding of a polyhedron without local cut points in a 3-manifold can be obtained by applying a pseudo isotopy of the manifold to a tame embedding of the polyhedron. Keldyš [13, 14] has a direct and more elegant proof of this fact for the special case where the polyhedron is a 2-manifold.

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We assume that the reader is familiar with [5] and [6]. The intro duction in [6] serves also as an introduction for this paper. Most of ou notation and terminology is taken from [6]. We use "$*$" as a symbol for join. Thus $P*K$ denotes the join of polyhedra (or complexes) P and K. Relative regular neighborhoods are used here. The theory of relative regular neighborhoods is developed in [11], where existence and uniqueness theorems are given. The uniqueness theorem in [11] is false in general (see [12, 21]) but is true for manifolds of dimension less than four.

We find it convenient to introduce the following definition for dealing with embeddings of non-compact polyhedra in 3-manifolds. A quadruple (X, μ_1, μ_2, n) has *Property R* if X is a separable metric space, μ_1 and μ_2 are continuous, non-negative, real functions on X, and n is a positive integer such that for each sequence of points $x_1, \dots, x_n$ of X where $\varrho(x_i, x_{i+1}) \leqslant \mu_2(x_i)\,(1 \leqslant i < n)$, $\mu_2(x_n) \leqslant \mu_1(x_1)$.

Lemma 4.2 and Theorem 5.1 represent the heart of this paper and present the most difficult reading here. There are two things which the reader can do to make the reading task easier. The first is to be familiar with the proof of Theorem 7.1 of [6]. This theorem is more simple and it motivates many of the steps in the proofs of Lemma 4.2 and Theorem 5.1. The second thing the reader can do is to first read the two proofs for the case $L = \emptyset$. For such a reading many bothersome constructions can be ignored.

2. Some piercing and separation lemmas. We omit most proofs in this section.

LEMMA 2.1. *Suppose D is a disk, $A_1, \dots, A_m$ are arcs, $p \in \operatorname{Int}(D)$ is an endpoint of each A_i, and $(\bigcup A_i) \cap D = P$.*

Suppose f is a homeomorphism of $D \cup (\bigcup A_i)$ into E^3 such that $f(A_1), \dots, f(A_m)$ all abut on the same side of $f(D)$.

There is a $\delta > 0$ such that if f_0 and f_1 are homeomorphisms of $D \cup (\bigcup A_i)$ into E^3 which agree on D and for which $d(f, f_e) < \delta$ $(e = 0, 1)$ then $f_0(A_1), \dots, f_0(A_m)$, $f_1(A_1), \dots, f_1(A_m)$ all abut on the same side of $f_0(D)$.

LEMMA 2.2. *Suppose D is a disk, A is an arc such that $p = A \cap D = \operatorname{Int}(A) \cap \operatorname{Int}(D)$ is a point, and f is a homeomorphism of $D \cup A$ into E^3.*

There is a $\delta > 0$ such that if f' is a homeomorphism of $D \cup A$ into E^3 for which $d(f, f') < \delta$, then $f'(A)$ pierces $f'(D)$ at $f'(p)$ if and only if $f(A)$ pierces $f(D)$ at $f(p)$.

LEMMA 2.3. *Suppose K is a finite, connected polyhedron of dimension at least one, L is a subpolyhedron of K, and v is a point joinable to K.*

Suppose f is a homeomorphism of $v*K$ into E^3 such that for each arc t in $v*K$ which intersects $v*L$ in a single point of $\operatorname{Int}(t)$ and each disk D in $v*L$ whose interior contains $t \cap v*L$, $f(t)$ does not pierce $f(D)$.

There is a $\delta > 0$ such that if f' is a homeomorphism of $v*K$ into E^3 for which $d(f, f') < \delta$, then for each arc t in $v*K$ which intersects $v*L$ in a single point of $\operatorname{Int}(t)$ and each disk D in $v*L$ whose interior contains $t \cap v*L$, $f'(t)$ does not pierce $f'(D)$.

LEMMA 2.4. *Suppose $D_1, \dots, D_m, \dots, D_n\,(m < n)$ are polyhedral disks such that each $D_i \cap D_j$ $(i \neq j)$ is an arc A_{ij} in $\operatorname{Bd}(D_i) \cap \operatorname{Bd}(D_j)$. Set $K = \bigcup D_i$ and $L = \bigcup_{i>m} D_i$. Suppose A is an arc in $\bigcap_{i,j>m} \operatorname{Int}(A_{ij})$ and each $A_{ij} \subset A$ where either $i \leqslant m$ or $j \leqslant m$. Suppose that $\bigcap A_{ij}$ is an arc B in A.*

Suppose f is a homeomorphism of K into E^3 such that for each arc t in K which intersects L in a single point of $\operatorname{Int}(t)$ and each disk D in L whose interior contains $t \cap L$, $f(t)$ does not pierce $f(D)$.

There is a $\delta > 0$ such that if f_0 and f_1 are pwl homeomorphisms of K into E^3 which agree on L and for which $d(f, f_e) < \delta$ $(e = 0, 1)$, then there is a polyhedral cube C in E^3 containing $f_0(L)$ so that

$$f_0\big(L - \operatorname{Cl}(K-L)\big) \subset \operatorname{Int}(C) \quad \textit{and} \quad \big(f_0(K) \cup f_1(K)\big) \cap C = f_0(L)\,.$$

Proof. We omit a proof for the case $n = m+1$. By applying Lemmas 2.1 and 2.2 and possibly reordering the disks $D_{m+1}, \dots, D_n$ we find that δ can be required to be sufficiently small so that if f_0 and f_1 are as in the hypothesis of the lemma then each $f_0(D_i) = f_1(D_i)$ and each $f_1(D_i)$ $(i \leqslant m)$ abuts on one side of $f_0(D_{m+1} \cup D_n)$ and each $f_0(D_i)$ $(m+1 < i < n)$ abuts on the other. By applying Lemma 2.4 of [5] for the disk $f_0(D_{m+1} \cup D_n)$ and fattening up $f_0(L)$ at points of $f_0\big(L - \operatorname{Cl}(K-L)\big)$ we find a polyhedral 3-manifold M whose interior contains $f_0\big(L - \operatorname{Cl}(K-L)\big)$ and which intersects $f_0(K) \cup f_1(K)$ in exactly $f_0(L)$. For C we take a regular neighborhood in M of the collapsible polyhedron $f_0(L)$.

LEMMA 2.5. *Suppose S is a polyhedral 2-sphere, K is a polyhedron in S, and L is a connected subpolyhedron of K with dimension at least one.*

Suppose L does not separate K in S, and suppose for each arc t in K such that $t \cap L$ is an interior point of t and each arc r of L whose interior contains the point $t \cap L$, t does not pierce r in S.

There is a polyhedral disk D in S containing L such that

$$L - \operatorname{Cl}(K-L) \subset \operatorname{Int}(D) \ \textit{and} \ D \cap K = L\,.$$

3. Isotopies which modify embeddings of cones. We omit most proofs in this section.

LEMMA 3.1. *Suppose K is a finite polyhedron, v is a point joinable to K, B is a compact pwl 3-manifold, and f is a pwl homeomorphism of $v*K$ into B such that $f(v*K) \cap \operatorname{Bd}(B) = f(K)$ and B collapses to $f(v*K)$.*

There is a pwl homeomorphism φ of the pair $\big(B, f(v*K)\big)$ onto a pair $(\tau, b*P)$ where τ is a 3-simplex with barycenter b and P is a polyhedron *in* $\mathrm{Bd}(\tau)$ *so that the composition* φf *is the join of the restrictions* $\varphi f|v\colon v\to b$ *and* $\varphi f|K\colon K\to P$.

Lemma 3.2. *Suppose* K *is a connected polyhedron,* v *is a point joinable to* K, B *is a pwl* 3*-cell, and* f *is a pwl homeomorphism of* $v*K$ *into* B *such that* $f(v*K)\cap \mathrm{Bd}(B)=f(K)$.

Then B *collapses to* $f(v*K)$.

By applying Lemmas 3.1 and 3.2 we get the following lemma. The corollary is obtained by making use of Lemma 2.5.

Lemma 3.3. *Suppose* K *is a connected polyhedron,* v *is a point joinable to* K, B *is a pwl* 3*-cell,* f *is a pwl homeomorphism of* $v*K$ *into* B *such that* $f(v*K)\cap\mathrm{Bd}(B)=f(K)$, *and* g *is pwl homeomorphism of* $f(v*K)$ *into* B *such that* g *is the identity on* $f(K)$ *and* $gf(v*K)\cap\mathrm{Bd}(B)=f(K)$.

Then there is a pwl isotopy $H_t(0\leqslant t\leqslant 1)$ *of* B *onto itself such that* H_t *is the identity on* $\mathrm{Bd}(B)$ *and* $H_1f=gf$.

Furthermore if L *is a subpolyhedron of* K *(possibly empty) and* g *is the identity on* $f(v*L)$ *then* H_t *can be constructed so that it is the identity on* $f(v*L)\cup O$ *where* O *is an open polyhedron in* B *containing* $f\big(v*L-v*\mathrm{Cl}(K-L)\big)$.

Corollary 3.3. *Suppose* L *in Lemma* 3.3 *is connected and of dimension at least one,* $f(L)$ *does not separate* $f(K)$ *in* $\mathrm{Bd}(B)$, *and* $L-\mathrm{Cl}(K-L)\neq\emptyset$.

Suppose for each arc t *in* K *such that* $t\cap L$ *is an interior point of* t *and each arc* r *in* L *whose interior contains the point* $t\cap L$, $f(t)$ *does not pierce* $f(r)$ *in* $\mathrm{Bd}(B)$.

Then H_t *can be chosen so that* $\mathrm{Cl}(O)$ *is a pwl* 3*-cell and* $\mathrm{Cl}(O)\cap\mathrm{Bd}(B)$ *is a* 2*-cell.*

Lemma 3.4. *Suppose* K *is a connected polyhedron,* v *is a point joinable to* K, M *is a pwl* 3*-manifold,* f *is a homeomorphism of* $v*K$ *into* $\mathrm{Int}(M)$, *and* $\varepsilon>0$.

There is a $\delta>0$ *such that if* f_0 *and* f_1 *are pwl homeomorphisms of* $v*K$ *into* M *where* $d(f,f_e)<\delta\,(e=0,1)$ *and* f_0 *agrees with* f_1 *on the complement of the inverse under* f^{-1} *of a* δ*-neighborhood of* $f(v)$, *then there is a pwl isotopy* $H_t(0\leqslant t\leqslant 1)$ *of* M *onto itself so that* $H_1f_0=f_1$ *and* H_t *is the identity on the complement of an* ε*-neighborhood of* $f(v)$.

Furthermore if L *is a subpolyhedron of* K *(possibly empty) such that* f_0 *agrees with* f_1 *on* $v*L$ *then* H_t *can be chosen so that it is also the identity on* $f_0(v*L)\cup O$ *where* O *is an open polyhedron in* M *which contains* $f_0\big(v*L-v*\mathrm{Cl}(K-L)\big)$.

Proof. There is a neighborhood of $f(v*K)$ in M which can be pwl embedded in E^3 under a uniformly continuous homeomorphism [16] so it is sufficient to consider the case $M=E^3$.

Let C be a pwl 3-cell of diameter less than ε such that $f(v)\in\mathrm{Int}(C)$ and $f(K)\subset\mathrm{Ext}(C)$. Let $a>0$ be so small that the image under f of each point $ty+(1-t)v\,(y\in K,\ 0\leqslant t\leqslant a)$ is contained in $\mathrm{Int}(C)$. Let $K(a)$ denote the polyhedron in $v*K$ whose points have the form $ay+(1-a)v\,(y\in K)$. Set $P(a)=v*K(a)$ and $T(a)=\mathrm{Cl}\big(v*K-P(a)\big)$.

Choose a positive number δ so small that a 3δ-neighborhood of $f(v)$ misses $f\big(T(a)\big)$, a δ-neighborhood of $f\big(P(a)\big)$ is contained in $\mathrm{Int}(C)$, and a δ-neighborhood of $f(K)$ is contained in $\mathrm{Ext}(C)$.

Let f_0 and f_1 be pwl homeomorphisms of $v*K$ into E^3 such that $d(f,f_e)<\delta\,(e=0,1)$ and f_0 agrees with f_1 on the complement of the inverse under f^{-1} of a δ-neighborhood of $f(v)$. If a subpolyhedron L is defined so that f_0 agrees with f_1 on $v*L$ and L is not empty set $T_a(a)=T(a)\cup v*L$. If no L is given or if L is given to be the empty polyhedron (in which case $f_0(v)=f_1(v)$) set $T_a(a)=T(a)$.

From the choice of δ, f_0 agrees with f_1 in a neighborhood of $T(a)$, $f_0(K)\subset\mathrm{Ext}(C)$, and $f_e\big(P(a)\big)\subset\mathrm{Int}(C)\ (e=0,1)$; therefore there is a regular neighborhood M_1 in E^3 of $f_0\big(P(a)\big)\cup f_1\big(P(a)\big)$ modulo $f_0\big(T(a)\big)$ such that $M_1\subset\mathrm{Int}(C)$. Let B_1 be a regular neighborhood of $f_1\big(P(a)\big)$ in M_1 such that $B_1\cap\mathrm{Bd}(M_1)$ is a regular neighborhood of $f_1\big(K(a)\big)=f_0\big(K(a)\big)$ in $\mathrm{Bd}(M_1)$. Choose a triangulation T of M_1 in which $f_0\big(P(a)\big)$, $f_1\big(P(a)\big)$, and B_1 underlie subcomplexes. Let T'' denote the second barycentric subdivision of T. Set $B_0=N\big(f_0(P(a)),T''\big)$. It is a regular neighborhood of $f_0\big(P(a)\big)$ in M_1. Both B_0 and B_1 are pwl 3-cells [22, 23]. Because both $\mathrm{Bd}(B_1)$ and $f_0\big(P(a)\big)$ underlie subcomplexes of T, $B_0\cap\mathrm{Bd}(B_1)$ is a regular neighborhood of $f_0\big(P(a)\big)\cap\mathrm{Bd}(B_1)$ in $\mathrm{Bd}(B_1)$. Let $E_0\subset\mathrm{Int}\big(B_1\cap\mathrm{Bd}(M_1)\big)$ denote the disk-with-holes which is that component of $B_0\cap\mathrm{Bd}(B_1)$ containing $f_0\big(K(a)\big)$. Since $E_0\subset\mathrm{Bd}(M_1)$, $E_0\subset\mathrm{Bd}(B_0)$.

The components of $\mathrm{Bd}(B_1)-\mathrm{Int}(E_0)$ are disks. Shrink these slightly, push them slightly into $\mathrm{Int}(B_1)$, and fatten up the pushed disks into mutually exclusive polyhedral cubes in $\mathrm{Int}(B_1)$ which miss $f_1(v*K)$. Do this so that if $R_1,\ldots,R_i,\ldots$ are the boundaries of the polyhedral cubes, then each R_i is in general position with respect to $\mathrm{Bd}(B_0)$, and if U_0 denotes the component of $\mathrm{Bd}(B_0)-\bigcup R_i$ containing E_0, then $U_0-E_0\subset\mathrm{Int}(B_1)$.

Use ([6], Lemma 2.9) to find a pwl isotopy $H_t^1\ (0\leqslant t\leqslant 1)$ of E^3 so that each component of $H_1^1\big(\mathrm{Bd}(B_0)-\mathrm{Cl}(U_0)\big)$ is contained in some $\mathrm{Int}(R_i)$ and H_t^1 is the identity on $\mathrm{Cl}(U_0)\cup(E^3-C)\cup O_1$ where O_1 is an open polyhedron containing $f_0\big(T_a(a)\big)$. Now $H_1^1\big(\mathrm{Bd}(B_0)\big)-E_0\subset\mathrm{Int}(B_1)$ so $H_1^1(B_0)\subset B_1$ and $H_1^1(B_0)\cap\mathrm{Bd}(B_1)=E_0$.

We have $f_0\big(T(a)\big)\cap B_1=f_0\big(K(a)\big)$ and $H_1^1f_0\big(P(a)-K(a)\big)\subset\mathrm{Int}(B_1)$. From Lemma 3.3 there is a pwl isotopy $H_t^2\ (0\leqslant t\leqslant 1)$ of E^3 onto itself so that $H_1^2H_1^1f_0=f_1$ and H_t^2 is the identity on $f_0\big(T_a(a)\big)\cup(E^3-B_1)\cup O_2$ where O_2 is an open polyhedron in E^3 containing $f_0\big(T_a(a)-\mathrm{Cl}(v*K-T_a(a))\big)$.

If no L has been provided set $H_t = H^1_{2t}(0 \leqslant t \leqslant 1/2)$ and $H_t = H^2_{2(t-1/2)} H_{1/2}(1/2 \leqslant t \leqslant 1)$. In this case $H_1 f_0 = f_1$ and H_t is the identity on the complement of C which is contained in the complement of an ε-neighborhood of $f(v)$.

If a non-empty L has been provided define H_t as in the previous paragraph. The statement there holds, and in addition H_t is the identity on $f_0(v * L) \cup O$ where O is the open polyhedron $O_1 \cap O_2$ which contains $f_0\big(v * L - v * \mathrm{Cl}(K-L)\big)$.

Finally if L is given to be the empty polyhedron so that $f_0(v) = f_1(v)$ use ([6], Lemma 2.3) to find a pwl isotopy $H_t(0 \leqslant t \leqslant 1)$ of E^3 which is the identity on $f_0(v)$ and the complement of C so that $H_1 = H^2_1 H^1_1$. Then $H_1 f_0 = f_1$ and H_t is the identity on $f_0(v)$ and on the complement of an ε-neighborhood of $f(v)$.

COROLLARY 3.4. *If the L in the hypothesis of Lemma 3.4 is provided in advance and is both connected and of dimension at least one, if $L - \mathrm{Cl}(K-L) \neq \emptyset$, and if for each arc t in $v * K$ such that $t \cap v * L$ is an interior point of t and each disk D in $v * L$ whose interior contains $t \cap v * L$, $f(t)$ does not pierce $f(D)$, δ can be chosen so that if f_0 agrees with f_1 on $v * L$ then H_t can be constructed so that $\mathrm{Cl}(O)$ is a pwl 3-cell containing $f_0(L)$.*

Proof. The piercing condition allows us to use Lemma 2.3 to place an additional restriction on δ in the proof of Lemma 3.4. In turn, this allows us to use the stronger Corollary 3.3 in place of Lemma 3.3 to obtain an O_2 such that $\mathrm{Cl}(O_2) \cap B_1$ is a pwl 3-cell which intersects $\mathrm{Bd}(B_1)$ in a disk. Instead of taking O to be $O_1 \cap O_2$ choose a regular neighborhood M_0 of $f_0(v * L)$ in the pwl 3-manifold $O_1 \cap \big((E^3 - B_1) \cup (\mathrm{Cl}(O_2) \cap B_1)\big)$ and take $\mathrm{Int}(M_0)$ for O.

4. Modifying a construction used in [6]. The proof of Theorem 5.1 of Section 5 requires a slight modification of the construction in ([6], Sec. 6) which we make here. This modification helps us to guarantee that the isotopies we build in Section 5 leave certain polyhedra pointwise fixed.

First we define a property, called *Property Q'*, as follows. Suppose D is a disk in E^3, L is a set in E^3 which is homeomorphic to a finite polyhedron, $D \cap L$ is a finite collection of mutually exclusive arcs in $\mathrm{Bd}(D)$, X is a tame Sierpiński curve normally situated in D, η is a positive number, and T is a rectilinear triangulation of E^3 with mesh less than η and i-skeleton T_i such that (D, X, T_2, η) has Property Q. We say (D, L, X, T_2, η) has Property Q' if $\mathrm{Bd}(D \cap L)$ misses T_2 and if for each 2-simplex Δ of T and each arc component t of $D \cap \Delta$ which intersects both $\mathrm{Bd}(\Delta)$ and L there is a null sequence $I_1(t), I_2(t), \ldots$ of mutually exclusive line segments in $\mathrm{Int}(\Delta)$ which converge to the point $t \cap L$ so that each $I_k(t) \cap (D \cup L)$ is a single point on t where $I_k(t)$ pierces D.

LEMMA 4.1. *Suppose D is a polyhedral disk, L is a finite polyhedron whose intersection with D is a finite collection of mutually exclusive arcs in $\mathrm{Bd}(D)$, f is a homeomorphism of $D \cup L$ into E^3, and η is a positive number.*

There is a tame Sierpiński curve X normally situated in $f(D)$, there is a rectilinear triangulation T of E^3 with mesh less than η and i-skeleton T_i, and there is an η-homeomorphism h of E^3 such that

$$\big(hf(D),\ hf(L),\ h(X),\ T_2,\ \eta\big) \text{ has Property } Q'.$$

Proof. Use ([5], Theorem 6.1) to find a tame Sierpiński curve X normally situated in $f(D)$, a rectilinear triangulation T of E^3 with mesh less than $\eta/3$ and i-skeleton T_i, and an $\eta/3$-homeomorphism h' of E^3 such that $\big(h'f(D), h'(X), T_2, \eta/3\big)$ has Property Q. We can assume that $h'f\big(\mathrm{Bd}(D \cap L)\big)$ misses T_2. From the definition of Property Q there is an $\eta/3$-homeomorphism φ' of $h'f(D)$ onto a polyhedral disk E in general position with respect to T_2 so that φ' is the identity on $h'(X)$ and $E \cap T_2 = h'(X) \cap T_2$.

For each 2-simplex Δ of T and each component t of $h'f(D) \cap \Delta$ which intersects both $\mathrm{Bd}(\Delta)$ and $h'f(L)$ use ([4], Theorem 5.1) to find a null sequence of mutually exclusive arcs $z_1(t), \ldots, z_k(t), \ldots$ in Δ which converge to the point $t \cap h'f(L)$ so that each $z_k(t) \cap h'f(D \cup L)$ is a single point on t where $z_k(t)$ pierces $h'f(D)$.

Let h'' be an $\eta/3$-homeomorphism of E^3 which is the identity on a neighborhood of T_1 and on the polyhedral disk $E = \varphi' h' f(D)$ and which leaves each simplex of T invariant so that in each $h''\big(z_k(t)\big)$ there is a line segment $I_k(t)$ that pierces $h''h'(D)$ at the point $z_k(t) \cap t$. The existence of such a homeomorphism follows from the two dimensional Schoenflies theorem. Let h denote the η-homeomorphism $h''h'$ and φ the η-homeomorphism $h''\varphi'(h'')^{-1}$ of $hf(D)$ onto E. Since h'' is the identity on $h'(X)$, φ is the identity on $h(X)$; thus $\big(hf(D), h(X), T_2, \eta\big)$ has Property Q and $\big(hf(D), hf(L), h(X), T_2, \eta\big)$ has Property Q'.

Consider now a polyhedral disk D and a finite collection $\{L_n\}$ (possibly empty) of mutually exclusive polyhedra such that each $D \cap L_n$ is an arc in $\mathrm{Bd}(D)$ and either L_n is an arc whose endpoints miss D or L_n is the sum $\bigcup_j L_{nj}$ over a finite collection of polyhedral disks $\{L_{nj}\}$ $(j \geqslant 1)$ for which every $L_{ni} \cap L_{nj}$ $(i \neq j)$ is an arc A_n in $\mathrm{Bd}(L_{ni}) \cap \mathrm{Bd}(L_{nj})$ whose interior contains $D \cap L_n$. Set $L = \bigcup L_n$, and let $L(1)$ denote the sum of the L_n's which are arcs together with the sum of the A_n's.

Let f be a homeomorphism of $D \cup L$ into E^3 and $\eta > 0$. Use Lemma 4.1 to find a tame Sierpiński curve X normally situated in $f(D)$, a triangulation T of E^3 with mesh less than η and i-skeleton T_i, and an η-homeomorphism h of E^3 such that $\big(hf(D), hf(L), h(X), T_2, \eta\big)$ has Property Q'. In the rest of the construction here we take advantage of the fact that

$(hf(D), h(X), T_2, \eta)$ has Property Q by mimicking the steps in ([6], Sec. 6) and so make the lemmas there applicable to the construction here.

As in ([6], Sec. 6) define a finite graph G_{IV} to be the sum of the components of $h(X) \cap T_2$ which intersect T_1, and let G_{IVO} denote the subgraph $\mathrm{Cl}(G_{IV} \cap hf(\mathrm{Int}(D)))$. Let $\Delta_1, \ldots, \Delta_j, \ldots$ denote the 2-simplexes of $T, t_1, \ldots, t_i, \ldots$ the arc components of the $G_{IVO} \cap \Delta_j$'s, and $p_1, \ldots, p_k, \ldots$ the points of $hf(D) \cap T_1$. For each t_i which intersects $hf(L)$ set $q_i = t_i \cap hf(L)$. Choose subdivisions T', T'', and T''' of T as in [6] so that $N(G_{IVO} \cap hf(L), T')$ fails to intersect T_1 and so that a neighborhood of $N(G_{IVO}, T'') \cap hf(L)$ in E^3 is contained in $N(G_{IVO} \cap hf(L), T')$. Note that no $N(p_k, T'')$ intersects $hf(L)$. Further, because of the fullness condition on T' in [6], $N(t_i, T') \cap N(t_j, T') \neq \emptyset$ only if $t_i \cap t_j \neq \emptyset$.

Call a collection $\{F(t_i)\}$ of polyhedral disks a *special collection of disks* if for each Δ_j and each t_i in Δ_j,

(1) $t_i \subset F(t_i)$,

(2) $F(t_i) \subset N(t_i, T'') \cap \Delta_j$,

(3) $N(t_i \cap \mathrm{Bd}(\Delta_j), T'') \cap \Delta_j \subset F(t_i)$, and

(4) $\mathrm{Bd}(F(t_i)) \cap \mathrm{Int}(\Delta_j)$ fails to contain a point of $hf(D)$.

LEMMA 4.2. *Suppose $\delta_1 > 0$.*

There is a special collection of disks $\{F(t_i)\}$ and there is a $\delta > 0$ such that if f_0 and f_1 are pwl homeomorphisms of $D \cup L$ into E^3 which agree on L and for which $d(f, f_e) < \delta$ $(e = 0, 1)$, then there is a pwl 3η-homeomorphism h_I of E^3 and there are mutually exclusive polyhedral cubes B_n in E^3 containing the two dimensional $f_0(L_n)$'s so that

1. $d(hf, h_I f_e) < \delta_1$ $(e = 0, 1)$,

2. $h_I f_e(D)$ $(e = 0, 1)$ *and* $h_I(\bigcup B_n)$ *are in general position with respect to* T_2,

3. *the cardinality of* $h_I f_1(D) \cap T_1$ *is the same as the cardinality of* $hf(D) \cap T_1$,

4. $h_I f_e(D)$ $(e = 0, 1)$ *fails to intersect* $(\bigcup \mathrm{Bd}(F(t_i))) - T_1$,

5. $h_I f_0(L(1)) \cap F(t_i)$ *is empty or a single point accordingly as* $t_i \cap hf(L)$ *is empty or a single point,*

6. *for each two dimensional* L_n, $h_I f_0(L_n - D) \subset h_I(\mathrm{Int}(B_n))$,

7. $h_I(f_0(L) \cup (\bigcup B_n)) \cap N(G_{IVO}, T'') \subset N(G_{IVO} \cap hf(L), T')$, *and*

8. *no component of any* $h_I(B_n) \cap F(t_i)$ *lies in* $\mathrm{Int}(F(t_i))$.

Proof. For each q_i let $O(q_i)$ be an open 3-cell of diameter less than $\delta_1/4$ which contains q_i so that $O(q_i) \cap T_2$ is an open 2-cell and $O(q_i) \subset N(q_i, T''')$. For each t_i select a polyhedron $L(t_i)$ as follows. If t_i misses $hf(L)$ let $L(t_i)$ denote the empty polyhedron. If L_n is an arc and t_i intersects $hf(L_n)$ let $L(t_i)$ be an arc in $\mathrm{Int}(D \cap L_n)$ such that $q_i \in hf(\mathrm{Int}(L(t_i)))$ and $hf(L(t_i)) \subset O(q_i)$. If L_n is two dimensional and t_i intersects $hf(L_n)$ let $L(t_i)$ be a polyhedron in L_n such that $D \cap L(t_i)$ is an arc in $\mathrm{Int}(D \cap L_n)$, $q_i \in hf(\mathrm{Int}(D \cap L(t_i)))$, each $L(t_i) \cap L_{nj}$ is a disk in L_{nj} whose intersection with $\mathrm{Bd}(L_{nj})$ is the arc $D \cap L(t_i)$, and $hf(L(t_i)) \subset O(q_i)$. For each t_i let $L'(t_i)$ denote the polyhedron $\mathrm{Cl}(L - L(t_i))$. In each $O(q_i)$ let $O'(q_i)$ be an open 3-cell containing q_i so that $\mathrm{Cl}(O'(q_i)) \subset O(q_i)$ and $\mathrm{Cl}(O'(q_i))$ fails to intersect $hf(L'(t_i))$.

For each q_i let $D(t_i)$ be a disk normally situated in D such that

(1) $q_i \in hf(\mathrm{Int}(D(t_i) \cap \mathrm{Bd}(D)))$,

(2) $hf(D(t_i)) \cap t_i$ is a spanning arc w_i of $hf(\mathrm{Bd}(D(t_i)))$,

(3) $hf(\mathrm{Cl}(\mathrm{Bd}(D(t_i)) \cap \mathrm{Int}(D))) \subset I(h(X), hf(D))$, and

(4) $hf(D(t_i)) \subset O'(q_i)$.

Notice that $D(t_i)$ is not necessarily polyhedral. Condition 3 can be achieved because each $t_i \subset I(h(X), hf(D))$. Set $s(t_i) = D(t_i) \cap \mathrm{Bd}(D)$ and $r(t_i) = \mathrm{Bd}(D(t_i)) - \mathrm{Int}(s(t_i))$. Conditions 3 and 4 on $D(t_i)$ show $hf(r(t_i)) \cap T_2$ is a single point where $hf(r(t_i))$ pierces some 2-simplex of T and $s(t_i) \subset \mathrm{Int}(D \cap L(t_i))$. Use the definition of Property Q' to find a line segment I_i in $O'(q_i) \cap T_2$ such that $I_i \cap hf(D \cup L)$ is a single point in $\mathrm{Int}(w_i)$ where I_i pierces $hf(D)$. Use a construction like the one in the proof of Theorem 4.1 of [4] to find a special collection of disks $\{F(t_i)\}$ so that for each t_i which intersects $hf(L)$,

(1) $hf(L(1)) \cap F(t_i) \subset hf(s(t_i))$,

(2) $F(t_i) \cap hf(L'(t_i)) = \emptyset$,

(3) I_i is contained in $F(t_i)$ and spans $\mathrm{Bd}(F(t_i))$, and

(4) $hf(L)$ intersects only one component of $F(t_i) - I_i$.

If t_i intersects $hf(L)$ let $E(t_i)$ denote the closure of the component of $F(t_i) - I_i$ which intersects $hf(L)$.

Let δ_2 be a positive number such that

(1) a $5\delta_2$-neighborhood of each $hf(D(t_i))$ is contained in $O'(q_i)$,

(2) a $5\delta_2$-neighborhood of $hf(L)$ intersects $N(G_{IVO}, T'')$ in a subset of $N(G_{IVO} \cap hf(L), T')$,

(3) for each t_i a $5\delta_2$-neighborhood of $hf(L'(t_i))$ misses $F(t_i)$,

(4) if t_i intersects $hf(L)$ a $5\delta_2$-neighborhood of $hf(L(t_i))$ is contained in $O(q_i)$, a $5\delta_2$-neighborhood of $hf(L'(t_i))$ fails to intersect $O'(q_i)$, a $5\delta_2$-neighborhood of $hf(L(1) - s(t_i))$ misses $F(t_i)$, a $5\delta_2$-neighborhood of $hf(s(t_i))$ misses $\mathrm{Cl}(F(t_i) - E(t_i))$, and a $5\delta_2$-neighborhood of $hf(r(t_i))$ misses $T_2 - \mathrm{Int}(F(t_i) - E(t_i))$,

(5) a $5\delta_2$-neighborhood of $hf(D)$ misses $(\bigcup \mathrm{Bd}(F(t_i))) - T_1$, and

(6) for each two dimensional L_n, δ_2 is subject to the restrictions on δ in Lemma 2.4 for the substitution

$$(1 \to m,\ D \to D_1,\ L_{nj} \to D_{j+1},\ hf \to f)\,.$$

Lemmas 5.1 and 5.2 of [6] provide a positive number δ_3 such that if f'' is a pwl homeomorphism of D into E^3 for which $d(hf, f'') < \delta_3$ and $f''(D)$ is in general position with respect to T_2, then there is a pwl δ_2-homeomorphism h'' of E^3 so that $h''f''(D)$ is in general position with respect to T_2, the cardinality of $h''f''(D) \cap T_1$ is the same as the cardinality of $hf(D) \cap T_1$, and each $h''f''(D) \cap I_i$ is a single point where I_i pierces $h''f''(D)$. Further, as in ([6], Lemma 6.1), there is a positive number δ such that if h' is a homeomorphism of E^3 for which $d(h, h') < \delta_3/2$ and f' is a homeomorphism of $D \cup L$ into E^3 for which $d(f, f') < \delta$ then $d(hf, h'f') < \delta_3$.

Let f_0 and f_1 be pwl homeomorphisms of $D \cup L$ into E^3 which agree on L and for which $d(f, f_e) < \delta$ $(e = 0, 1)$. Use [2, 17] to find a pwl homeomorphism h_a of E^3 such that $d(h, h_a) < \delta_3/2$ and $h_a f_1(D)$ is in general position with respect to T_2. We have $d(hf, h_a f_e) < \delta_3$ $(e = 0, 1)$.

From Condition 6 on δ_2 we can find for each two dimensional L_n a polyhedral cube C_n containing $h_a f_0(L_n)$ such that $h_a f_0(L_n - D) \subset \operatorname{Int}(C_n)$ and $C_n \cap h_a\big(f_0(D) \cup f_1(D)\big) = h_a f_0(D \cap L_n)$. We can assume that the C_n's are mutually exclusive. For each two dimensional L_n and each $L(t_i)$ in L_n let $C(t_i)$ be a regular neighborhood of $h_a f_0\big(L(t_i)\big)$ modulo $h_a f_0\big(\operatorname{Cl}\big(L_n - L(t_i)\big)\big)$ in C_n such that $C(t_i)$ is contained in a δ_2-neighborhood of $h_a f_0\big(L(t_i)\big)$ and $C(t_i)$ intersects $\operatorname{Bd}(C_n)$ in a disk. Then in each pwl 3-manifold $M_n = \operatorname{Cl}\big(C_n - \bigcup C(t_i)\big)$ choose a regular neighborhood C'_n of $h_a f_0(L_n) \cap M_n$ which is contained in a δ_2-neighborhood of $h_a f_0(L_n) \cap M_n$ so that $C'_n \cup \Big(C_n \cap \big(\bigcup C(t_i)\big)\Big)$ is a pwl 3-manifold and C'_n collapses to $\big(M_n \cap h_a f_0(L_n)\big) \cup \Big(C'_n \cap \big(\bigcup C(t_i)\big)\Big)$. For each $C(t_i)$ in C_n, $C'_n \cap C(t_i)$ is connected. Each $B'_n = C'_n \cup \Big(C_n \cap \big(\bigcup C(t_i)\big)\Big)$ is a regular neighborhood of the collapsible polyhedron $h_a f_0(L_n)$ modulo $h_a\big(f_0(D) \cup f_1(D)\big)$ and so is a ball [11].

Because $d(hf, h_a f_1) < \delta_3$ there is a pwl δ_2-homeomorphism h_b of E^3 such that $h_b h_a f_1(D)$ is in general position with respect to T_2, the cardinality of $h_b h_a f_1(D) \cap T_1$ is the same as the cardinality of $hf(D) \cap T_1$, and each $h_b h_a f_1(D) \cap I_i$ is a single point where I_i pierces $h_b h_a f_1(D)$. We can assume that $h_b h_a f_0(D)$ and $h_b\big(\bigcup B'_n\big)$ are in general position with respect to T_2. We have $d(hf, h_b h_a f_e) < 2\delta_2$.

Consider an arc t_i. From Conditions 2 and 5 on δ_2, if t_i fails to intersect $hf(L)$, $N(t_i, T'')$ fails to intersect $h_b h_a f_0(L) \cup h_b\big(\bigcup B'_n\big)$ and $h_b h_a\big(f_0(D) \cup f_1(D)\big)$ fails to intersect $\operatorname{Bd}\big(F(t_i)\big) - T_1$. From the conditions on δ_2, if t_i intersects $hf(L)$, then $\big(h_b h_a f_0(L) \cup h_b\big(\bigcup B'_n\big)\big) \cap \big(N(t_i, T'')\big)$

$\subset N(q_i, T')$, $h_b\Big(\operatorname{Cl}\big(\big(\bigcup B'_n\big) - C(t_i)\big) \cup h_b h_a\big(L'(t_i)\big)\Big)$ misses $O'(q_i) \cup F(t_i)$, $h_b h_a f_0\Big(\big(L(1) - s(t_i)\big)\Big)$ misses $F(t_i)$, $h_b h_a\big(f_0(D) \cup f_1(D)\big)$ misses $\operatorname{Bd}\big(F(t_i)\big) - T_1$, and $h_b h_a f_1\big(D(t_i)\big) \subset O'(q_i)$. Furthermore for each $D(t_i)$ the endpoints of $h_b h_a f_1\big(r(t_i)\big)$ lie in the interiors of different 3-simplexes of T, $h_b h_a f_1\big(r(t_i)\big) \cap T_2 \subset \operatorname{Int}\big(F(t_i) - E(t_i)\big)$, and $h_b h_a f_0\big(s(t_i)\big)$ misses $\operatorname{Cl}\big(F(t_i) - E(t_i)\big)$.

Consider next the components of a $h_b h_a f_1\big(D(t_i)\big) \cap T_2$. They are subsets of arcs and simple closed curves so they are arcs and simple closed curves. Note that $h_b h_a f_1(D) \cap I_i$ is a single point, $\operatorname{Bd}\big(F(t_i)\big) - T_1$ misses $h_b h_a f_1(D)$, I_i separates $F(t_i)$, and T_2 separates the two points of $h_b h_a f_1\big(\operatorname{Bd}\big(r(t_i)\big)\big)$; thus there is exactly one component of $h_b h_a f_1\big(D(t_i)\big) \cap T_2$ which runs from $h_b h_a f_1\big(r(t_i)\big)$ to $h_b h_a f_1\big(s(t_i)\big)$, and since $h_b h_a f_1\big(D(t_i)\big)$ does not intersect $\operatorname{Bd}\big(F(t_i)\big)$ this component is contained in $\operatorname{Int}\big(F(t_i)\big)$. See Figure 4.1.

Use ([19], Cor. 1 to Theorem 1) to define a pwl homeomorphism h_c of E^3 which is the identity on $h_b h_a f_1\Big(\operatorname{Cl}\big(D - \bigcup D(t_i)\big)\Big) \cup \Big(\bigcup \operatorname{Bd}\big(F(t_i)\big)\Big)$ and on the complement of $\bigcup O'(q_i)$ so that $h_c h_b h_a f_e(D)$ $(e = 0, 1)$ and $h_c h_b\big(\bigcup B'_n\big)$ are in general position with respect to T_2 and so that h_c pushes each $h_b h_a f_1\big(D(t_i)\big)$ back onto the subdisk as indicated in Figure 4.1. For each t_i that intersects $hf(L)$, $h_c h_b h_a f_0\big(L(1)\big) \cap F(t_i)$ is a single point. Now h_c is a $\delta_1/4$-homeomorphism so $d(hf, h_c h_b h_a f_e) < 2\delta_2 + \delta_1/4 < 3\delta_1/4$ $(e = 0, 1)$.

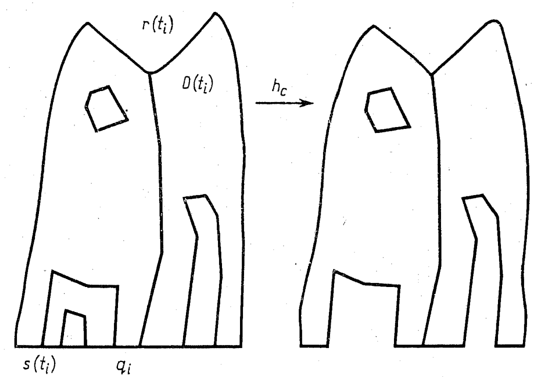

Fig. 4.1

For a t_i and a two dimensional L_n consider the intersection $F(t_i) \cap h_c h_b(B'_n)$. Conditions 2 and 4 on δ_2 show that this intersection is empty if t_i misses $hf(L_n)$ and is contained in $O(q_i)$ if t_i intersects $hf(L_n)$. Furthermore in the second case $h_c h_b\Big(\operatorname{Cl}\big(\big(\bigcup B'_n\big) - C(t_i)\big)\Big)$ misses $\big(F(t_i)\big)$. Suppose J is a component of $F(t_i) \cap h_c h_b(\mathrm{Bd}(B'_n))$ which is contained in $\mathrm{Int}(F(t_i))$. Suppose J contained a point of $h_c h_b h_a f_0(L(1))$. The only such point possible would be $h_c h_b h_a f_0(s(t_i)) \cap F(t_i)$ so $h_c h_b h_a f_0(s(t_i))$ would cross J on the 2-sphere $h_c h_b(\mathrm{Bd}(C(t_i)))$. But in that case J would separate on $h_c h_b(\mathrm{Bd}(C(t_i)))$ the connected set $h_c h_b(C'_n \cap C(t_i))$. Thus J bounds a disk on $h_c h_b(\mathrm{Bd}(C(t_i)))$ which misses both $h_c h_b(C'_n)$ and $h_c h_b h_a f_0(L(1))$.

Just as in Step 3 of the proof of Theorem 7.1 of [6] we find a pwl homeomorphism h_d of E^3 which is the identity on $h_c h_b(\bigcup C'_n) \cup (\bigcup \mathrm{Bd}(F(t_i))) \cup \cup h_c h_b h_a f_0(L(1))$ and on the complement of $\bigcup O(q_i)$ so that $h_d \ldots h_a(f_0(D) \cup f_1(D))$ and $h_d h_c h_b(\bigcup B'_n)$ are in general position with respect to T_2 and no component of any $h_d h_c h_b(B'_n) \cap F(t_i)$ lies in $\mathrm{Int}(F(t_i))$. The homeomorphism h_d is a $\delta_1/4$-homeomorphism of E^3.

Define h_{I} by the rule $h_{\mathrm{I}} = h_d h_c h_b h_a$, and for each B'_n set $B_n = h_a^{-1}(B'_n)$. Condition 6 is then satisfied in the conclusion of the lemma. The homeomorphism h_{I} is an $\eta + \delta_2 + \delta_1 < 3\eta$-homeomorphism of E^3 and $d(hf, h_{\mathrm{I}} f_e) < 3\delta_1/4 + \delta_1/4 = \delta_1$ $(e = 0, 1)$ so Condition 1 is satisfied. Because h_c and h_d are the identity on the complement of $\bigcup O(q_i)$, $h_{\mathrm{I}} f_1(D) \cap T_1 = h_b h_a f_1(D) \cap T_1$ so Condition 3 is satisfied. One can check the construction of the individual homeomorphisms composing h_{I} to verify that the other conditions are satisfied in the conclusion of the lemma.

5. Small isotopies of E^3 which transform one embedding of a disk into another. Here we give a proof of Theorem 5.1, a restricted isotopy theorem, which is closely related to the proof of Theorem 7.1 of [6]. We use this restricted theorem to prove the general theorem in much the same way as we use Theorem 7.1 of [6] to prove the general cartesian product theorem there.

Lemma 5.1, which follows, is used in the fifth step of the proof of Theorem 5.1. To prove it one can employ ([6], Lemma 3.1) together with two dimensional techniques analogous to the three dimensional ones used in the first four steps of the proof of Theorem 5.1.

LEMMA 5.1. *Suppose D is a polyhedral disk and $R \subset \mathrm{Bd}(D)$ is either the empty set or a 1-manifold with boundary. Suppose W is a polyhedral subdisk of D whose intersection with each component of R is an arc in the interior of that component and the closure of whose complement in D is made up of finitely many (possibly zero) disks normally situated in D.*

Suppose f is a homeomorphism of D into E^3 and $\varepsilon > 0$.

There is a $\delta > 0$ such that if f' is a pwl homeomorphism of D into E^3 for which $d(f, f') < \delta$, if h is a pwl δ-homeomorphism of $f'(W)$ into $f'(D)$ which is the identity on $f'(W \cap R)$ and sends $f'(W - R)$ into $f'(D - R)$, and if L is a polyhedron in E^3 whose intersection with $f'(D)$ is contained in $f'(R)$, then there is a pwl ε-isotopy H_t $(0 \leqslant t \leqslant 1)$ of E^3 which is the identity on L so that $H_1 hf' = f'|W$.

THEOREM 5.1. *Suppose D is a polyhedral disk and $\{L_n\}$ is a finite collection (possibly empty) of mutually exclusive polyhedra such that each $D \cap L_n$ is an arc in $\mathrm{Bd}(D)$ and either L_n is an arc whose endpoints miss D or L_n is the sum $\bigcup_j L_{nj}$ over a finite collection of polyhedral disks $\{L_{nj}\}$ where every $L_{ni} \cap L_{nj}$ $(i \neq j)$ is an arc A_n in $\mathrm{Bd}(L_{ni}) \cap \mathrm{Bd}(L_{nj})$ whose interior contains $D \cap L_n$. Set $L = \bigcup L_n$, and let $L(1)$ denote the sum of the L_n's which are arcs together with the sum of the A_n's.*

Suppose W is a polyhedral subdisk of D such that $\mathrm{Cl}(D - W)$ consists of finitely many (possibly zero) mutually exclusive disks normally situated in D and for each L_n, $W \cap L_n$ is an arc in $\mathrm{Int}(D \cap L_n)$.

Suppose f is a homeomorphism of $D \cup L$ into E^3 and $\varepsilon > 0$.

There is a $\delta > 0$ such that if f_0 and f_1 are pwl homeomorphisms of $D \cup L$ into E^3 which agree on L and for which $d(f, f_e) < \delta$ $(e = 0, 1)$, then there is a pwl ε-isotopy $H_t (0 \leqslant t \leqslant 1)$ of E^3 onto itself and there are mutually exclusive polyhedral cubes B_n containing the two dimensional $f_0(L_n)$'s so that

1. *for each two dimensional L_n, $f_0(L_n - D) \subset \mathrm{Int}(B_n)$ and $B_n \cap \cap (f_0(D) \cup f_1(D)) = f_0(D \cap L_n)$,*
2. *H_t is the identity on $f_0(L) \cup (\bigcup B_n)$ and on the complement of an ε-neighborhood of $f_0(D)$, and*
3. *$H_1 f_0 = f_1$ on $W \cup L$.*

Proof. Note that if f is pwl and $L = \emptyset$ the theorem is a special case of Theorem 2 of [20].

The proof is carried out in five steps. Just as in the proof of Theorem 7.1 of [6] the choice of a particular epsilon, eta, or delta is often provisional on conditions to be introduced later in the proof.

Step 1. A special graph on $f(D)$. In each $\mathrm{Int}(D \cap L_n)$ choose an arc R_n whose interior contains $W \cap L_n$. Set $R = \bigcup R_n$. Let ε_1 be a positive number less than one fifth the distance from $f(W \cap L)$ to $f(L(1) - R)$. From Lemma 4.1 and ([6], Sec. 6) there is a positive number $\eta < \varepsilon_1/2$, a tame Sierpiński curve X normally situated in $f(D)$, a rectilinear triangulation T of E^3 with mesh less than η and i-skeleton T_i, and an η-homeomorphism h of E^3 so that $(hf(D), hf(L), h(X), T_2, \eta)$ has Property Q'. Furthermore if G_{IV} denotes the graph which consists of $hf(\mathrm{Bd}(D))$ together with the components of $hf(D) \cap T_2$ that intersect T_1, and if G_{III} denotes the graph $h^{-1}(G_{\mathrm{IV}})$, then there is a finite collection $D_1^{\mathrm{I}}, \ldots, D_m^{\mathrm{I}}, \ldots$ of ε_1-disks which are normally situated in $f(D)$, whose interiors are mutually exclusive, and no two of which intersect in a disconnected set so that $G_{\rm I} = \bigcup \mathrm{Bd}(D_m^{\rm I})$ is a stable subgraph of $G_{\rm III}$. Set $G_{\rm II} = h(G_{\rm I})$, $G_{\rm IVO} = \mathrm{Cl}\big(G_{\rm IV} \cap hf(\mathrm{Int}(D))\big)$, and $G_{\rm IIO} = \mathrm{Cl}\big(G_{\rm II} \cap hf(\mathrm{Int}(D))\big)$.

For each $D_m^{\rm I}$ let $D_m^{\rm II}$ denote the $2\varepsilon_1$-disk $h(D_m^{\rm I})$. We assume ε_1 is sufficiently small so there are at least two points of $G_{\rm IVO}$ on $hf(\mathrm{Bd}(D))$. Let $\Delta_1, \ldots, \Delta_j, \ldots$ denote the 2-simplexes of T, $t_1, \ldots, t_i, \ldots$ the arcs which are the components of the $(G_{\rm IVO} \cap \Delta_j)$'s, and $p_1, \ldots, p_k, \ldots$ the points of $hf(D) \cap T_1$.

Step 2. Converting pwl approximations to f into special pwl approximations to hf. Let T', T'', and T''' be subdivisions of T subject to the conditions indicated in ([6], Sec. 6) and Section 4 here. Let ε_2 be a positive number less than the diameter of each simplex of T''' that intersects $hf(D)$. Choose a positive number ε_3 so small that $4\varepsilon_3$ is subject to the restrictions on δ in Lemma 5.3 of [6] for the system $\big(hf(D), h(X), T, \eta\big)$, the graphs $G_{\rm IVO}$ and $G_{\rm IV}$, and the number ε_2. Let δ_1 be a positive number subject to the conditions on δ in Lemma 5.2 of [6] for $hf(D)$ and ε_3.

Find a special collection of disks $\{F(t_i)\}$ and a positive number δ from Lemma 4.2 for the system $\big(hf(D),\ hf(L),\ h(X),\ T,\ \eta\big)$, the graphs $G_{\rm IV}$ and $G_{\rm IVO}$, the p_k's, the subdivisions T', T'', and T''', and the positive number δ_1.

Let f_0 and f_1 be pwl homeomorphisms of $D \cup L$ into E^3 which agree on L so that $d(f, f_e) < \delta\ (e = 0, 1)$.

From Lemma 4.2 there is a pwl 3η-homeomorphism $h_{\rm I}$ of E^3 and there are mutually exclusive polyhedral cubes B_n containing the two dimensional $f_0(L_n)$'s so that the eight statements in the conclusion of the lemma are valid. There is no loss in supposing that the points p_k are exactly the points of $h_{\rm I} f_1(D) \cap T_1$.

Since $\delta_1 < \varepsilon_3$ Lemma 5.3 of [6] provides a homeomorphism π_1' of $G_{\rm IV}$ onto the graph $G_{\rm IV}^1 = G_{\rm IVO}^1 \cup h_{\rm I} f_1\big(\mathrm{Bd}(D)\big)$ where $G_{\rm IVO}^1$ consists of the components of $h_{\rm I} f_1(D) \cap T_2$ which intersect T_1. For each p_k, $\pi_1'(p_k) = p_k$, for each Δ_j and each t_i in Δ_j, $\pi_1'(t_i)$ is an arc component of $G_{\rm IVO}^1 \cap \Delta_j$ that lies in an ε_2-neighborhood of t_i, and for each arc s which is the closure of a component of $hf\big(\mathrm{Bd}(D)\big) - G_{\rm IVO}$, $\pi_1'(s)$ is the closure of a component of $h_{\rm I} f_1\big(\mathrm{Bd}(D)\big) - G_{\rm IVO}^1$ and it lies in an ε_2-neighborhood of s.

Lemma 5.2. of [6] provides a pwl isotopy $H_t^1\ (0 \leqslant t \leqslant 1)$ of E^3 so that $H_1^1 h_{\rm I} f_0(D)$ is in general position with respect to T_2, the cardinality of $H_1^1 h_{\rm I} f_0(D) \cap T_1$ is the same as the cardinality of $hf(D) \cap T_1$, and H_t^1 is the identity on the complement of an ε_3-neighborhood of $\bigcup p_k$. Again we can suppose that the points p_k are exactly the points of $H_1^1 h_{\rm I} f_0(D) \cap T_1$. Note that by Condition 7 in Lemma 4.2, by the choice of ε_3, and by Condition 3 in the definition of special collection of disks, H_t^1 is a $2\varepsilon_3$-isotopy which is the identity on $h_{\rm I}\big(f_0(L) \cup (\bigcup B_n)\big) \cup \Big(\big(\bigcup \mathrm{Bd}\big(F(t_i)\big)\big) - T_1\Big)$. Furthermore $d(hf, H_1^1 h_{\rm I} f_0) < \delta_1 + 2\varepsilon_3 < 3\varepsilon_3$.

A second application of Lemma 5.3 of [6] provides a homeomorphism π_0' of $G_{\rm IV}$ onto the graph $G_{\rm IVO}^0 \cup H_1^1 h_{\rm I} f_0\big(\mathrm{Bd}(D)\big)$ where $G_{\rm IVO}^0$ consists of the components of $H_1^1 h_{\rm I} f_0(D) \cap T_2$ which intersect T_1. For each p_k, $\pi_0'(p_k) = p_k$, for each Δ_j and each t_i in Δ_j, $\pi_0'(t_i)$ is an arc component of $G_{\rm IVO}^0 \cap \Delta_j$ that lies in an ε_2-neighborhood of t_i, and for each arc s which is the closure of a component of $hf\big(\mathrm{Bd}(D)\big) - G_{\rm IVO}$, $\pi_0'(s)$ is a component of $H_1^1 h_{\rm I} f_0\big(\mathrm{Bd}(D)\big) - G_{\rm IVO}^0$ and it lies in an ε_2-neighborhood of s.

Define r_m's, $L(r_m)$'s, $N_O(r_m, T''')$'s, and $N_U(r_m, T''')$'s as in ([6], Sec. 6). Let ε_4 be a positive number. Lemma 6.6 of [6] shows that we can require δ_1, ε_1, and ε_2 to be sufficiently small so $\pi_0'|G_{\rm II}$ and $\pi_1'|G_{\rm II}$ can be extended to ε_4-homeomorphisms π_0 and π_1 of $hf(D)$ onto $H_1^1 h_{\rm I} f_0(D)$ and $h_{\rm I} f_1(D)$ such that each $\pi_e(D_m^{\rm II})\ (e = 0, 1)$ has diameter less than ε_4; furthermore for each $D_m^{\rm II}$, $N_O(r_m, T''') \cup \pi_e(r_m)$ contains a neighborhood of $\pi_e(r_m)$ in $\pi_e(D_m^{\rm II})\ (e = 0, 1)$. Let $G_{\rm II}^e$ and $G_{\rm IIO}^e\ (e = 0, 1)$ denote the respective graphs $\pi_e(G_{\rm II})$ and $\pi_e(G_{\rm IIO})$. Let $G_{\rm IIO}^{W0}$ denote the intersection of $G_{\rm IIO}^0$ with the sum of all $\pi_0(D_m^{\rm II})$'s that intersect $H_1^1 h_{\rm I} f_0(W)$.

We say that $D_m^{\rm II}$ is of Type 1, 2, 3, or 4 if $\pi_0(D_m^{\rm II}) \cap H_1^1 h_{\rm I} f_0(W) \neq \emptyset$ and

(1) $D_m^{\rm II} \cap hf\big(\mathrm{Bd}(D)\big) = \emptyset$,

(2) $D_m^{\rm II} \cap hf\big(\mathrm{Bd}(D)\big) \neq \emptyset$, but $D_m^{\rm II} \cap hf(R) = \emptyset$,

(3) $D_m^{\rm II} \cap hf(R) \neq \emptyset$, but $D_m^{\rm II}$ does not intersect a two dimensional $hf(L_n)$, or

(4) $D_m^{\rm II}$ intersects a two dimensional $hf(L_n)$.

The conditions on ε_1 show that $\pi_e(D_m^{\rm II}) \cap h_{\rm I} f_0(L) \neq \emptyset\ (e = 0, 1)$ if and only if $D_m^{\rm II}$ is of Type 3 or 4, and in this case $\pi_e(D_m^{\rm II}) \cap h_{\rm I} f_0(L) \subset h_{\rm I} f_0\big(\mathrm{Int}(R)\big)$.

Step 3. An isotopy which takes $G_{\rm IIO}^{W0}$ into $G_{\rm IIO}^1$; special regular neighborhoods of $\pi_1(D_m^{\rm II})$'s. Consider an arc t_i in $G_{\rm IIO}^{W0}$. From Condition 4 in Lemma 4.2, Conditions 3 and 4 in the definition of special collection of disks, and the construction of H_t^1, $\big(H_1^1 h_{\rm I} f_0(D) \cup h_{\rm I} f_1(D)\big) \cap \mathrm{Bd}\big(F(t_i)\big) = t_i \cap \mathrm{Bd}\big(F(t_i)\big)$ and $F(t_i) \cap T_1$ contains a neighborhood of $t_i \cap T_1$ in T_1 so $\pi_e(t_i) \subset F(t_i)\ (e = 0, 1)$.

From Condition 7 in Lemma 4.2, $h_{\rm I}\big(f_0(L) \cup (\bigcup B_n)\big) \cap F(t_i) \neq \emptyset$ only if $t_i \cap hf(R) \neq \emptyset$. In this case $F(t_i) \cap h_{\rm I} f_0\big(L(1)\big)$ is the single point $\pi_0\big(t_i \cap hf(R)\big)$, and $h_{\rm I}(\bigcup B_n) \cap F(t_i)$ has no component which lies in $\mathrm{Int}\big(F(t_i)\big)$. See Figure 5.1.

From the remarks in the preceding paragraph there is a pwl η-isotopy $H_t^2\ (0 \leqslant t \leqslant 1)$ of E^3 such that for each t_i in $G_{\rm IIO}^{W0}$, $H_1^2 \pi_0(t_i) = \pi_1(t_i)$, H_t^2 leaves each simplex of T invariant, and H_t^2 is the identity on simplexes of T which do not intersect $G_{\rm IIO}^{W0}$, on $T_2 - \bigcup \mathrm{Int}\big(F(t_i)\big)$, and on $h_{\rm I}\big(f_0(L) \cup (\bigcup B_n)\big)$.

Because H_t^2 leaves each simplex of T invariant $H_1^2\pi_0(hf(D))$ is in general position with respect to T_2.

Let T^{iv} be a subdivision of T''' in which each $h_{\mathrm{I}}(B_n)$, each $H_1^2\pi_0(D_m^{\mathrm{II}})$, and each $\pi_1(D_m^{\mathrm{II}})$ underlies a full subcomplex and which has fine enough mesh so that each $N(\pi_1(D_m^{\mathrm{II}}), T^{\mathrm{iv}})$ has diameter less than ε_4. Let T^{v} be a first derived subdivision of T^{iv}. Consider the regular neighborhoods $N(\pi_1(D_m^{\mathrm{II}}), T^{\mathrm{v}})$. They are pwl 3-cells. Notice that

$$N(\pi_1(D_p^{\mathrm{II}}), T^{\mathrm{v}}) \cap N(\pi_1(D_q^{\mathrm{II}}), T^{\mathrm{v}}) = N(\pi_1(r_p \cap r_q), T^{\mathrm{v}}) \quad (p \neq q) .$$

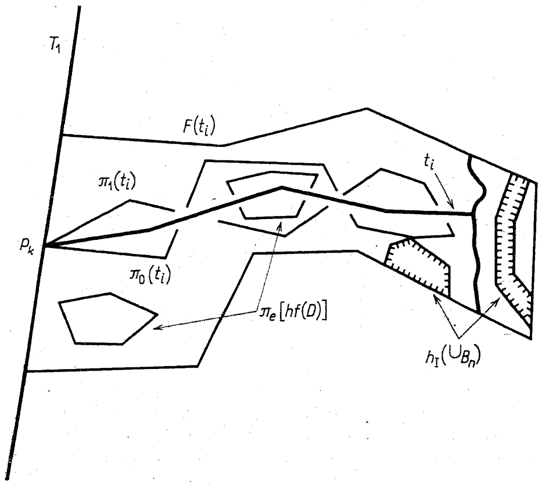

Fig. 5.1

From our application of Lemma 6.6 of [6] it follows that for each D_m^{II}, $\pi_e(D_m^{\mathrm{II}}) \cap N(\pi_e(r_m), T^{\mathrm{v}}) \cap N_U(r_m, T''') = \emptyset$ $(e = 0, 1)$. Among the (D_m^{II})'s of Type 1–4 only for a D_m^{II} of Type 4 does $\pi_1(D_m^{\mathrm{II}})$ intersect $h_{\mathrm{I}}(\bigcup B_n)$, and in this case $\pi_1(D_m^{\mathrm{II}}) \cap h_{\mathrm{I}}(\bigcup B_n)$ is an arc, $N(\pi_1(D_m^{\mathrm{II}}), T^{\mathrm{v}}) \cap h_{\mathrm{I}}(\bigcup \mathrm{Bd}(B_n))$ is a 2-cell, and $N(\pi_1(D_m^{\mathrm{II}}), T^{\mathrm{v}}) \cap h_{\mathrm{I}}(\bigcup \mathrm{Bd}(B_n)) \cap L(r_m)$ is a pair of arcs. Recall from Condition 2 in Lemma 4.2 that each $h_{\mathrm{I}}(B_n)$ is in general position with respect to T_2.

For each D_m^{II} of Type 1–4 let C_m denote the sum of all simplexes of T^{v} in $N(\pi_1(D_m^{\mathrm{II}}), T^{\mathrm{v}})$ which fail to intersect $(N(r_m, T^{\mathrm{v}}) \cap N_U(r_m, T''')) \cup h_{\mathrm{I}}(\bigcup \mathrm{Int}(B_n))$. For a D_m^{II} of Type 1 or 4, C_m is obtained from the regular neighborhood $N(\pi_1(D_m^{\mathrm{II}}), T^{\mathrm{v}})$ of $\pi_1(D_m^{\mathrm{II}})$ by removing a solid torus which intersects $\mathrm{Bd}(N(\pi_1(D_m^{\mathrm{II}}), T^{\mathrm{v}}))$ in an annulus homotopic to the center line of the torus; thus C_m is a pwl 3-cell of diameter less than ε_4 whose boundary is spanned by $\pi_1(D_m^{\mathrm{II}})$. Similarly, for a D_m^{II} of Type 2 or 3, C_m is a pwl 3-cell of diameter less than ε_4 which contains $\pi_1(D_m^{\mathrm{II}})$ so that $\pi_1(D_m^{\mathrm{II}}) \cap \mathrm{Bd}(C_m) = \pi_1(r_m)$. The $C_{m'}$'s have mutually exclusive interiors which fail to intersect $h_{\mathrm{I}}(\bigcup B_n)$. If a D_m^{II} is of Type 1 or 4 then a neighborhood of $H_1^2\pi_0(\mathrm{Bd}(D_m^{\mathrm{II}})) = \pi_1(\mathrm{Bd}(D_m^{\mathrm{II}}))$ in $H_1^2\pi_0(D_m^{\mathrm{II}})$ is contained in C_m. If a D_m^{II} is of Type 2 or 3 then a neighborhood of $H_1^2\pi_0(r_m) = \pi_1(r_m)$ in $H_1^2\pi_0(D_m^{\mathrm{II}})$ is contained in C_m.

Step 4. Pushing $H_1^2H_1^1h_{\mathrm{I}}f_0(W)$ into $h_{\mathrm{I}}f_1(D)$. For a D_m^{II} of Type 1 or 4 choose two polyhedral disks in $\mathrm{Bd}(C_m)$ which do not intersect $\pi_1(\mathrm{Bd}(D_m^{\mathrm{II}}))$ and whose interiors contain $(H_1^2\pi_0(D_m^{\mathrm{II}}) - \pi_1(\mathrm{Bd}(D_m^{\mathrm{II}})) \cap \mathrm{Bd}(C_m))$. Push these disks slightly into $\mathrm{Int}(C_m)$ and fatten them up into disjoint polyhedral cubes in $\mathrm{Int}(C_m)$ with boundaries S_{m1} and S_{m2} which fail to intersect $h_{\mathrm{I}}f_0(L)$ and which are in general position with respect to $H_1^2\pi_0(D_m^{\mathrm{II}})$ so that if U_m denotes the component of $H_1^2\pi_0(D_m^{\mathrm{II}}) - (S_{m1} \cup S_{m2})$ containing $\pi_1(\mathrm{Bd}(D_m^{\mathrm{II}})) = H_1^2\pi_0(\mathrm{Bd}(D_m^{\mathrm{II}}))$ then $U_m - \pi_1(\mathrm{Bd}(D_m^{\mathrm{II}})) \subset \mathrm{Int}(C_m)$.

Similarly for a D_m^{II} of Type 2 or 3 find a single polyhedral 2-sphere S_{m1} in $\mathrm{Int}(C_m)$ which fails to intersect $h_{\mathrm{I}}f_0(L)$ and is in general position with respect to $H_1^2(\pi_0(D_m^{\mathrm{II}}))$ so that if U_m denotes the component of $H_1^2\pi_0(D_m^{\mathrm{II}}) - S_{m1}$ containing $\pi_1(r_m)$ then $U_m - \pi_1(r_m) \subset \mathrm{Int}(C_m)$.

Use Lemma 2.9 of [6] to find a pwl $13\varepsilon_4$-isotopy H_t^3 $(0 \leqslant t \leqslant 1)$ of E^3 which is the identity on $h_{\mathrm{I}}(f_0(L) \cup (\bigcup B_n)) \cup (\bigcup U_m)$ and on the complement of a $3\varepsilon_4$-neighborhood of $\bigcup C_m$ so that for each D_m^{II} of Type 1–4, $H_1^3H_1^2\pi_0(D_m^{\mathrm{II}}) - \mathrm{Cl}(U_m) \subset \bigcup_j \mathrm{Int}(S_{mj})$. For each D_m^{II} of Type 1 or 4, $H_1^3H_1^2\pi_0(D_m^{\mathrm{II}} - \mathrm{Bd}(D_m^{\mathrm{II}})) \subset \mathrm{Int}(C_m)$, and for each D_m^{II} of Type 2 or 3, $H_1^3H_1^2\pi_0(D_m^{\mathrm{II}} - r_m) \subset \mathrm{Int}(C_m)$.

Use ([6], Lemma 2.6) for D_m^{II}'s of Type 1 and 4 and Lemma 3.3 for D_m^{II}'s of Type 2 and 3 to find a pwl ε_4-isotopy H_t^4 $(0 \leqslant t \leqslant 1)$ of E^3 which is the identity on $h_{\mathrm{I}}f_0(L)$ and on the complement of $\bigcup \mathrm{Int}(C_m)$ so that for each D_m^{II} of Type 1–4, $H_1^4H_1^3H_1^2\pi_0(D_m^{\mathrm{II}}) = \pi_1(D_m^{\mathrm{II}})$. Now $H_1^4 \ldots H_1^1h_{\mathrm{I}}f_0(W) \subset f_1(D)$.

Step 5. The isotopy H_t. Consider the pwl homeomorphism $h_{\mathrm{I}}^{-1}H_1^4 \ldots H_1^1h_{\mathrm{I}}f_0f_1^{-1}$ of $f_1(W)$. It is a $2\delta + 3\eta + 2\varepsilon_3 + \eta + 13\varepsilon_4 + \varepsilon_4 + 3\eta$ or $25\varepsilon_4$-homeomorphism of $f_1(W)$. Lemma 5.1 shows that ε_4 can be required to be sufficiently small so $\varepsilon_4 < \varepsilon/100$ and so that there is a pwl $\varepsilon/2$-isotopy H_t^5 $(0 \leqslant t \leqslant 1)$ of E^3 which is the identity on $f_0(L) \cup (\bigcup B_n)$ and on the complement of an ε-neighborhood of $f(D)$ such that $H_1^5h_{\mathrm{I}}^{-1}H_1^4 \ldots H_1^1h_{\mathrm{I}}f_0f_1^{-1}f_1 = f_1$ on $W \cup L$.

Define the promised isotopy H_t by the rule $H_0 = I$, $H_t = h_{\mathrm{I}}^{-1} \circ H^j_{5(t-(j-1)/5)}h_{\mathrm{I}}H_{(j-1)/5}$ $((j-1)/5 \leqslant t \leqslant j/5,\ j \leqslant 4)$ and $H_t = H^5_{5(t-4/5)}H_{4/5}$ $(4/5 \leqslant t \leqslant 1)$. Now $H_1f_0 = f_1$ on $W \cup L$, and H_t is the identity on

$f_0(L) \cup (\bigcup B_n)$. Furthermore H_t is a $(2\varepsilon_3+2(3\eta))+(\eta+2(3\eta))+ +(13\varepsilon_4+2(3\eta))+(\varepsilon_4+2(3\eta))+\varepsilon/2$ or $41\varepsilon/100+50\varepsilon/100<\varepsilon$ — isotopy of E^3. Finally H_t $(0 \leqslant t \leqslant 4/5)$ is the identity on the complement of the pre-image under h_1^{-1} of a $3\varepsilon_4$-neighborhood of $hf(D)$. But such a set is contained in a $3\varepsilon_4+2(3\eta)<\varepsilon$-neighborhood of $hf(D)$; thus H_t is the identity on the complement of an ε-neighborhood of $f(D)$.

This completes the proof of the theorem.

The polyhedron $D \cup L$ is collapsible so for any embedding of it into the interior of a pwl 3-manifold there is a polyhedral cube-with-handles which contains a neighborhood of the embedding in the 3-manifold [16]. Since a polyhedral cube-with-handles can be pwl embedded in E^3 under a uniformly continuous homeomorphism we have the following corollary:

Corollary 5.1. *Theorem 5.1 remains true if E^3 is replaced by an arbitrary pwl 3-manifold M provided that f takes $D \cup L$ into* $\mathrm{Int}(M)$.

6. Piecing together isotopies.

Lemma 6.1. *Suppose $\Delta_1, \ldots, \Delta_m, \ldots, \Delta_n$ $(0 \leqslant m \leqslant n)$ are 2-simplexes whose pairwise intersections are all the same, a 1-simplex σ which is a face of each Δ_i. Set $K = \bigcup \Delta_i$.*

Suppose A is an arc in $\mathrm{Int}(\sigma)$. For each Δ_i let E_i be a polyhedral disk in $\mathrm{Int}(\Delta_i)$ and F_i a polyhedral disk in Δ_i such that $E_i \cap F_i$ is an arc B_i and $F_i \cap \mathrm{Bd}(\Delta_i) = A$.

Let L be a subpolyhedron of K which is σ, or $\bigcup_{i>m} \Delta_i$, or the empty polyhedron.

Suppose M is a pwl 3-manifold, f is a homeomorphism of K into $\mathrm{Int}(M)$, and $\varepsilon > 0$.

There is a $\delta > 0$ such that if f_0 and f_1 are pwl homeomorphisms of K into M which agree on $L \cup (\bigcup E_i)$ and for which $d(f, f_e) < \delta$ $(e = 0, 1)$, then there is a pwl ε-isotopy H_t $(0 \leqslant t \leqslant 1)$ of M onto itself so that $H_1 f_0$ agrees with f_1 on $L \cup (\bigcup E_i) \cup (\bigcup_{i \leqslant m} F_i)$ and H_t is the identity on the complement of an ε-neighborhood of $f(\bigcup_{i \leqslant m} F_i)$ and on $f_0(L \cup (\bigcup E_i)) \cup O$ where O is an open polyhedron containing $f_0(L - \mathrm{Cl}(K-L))$.

Proof. The isotopy H_t is constructed by piecing together isotopies H_t^i $(0 \leqslant t \leqslant 1, 1 \leqslant i \leqslant m)$ obtained from applications of Corollary 5.1. Each H_t^i brings $H_1^{i-1} \ldots H_1^1 f_0$ into agreement with f_1 on a neighborhood of F_i in Δ_i and is the identity on a neighborhood of $f_1(\bigcup_{j<i} F_j)$ in $f_1(\bigcup_{j<i} \Delta_i)$ and on $f_0(L \cup (\bigcup E_i)) \cup C_i$ where C_i is a polyhedral cube whose interior contains $f_0(L-\sigma)$ and C_i fails to intersect $(H_1^{i-1} \ldots H_1^1 f_0(F_i - \sigma)) \cup f_1(F_i - \sigma)$. The open polyhedron O is given by $\bigcap \mathrm{Int}(C_i)$.

Corollary 6.1. *If L is two dimensional and if for each arc t in K such that $t \cap L$ is a single point in $\mathrm{Int}(t)$ and each disk D in L whose interior contains $t \cap L$, $f(t)$ does not pierce $f(D)$, then for sufficiently small values of δ, H_t can be chosen so that $\mathrm{Cl}(O)$ is a polyhedral cube.*

Proof. By applying the full strength of Lemma 2.4 where it is used in the proof of Theorem 5.1 we find that δ can be required to be sufficiently small so that each C_i fails to intersect $(H_1^{i-1} \ldots H_1^1 f_0(F_j - \sigma)) \cup f_1(F_j - \sigma)$ $(i \leqslant j \leqslant m)$. Thus each C_i $(1 \leqslant i \leqslant m)$ contains a regular neighborhood of $f_0(L) = f_1(L)$ modulo $f_1(F_m)$.

Let T be a triangulation of E^3 in which $f_0(A)$ and each C_i underlies a subcomplex, and let T'' be the second barycentric subdivision of T. The polyhedron $C = N(f_0(L-A), T'')$ is a regular neighborhood of the collapsible polyhedron $f_0(L)$ modulo $f_1(F_m)$ [11] and so is a polyhedral cube. We have $C \subset \bigcap C_i$. In place of the O mentioned before take $\mathrm{Int}(C)$.

7. The general isotopy theorem for embeddings of polyhedra in interiors of 3-manifolds.

Theorem 7.1. *Suppose K is a polyhedron with no local cut points, L is a subpolyhedron of K, M is a pwl 3-manifold, f is a homeomorphism of K onto a closed subset of $\mathrm{Int}(M)$, and μ is a continuous, non-negative, real function on M which is positive on $f(\mathrm{Cl}(K-L))$.*

There is a continuous, positive, real function ν on K such that if f_0 and f_1 are pwl homeomorphisms of K into M which agree on L so that for each $x \in K$, $\varrho(f(x), f_e(x)) < \nu(x)$ $(e = 0, 1)$, then there is a pwl isotopy H_t $(0 \leqslant t \leqslant 1)$ of M so that $H_1 f_0 = f_1$, H_t is the identity on $f_0(L) \cup O$ where O is an open polyhedron containing $f_0(L - \mathrm{Cl}(K-L))$, and the track under H_t of each point x of M has diameter no greater than $\mu(x)$.

Proof. Let μ_1 be a continuous, non-negative, real function on M which is positive on $f(\mathrm{Cl}(K-L))$ so that $(M, \mu/4, \mu_1, 4)$ has Property R (see Introduction for a definition of Property R). Let T be a triangulation of K in which L underlies a full subcomplex and which has sufficiently fine mesh so that for each simplex s of T which intersects $\mathrm{Cl}(K-L)$ the diameter of $f(s)$ is less than the minimum value of μ_1 over $f(s)$. Let K_i denote the i-skeleton of T_K (considered as a polyhedron). Let $\tau_1, \ldots, \tau_i, \ldots$ denote the 3-simplexes, $\Delta_1, \ldots, \Delta_i, \ldots$ the 2-simplexes, $\sigma_1, \ldots, \sigma_j, \ldots$ the 1-simplexes, and $v_1, \ldots, v_k, \ldots$ the vertices of T. For each simplex s of T let $b(s)$ denote the barycenter of s, and let T' denote the first barycentric subdivision of T. Each $\mathrm{lk}(v_k, T')$ is connected.

Choose connected open sets $O(s)$ in $\mathrm{Int}(M)$ for the simplexes of T—first for the 3-simplexes, next for the vertices, then for the 1-simplexes, and finally for the 2-simplexes—so that

(1) for each τ_i and each $s \subset \tau_i$, $O(s) \subset O(\tau_i)$, for each v_k, $f(v_k) \in O(v_k)$, and for each 1- or 2-simplex s, $f(\mathrm{Int}(s)) \subset O(s)$,

(2) for each s that intersects $\mathrm{Cl}(K-L)$ the diameter of $O(s)$ is less than the greatest lower bound of μ_1 over $O(s)$,

(3) for each pair of simplexes s, s' of T which have dimensions less than three, $O(s)\cap O(s')\neq\emptyset$ only if s is a face of s' or vice versa, and $\mathrm{Cl}\big(O(s)\big)\cap\mathrm{Cl}\big(O(s')\big)\neq\emptyset$ only if $s\cap s'\neq\emptyset$,

(4) no $\mathrm{Cl}\big(O(s)\big)$ intersects $f\Big(\mathrm{Cl}\big(K-N(s,T')\big)\Big)$, and

(5) the union of sets $\mathrm{Cl}\big(O(s)\big)$ over any collection of simplexes s of T is a closed subset of M.

We assume that no $\mathrm{Cl}\big(O(s)\big)=M$.

For each v_k consider the cone $v_k * \mathrm{lk}(v_k, T')$. Let $\varepsilon(v_k)$ be a positive number less than half the distance from $f(v_k)$ to $M-O(v_k)$ so that for each $v_j\neq v_k$ and each simplex s of T' which contains v_k but not v_j, $2\varepsilon(v_k) < \varrho\Big(f(s), \mathrm{Cl}\big(O(v_j)\big)\Big)$, and for each τ_i which contains v_k, $2\varepsilon(v_k)<\varrho\big(f(\tau_i), M-O(\tau_i)\big)$. If $v_k\notin\mathrm{Cl}(K-L)$ set $\delta(v_k)=\varepsilon(v_k)$; otherwise substitute $\big(\mathrm{lk}(v_k,T')\cap K_2\to K, v_k\to v, M\to M, f\to f, \varepsilon(v_k)\to\varepsilon\big)$ in Lemma 3.4, to get an associated positive number $\delta(v_k)$.

If f' is a homeomorphism of K into M such that for each v_k and each simplex s of T which contains v_k, $x\in s\Rightarrow\varrho\big(f(x),f'(x)\big)<\delta(v_k)$, then for each v_k, $f'\Big(\mathrm{Cl}\big(K-N(v_k, T')\big)\Big)$ fails to intersect $\mathrm{Cl}\big(O(v_k)\big)$.

Let T'' be a derived subdivision of T' such that for each v_k, $f\big(N(v_k,T'')\big)\subset O(v_k)$ and has diameter less than $\delta(v_k)$, and for each σ_j, $f\Big(N\big(b(\sigma_j), T''\big)\Big)\subset O(\sigma_j)$. Now K is the sum $\bigcup N\big(b(s), T''\big)$ over the simplexes s of T (see [23], Ch. 3, (ii) on p. 14). For each Δ_i set $E_i=N\big(b(\Delta_i), T''\big)$, and for each σ_j that is a face of Δ_i set $F_{ij}=\Delta_i\cap N\big(b(\sigma_j), T''\big)$.

For each 1- and 2-simplex s of T let $O'(s)$ be a connected open subset of $O(s)$ such that $f\big(N(b(s), T''\big)\subset O'(s)$ and $\mathrm{Cl}\big(O'(s)\big)\big)\subset O(s)$.

For each σ_j let $\varepsilon(\sigma_j)$ be a positive number such that

(1) $2\varepsilon(\sigma_j)<\delta(v_k)$ for every $v_k\in N(\sigma_j, T)$,

(2) $2\varepsilon(\sigma_j)$ is less than the distance from $f\Big(N\big(b(\sigma_j), T''\big)\Big)$ to $M-O'(\sigma_j)$,

(3) for every Δ_i which has σ_j as a face and every σ_k which is not a face of Δ_i, $2\varepsilon(\sigma_j)$ is less than the distance from $f(\Delta_i)$ to $\mathrm{Cl}\big(O'(\sigma_k)\big)$, and

(4) for every Δ_i which has σ_j as a face $2\varepsilon(\sigma_j)$ is less than the distance from $f\big(\mathrm{Bd}(\Delta_i)-\mathrm{Int}(\sigma_j)\big)$ to $\mathrm{Cl}\big(O'(\sigma_j)\big)$.

If $\sigma_j\not\subset\mathrm{Cl}(K-L)$ set $\delta(\sigma_j)=\varepsilon(\sigma_j)$. If $\sigma_j\subset\mathrm{Cl}(K-L)$ let $\Delta_{i_1(j)}, \ldots, \Delta_{i_m(j)}, \ldots, \Delta_{i_n(j)}$ $\big(m=m(j)\leqslant n=n(j)\big)$ denote the 2-simplexes of T which have σ_j as a face. Choose the indexes so $\Delta_{i_p(j)}\subset L$ if and only if $p>m$. Substitute $\big(\Delta_{i_p(j)}\to\Delta_p, \bigcup_p\Delta_{i_p(j)}\to K, \; m\to m, \; \mathrm{Cl}\big(L\cap\big(\mathrm{Int}(\sigma_j)\cup\bigcup_{p>m}\mathrm{Int}(\Delta_{i_p(j)})\big)\big)\big)\to L, \; F_{i_p(j)j}\to F_p, \; E_{i_p(j)}\to E_p, \; M\to M, \; \varepsilon(\sigma_j)\to\varepsilon\big)$ in Lemma 6.1 to find an associated positive number $\delta(\sigma_j)$.

If f' is a homeomorphism of K into M such that for each σ_j and each Δ_i which has σ_j as a face, $x\in\Delta_i\Rightarrow\varrho\big(f(x),f'(x)\big)<\delta(\sigma_j)$, then for any simplex s of T, $f'(s)\cap\mathrm{Cl}\big(O'(\sigma_j)\big)\neq\emptyset$ only if σ_j is a face of s.

For each Δ_i let $\varepsilon(\Delta_i)$ be a positive number such that

(1) $2\varepsilon(\Delta_i)$ is less than $\delta(\sigma_j)$ for each σ_j which intersects Δ_i,

(2) $2\varepsilon(\Delta_i)$ is less than the distance from $f(E_i)$ to $M-O'(\Delta_i)$ and from $f\big(\mathrm{Bd}(\Delta_i)\big)$ to $\mathrm{Cl}\big(O'(\Delta_i)\big)$, and

(3) $2\varepsilon(\Delta_i)$ is less than the distance from $f(\Delta_i)$ to $\mathrm{Cl}\big(O'(\Delta_j)\big)$ for each $\Delta_j\neq\Delta_i$.

If $\Delta_i\subset L$ set $\delta(\Delta_i)=\varepsilon(\Delta_i)$; otherwise substitute $\big(E_i\to D, \emptyset\to L, E_i\to W, f\to f, M\to M, \varepsilon(\Delta_i)\to\varepsilon\big)$ in Corollary 5.1 to find an associated positive number $\delta(\Delta_i)$.

If f' is a homeomorphism of K into M such that for each Δ_i, $x\in\Delta_i\Rightarrow\varrho\big(f(x),f'(x)\big)<\delta(\Delta_i)$, then for each Δ_i, $f'(K_2)\cap\mathrm{Cl}\big(O'(\Delta_i)\big)\subset f'\big(\mathrm{Int}(\Delta_i)\big)$.

Let ν be a positive, continuous, real function on K such that for each simplex s of T the maximum value of ν on s is less than $\delta(s')$ for each simplex s' of T which intersects s.

Let f_0 and f_1 be pwl homeomorphisms of K into M which agree on L so that for each $x\in K$, $\varrho\big(f(x),f_e(x)\big)<\nu(x)$ $(e=0,1)$.

For each $\Delta_i\not\subset L$, $x\in\Delta_i\Rightarrow\varrho\big(f(x),f_e(x)\big)<\delta(\Delta_i)$ $(e=0,1)$ so from Corollary 5.1 and from our remarks about the $\varepsilon(\Delta_i)$'s and $\delta(\Delta_i)$'s there is a pwl $\varepsilon(\Delta_i)$-isotopy H_t^{1i} $(0\leqslant t\leqslant 1)$ of M such that $H_1^{1i}f_0=f_1$ on E_i and H_t^{1i} is the identity on an open polyhedron O_{1i} in M which contains $f_0(L)\cup\big(M-O'(\Delta_i)\big)$. Because the $\mathrm{Cl}\big(O'(\Delta_i)\big)$'s are mutually exclusive, because their sum is closed in M, and from the conditions on the diameters of the $O(s)$'s, we can define a locally pwl isotopy H_t^1 $(0\leqslant t\leqslant 1)$ of M by setting $H_t^1=H_t^{1i}$ on each $O'(\Delta_i)$ for which $\Delta_i\not\subset L$ and setting $H_t^1=I$ elsewhere. By ([6], Prop. 2.1), H_t^1 is a pwl isotopy. Further H_t^1 is the identity on the open polyhedron $O_1=\bigcap O_{1i}$ which contains $f_0(L)$, and the track under H_t^1 of each point x of M has diameter no greater than $\mu_1(x)$.

From Condition 1 on the $\varepsilon(\Delta_i)$'s we find that for each σ_j and each Δ_i which has σ_j as a face, $x\in\Delta_i\Rightarrow\varrho\big(f(x), H_1^1f_0(x)\big)<\delta(\sigma_j)$ and $\varrho\big(f(x),f_1(x)\big)<\delta(\sigma_j)$; thus from Lemma 6.1 and our remarks about the $\varepsilon(\sigma_j)$'s and $\delta(\sigma_j)$'s there is for each $\sigma_j\subset\mathrm{Cl}(K-L)$ a pwl isotopy H_t^{2j} $(0\leqslant t\leqslant 1)$ of M such that $H_1^{2j}H_1^1f_0=f_1$ on $\big(\bigcup_p F_{i_p(j)j}\big)$ and H_t^{2j} is the identity on $f_0(L\cap K_2)\cup H_1^1f_0\big(\bigcup_p E_{i_p(j)}\big)\cup O_{2j}$ where O_{2j} is an open polyhedron containing

$(M-O'(\sigma_j)) \cup f_0\big((L \cap K_2)-\mathrm{Cl}(K_2-(L \cap K_2))\big)$. Because for any simplex s of T, $H_1^1 f_0(s) \cap \mathrm{Cl}(O'(\sigma_j)) \neq \emptyset$ only if σ_j is a face of s, and because the $\mathrm{Cl}(O'(\sigma_j))$'s are mutually exclusive and have a closed sum in M, we can define a pwl isotopy H_t^{2a} $(0 \leqslant t \leqslant 1)$ as before so that $H_1^{2a} H_1^1 f_0 = f_1$ on $\mathrm{Cl}(K_2 - \bigcup N(v_k, T''))$, H_t^{2a} is the identity on $f_0(L \cap K_2) \cup O_{2a}$ where $O_{2a} = \bigcap O_{2j}$ is an open polyhedron containing $f_0\big((L \cap K_2)-\mathrm{Cl}(K_2-(L \cap K_2))\big)$, and the track under H_t^{2a} of each point x of M has diameter no greater than $\mu_1(x)$. Now H_t^{2a} is the identity on each $f_0(\mathrm{Bd}(\tau_i))$ where $\tau_i \subset L$ so we can change H_t^{2a} just on the images of these simplexes to get a pwl isotopy H_t^2 $(0 \leqslant t \leqslant 1)$ of M such that $H_1^2 H_1^1 f_0 = f_1$ on $K_2 - \bigcup N(v_k, T'')$, H_t^2 is the identity on $f_0(L) \cup O_2$ where O_2 is the open polyhedron $O_{2a} \cup \cup \big(\bigcup \{f_0(\mathrm{Int}(\tau_i)) | \tau_i \subset L\}\big)$ which contains $f_0(L-\mathrm{Cl}(K-L))$, and the track under H_t^2 of each point x of M has diameter no greater than $\mu_1(x)$.

From Condition 1 on the $\varepsilon(\sigma_j)$'s we find that for each v_k and each Δ_i which contains v_k, $x \in \Delta_i \Rightarrow \varrho(f(x), H_1^2 H_1^1 f_0(x)) < \delta(v_k)$ and $\varrho(f(x), f_1(x)) < \delta(v_k)$; thus from Lemma 3.4 and our remarks about the $\varepsilon(v_k)$'s and $\delta(v_k)$'s there is a pwl $\varepsilon(v_k)$-isotopy H_t^{3k} $(0 \leqslant t \leqslant 1)$ of M such that $H_1^{3k} H_1^2 H_1^1 f_0 = f_1$ on $K_2 \cap N(v_k, T')$ and H_t^{3k} is the identity on $f_0(L \cap K_2) \cup \cup O_{3k}$ where O_{3k} is an open polyhedron containing $(M - O(v_k)) \cup \cup f_0\big((L \cap K_2)-\mathrm{Cl}(K_2-(L \cap K_2))\big)$. From the conditions on the $\varepsilon(v_k)$'s we find that no $H_1^2 H_1^1 f_0\big(\mathrm{Cl}(K - N(v_k, T'))\big)$ intersects $\mathrm{Cl}(O(v_k))$. We define as before a pwl isotopy H_t^{3a} $(0 \leqslant t \leqslant 1)$ of M so that $H_1^{3a} H_1^2 H_1^1 f_0 = f_1$ on K_2, H_t^{3a} is the identity on $f_0(L \cap K_2) \cup O_{3a}$ where $O_{3a} = \bigcap O_{3k}$ is an open polyhedron containing $f_0\big((L \cap K_2)-\mathrm{Cl}(K_2-(L \cap K_2))\big)$, and the track of each point x of M under H_t^{3a} has diameter no greater than $\mu_1(x)$. Then, as before, we convert H_t^{3a} to a pwl isotopy H_t^3 which is the identity on $f_0(L) \cup O_3$ where $O_3 = O_{3a} \cup \big(\bigcup \{f_0(\mathrm{Int}(\tau_i)) | \tau_i \subset L\}\big)$ is an open polyhedron containing $f_0(L-\mathrm{Cl}(K-L))$ so that $H_1^3 H_1^2 H_1^1 f_0 = f_1$ on K_2.

For each $\tau_i \not\subset L$, $H_1^3 H_1^2 H_1^1 f_0(\tau_i) \subset O(\tau_i)$ and $H_1^3 H_1^2 H_1^1 f_0 = f_1$ on $\mathrm{Bd}(\tau_i)$; thus $H_1^3 H_1^2 H_1^1 f_0(\tau_i) = f_1(\tau_i)$. We apply the Alexander deformation theorem (stated as Lemma 2.3 in [6]) to find a pwl isotopy H_t^4 $(0 \leqslant t \leqslant 1)$ of M so that $H_1^4 H_1^3 H_1^2 H_1^1 f_0 = f_1$ and H_t^4 is the identity on the complement of $\bigcup \{f_1(\mathrm{Int}(\tau_i)) | \tau_i \not\subset L\}$. This shows that H_1^4 is the identity on $f_0(L) \cup O_4$ where O_4 is an open polyhedron containing $f_0(L-\mathrm{Cl}(K-L))$ and the track of each point x of M under H_t^4 has diameter no greater than $\mu_1(x)$.

Define H_t by the rule $H_0 = I$ and $H_t = H^i_{4(t-(i-1)/4)} H_{(i-1)/4}$ $\big((i-1)/4 \leqslant \leqslant t \leqslant i/4,\ 1 \leqslant i \leqslant 4\big)$. We have $H_1 f_0 = f_1$, H_t is the identity on $f_0(L) \cup O$ where $O = \bigcap O_i$ is an open polyhedron containing $f_0(L-\mathrm{Cl}(K-L))$, and for each point x of M the track of x under H_t has diameter no greater than $\mu_1(x) + \mu_1(H_1^1(x)) + \mu_1(H_1^2 H_1^1(x)) + \mu_1(H_1^3 H_1^2 H_1^1(x)) \leqslant 4(1/4\mu(x))$ by the definition of Property R.

COROLLARY 7.1. *Suppose the L in Theorem 7.1 has no point components and no local cut points, and suppose $L-\mathrm{Cl}(K-L) \neq \emptyset$.*

Suppose that for each $x \in L$ and each neighborhood $N(x)$ of $f(x)$ in M there is a connected subset $N_1(x)$ of $N(x)-f(L)$ which contains each component of $f(K-L) \cap N(x)$ whose closure contains $f(x)$.

Then ν can be chosen and H_t constructed so that $\mathrm{Cl}(O)$ is a polyhedral 3-manifold with boundary.

Proof. The condition on f in the hypothesis implies that for each arc t in K such that $t \cap L$ is an interior point of t and each disk D in L whose interior contains $t \cap L$, $f(t)$ does not pierce $f(D)$.

By using Corollary 6.1 in place of Lemma 6.1 and Corollary 3.4 in place of Lemma 3.4 in the proof of Theorem 7.1 we can choose ν and construct H_t^i $(i = 2, 3)$ so that $\mathrm{Cl}(O_{ia})$ is a polyhedral 3-manifold with boundary. Further, following the proofs of Corollaries 6.1 and 3.4, we find that there is a regular neighborhood M_1 in M of $f_0(L) = f_1(L)$ modulo $f_1(\mathrm{Cl}(K-L))$. We can assume $M_1 \subset O_1$. Notice that H_t^1 and H_t^4 are the identity on M_1. Since for each $\tau_i \subset L$, $f_0(\mathrm{Bd}(\tau_i)) \subset \mathrm{Cl}(O_{ja})$ $(j = 1, 2)$, we see that $\mathrm{Cl}(O_j)$ $(j = 1, 2)$ is a 3-manifold with boundary which contains a regular neighborhood of $f_0(L)$ modulo $f_1(\mathrm{Cl}(K-L))$.

Just as in the proof of Corollary 6.1 we construct a regular neighborhood M_* of $f_0(L)$ modulo $f_1(\mathrm{Cl}(K-L))$ which is contained in $M_1 \cap \mathrm{Cl}(O_2) \cap \cap \mathrm{Cl}(O_3)$. Then for the promised O we use $\mathrm{Int}(M_*)$.

The next theorem is a two dimensional analogue of Theorem 7.1. We omit a proof.

THEOREM 7.2. *Suppose M is a pwl 3-manifold, K is a polyhedron, K_a and L are subpolyhedra of K, f is a homeomorphism of K onto a closed subset of M such that $f(K) \cap \mathrm{Bd}(M) = f(K_a)$, and μ is a continuous, non-negative, real function on M which is positive on $f(\mathrm{Cl}(K_a - L))$.*

There is a positive, continuous, real function ν on K such that if f_0 and f_1 are pwl homeomorphisms of K into M which agree on L so that $f_e(K) \cap \mathrm{Bd}(M) = f_e(K_a)$ $(e = 0, 1)$ and for each point x of K, $\varrho(f(x), f_e(x)) < \nu(x)$ $(e = 0, 1)$, then there is a pwl isotopy H_t $(0 \leqslant t \leqslant 1)$ of M onto itself so that $H_1 f_0 = f_1$ on K_a, H_t is the identity on $f_0(L) \cup O$ where O is an open polyhedron containing $f_0(L-\mathrm{Cl}(K-L))$, and the track of each point x of M under H_t has diameter no greater than $\mu(x)$.

Furthermore if for each arc t in K_a such that $t \cap L$ is an interior point of t and each arc r of $L \cap K_a$ whose interior contains $t \cap L$, $f(t)$ does not pierce $f(r)$ in $\mathrm{Bd}(M)$, if $K_a \cap (L-\mathrm{Cl}(K-L)) \neq \emptyset$ and if $\mathrm{Cl}(K_a - L) \cap L$ has no point components, then H_t can be constructed so that $\mathrm{Cl}(O)$ and $\mathrm{Cl}(O) \cap \mathrm{Bd}(M)$ are polyhedral manifolds with boundary.

8. The general isotopy theorem for manifolds with boundary.

THEOREM 8.1. *Suppose M is a pwl 3-manifold, K is a polyhedron with no local cut points, K_a and L are subpolyhedra of K where K_a has no point components, and f is a homeomorphism of K onto a closed subset of M such that $f(K)\cap \mathrm{Bd}(M) = f(K_a)$. Let μ be a continuous, non-negative, real function on M which is positive on $f\big(\mathrm{Cl}(K-L)\big)$.*

There is a continuous, positive, real function ν on K such that if f_0 and f_1 are pwl homeomorphisms of K into M which agree on L so that $f_e(K)\cap \mathrm{Bd}(M) = f_e(K_a)$ $(e = 0, 1)$ and for each point x of K, $\varrho\big(f(x), f_e(x)\big) < \nu(x)$ $(e = 0, 1)$, then there is a pwl isotopy H_t $(0 \leqslant t \leqslant 1)$ of M onto itself so that $H_1 f_0 = f_1$, the track under H_t of each point x of M has diameter no greater than $\mu(x)$ and H_t is identity on $f_0(L)\cup O$ where O is an open polyhedron in M containing $f_0\big(L - \mathrm{Cl}(K-L)\big)$.

Furthermore if f_0 agrees with f_1 on K_a then H_t can be chosen so that it is the identity on $\mathrm{Bd}(M)$ and so that $\mathrm{Bd}(M) - f_0\Big(\mathrm{Cl}\big(K - (L\cup K_a)\big)\Big) \subset O$.

Proof. Disregard for a moment the pwl structure of $\mathrm{Bd}(M)$ and use [1, 2] to find a triangulation of the topological space $\mathrm{Bd}(M)$ which makes the restriction of f to K_a a pwl homeomorphism. Use this triangulation to define a polyhedron Q which contains K_a as a subpolyhedron and which intersects K in exactly K_a and to define a homeomorphism f' of $K\cup Q$ onto $f(K)\cup \mathrm{Bd}(M)$ so that $f' = f$ on K.

Define a pwl 3-manifold M' whose interior contains M by adding the pwl product $\mathrm{Bd}(M)\times[0,1)$ to M with the identification $y\in \mathrm{Bd}(M) \Rightarrow y = (y, 0)$. Give M' a metric which extends the metric on M, and extend μ to a continuous, non-negative, real function μ' on M'.

Let μ_2' be a continuous, non-negative, real function on M' which is positive on $f\big(\mathrm{Cl}(K-L)\big)$ so that $(M', \tfrac{1}{2}\mu', \mu_2', 2)$ has Property R. Substitute $(M'\to M,\ K\cup Q\to K,\ L\cup Q\to L,\ f'\to f,\ \mu_2'\to\mu)$ in Theorem 7.1 to find an associated positive, continuous function ν_2' on $K\cup Q$.

Extend the function $\nu_2'(f')^{-1}$ to a continuous, non-negative, real function μ_{11} on M. Let μ_1 be a continuous, non-negative, real function on M which is positive on $f\big(\mathrm{Cl}(K_a - L)\big)$ so that $(M, \tfrac{1}{2}\mu_{11}, \mu_1, 2)$ has Property R and so that for each point x of M, $\mu_1(x) \leqslant \mu_2'(x)$. Substitute $(M\to M,\ K\to K,\ K_a\to K_a,\ L\to L,\ f\to f,\ \mu_1\to\mu)$ in Theorem 7.2 to find an associated positive, continuous function ν_1 on K.

Let μ_0 be a continuous, non-negative, real function on M which is positive on $\mathrm{Bd}(M)$ so that $(M, \tfrac{1}{2}\mu_{11}, \mu_0, 2)$ has Property R. Substitute $(M\to M,\ K\to K,\ K_a\to K_a,\ \emptyset\to L,\ f\to f,\ \mu_0\to\mu)$ in Theorem 7.2 to find an associated positive, continuous function ν_0 on K.

Let ν' be a positive, continuous function on $K\cup Q$ such that for each point x of K, $\nu'(x) < \nu_0(x), \nu_1(x)$ and for each point x of Q, $\nu'(x) < \mu_0\big(f'(x)\big)$. Let ν denote the restriction of ν' to K.

Now let f_0 and f_1 be pwl homeomorphisms of K into M which agree on L such that $f_e(K)\cap \mathrm{Bd}(M) = f_e(K_a)$ $(e = 0, 1)$ and for each point x of K, $\varrho\big(f(x), f_e(x)\big) < \nu(x)$ $(e = 0, 1)$. Use [1, 2] to find a pwl homeomorphism f_2' of $K\cup Q$ into M such that $f_2'(Q) = \mathrm{Bd}(M)$ and for each point x of $K\cup Q$, $\varrho\big(f'(x), f_2'(x)\big) < \nu'(x)$.

From the conditions on ν' there is a pwl homeomorphism h of M onto itself so that $hf_2' = f_1$ on K_a and for each point x of M, $\varrho\big(x, h(x)\big) \leqslant \mu_0(x)$. Extend f_1 to a pwl homeomorphism f_1' of $K\cup Q$ into M by setting f_1' equal to hf_2' on Q. If $x\in Q$, $\varrho\big(f'(x), f_1'(x)\big) < \nu'(x) + \mu_0\big(f_2'(x)\big) \leqslant \mu_0\big(f'(x)\big) + \mu_0\big(f_2'(x)\big) \leqslant \mu_{11}\big(f'(x)\big) = \nu_2'(x)$ by the definition of Property R and the fact that $\varrho\big(f'(x), f_2'(x)\big) < \mu_0\big(f'(x)\big)$. If $x\in K$, $\varrho\big(f'(x), f_1'(x)\big) < \nu(x) \leqslant \nu_2'(x)$.

From the conditions on ν_1 there is a pwl isotopy H_t^1 $(0\leqslant t\leqslant 1)$ of M onto itself such that $H_1^1 f_0$ agrees with f_1 on K_a, H_t^1 is the identity on $f_0(L)\cup O_1$ where O_1 is an open polyhedron containing $f_0\big(L - \mathrm{Cl}(K-L)\big)$, and the track of each point x of M under H_t^1 has diameter no greater than $\mu_1(x)$. If f_0 agrees with f_1 on K_a set $H_t^1 = I$. Extend $H_1^1 f_0$ to a pwl homeomorphism f_0' of $K\cup Q$ into M by setting f_0' equal to f_1' on Q. For each point x of $K\cup Q$ we have $\varrho\big(f'(x), f_0'(x)\big) < \nu'(x) + \mu_1\big(f_0'(x)\big) \leqslant \mu_{11}\big(f'(x)\big) = \nu_2'(x)$ just as for f_1'.

Since $\varrho\big(f'(x), f_e'(x)\big) < \nu_2'(x)$ $(e = 0, 1;\ x\in K\cup Q)$ there is by Theorem 7.1 a pwl isotopy H_t^2 $(0\leqslant t\leqslant 1)$ of M' onto itself such that $H_1^2 f_0' = f_1'$, H_t^2 is the identity on $f_0(L)\cup \mathrm{Bd}(M)\cup O_2$ where O_2 is an open polyhedron containing $f_0'\Big((L\cup Q) - \mathrm{Cl}\big((K\cup Q) - (L\cup Q)\big)\Big) = f_0\big(L - \mathrm{Cl}(K-L)\big) \cup \Big(\mathrm{Bd}(M) - f_0\Big(\mathrm{Cl}\big(K - (L\cup K_a)\big)\Big)\Big)$, and the track of each point x of M under H_t^2 has diameter no greater than $\mu_2'(x)$. Because H_t^2 is the identity on $\mathrm{Bd}(M)$, H_t^2 takes M onto itself.

Define the promised isotopy H_t by the rule $H_t = H_{2t}^1$ $(0\leqslant t\leqslant 1/2)$ and $H_t = H_{2(t-1/2)}^2 H_{1/2}$ $(1/2\leqslant t\leqslant 1)$. We have $H_1 f_0 = f_1$ and H_t is the identity on $f_0(L)\cup O$ where $O = O_1\cap O_2$ is an open polyhedron containing $f_0\big(L - \mathrm{Cl}(K-L)\big)$. If f_0 agrees with f_1 on K_a then H_t is the identity on $f_0(L)\cup \mathrm{Bd}(M)\cup O_2$ where O_2 contains $\mathrm{Bd}(M) - f_0\Big(\mathrm{Cl}\big(K - (L\cup K_a)\big)\Big)$. The track of any point x of M is no greater than $\mu_1(x) + \mu_2'\big(H_1^1(x)\big) \leqslant \mu_2'(x) + \mu_2'\big(H_1^1(x)\big) \leqslant \mu(x)$ since $\varrho\big(x, H_1^1(x)\big) \leqslant \mu_2'(x)$ and $(M', \tfrac{1}{2}\mu', \mu_2', 2)$ has Property R.

Here is a topological analogue of Theorem 8.1. Although a stronger version is valid which provides fixed point sets similar to those in Theorem 8.1 its proof requires modifications of [1, 18] which we do not wish to make here.

THEOREM 8.2. *Suppose M is a 3-manifold with boundary, K is a polyhedron with no local cut points, K_a is a subpolyhedron of K with no* *point components, and f is a homeomorphism of K onto a closed subset of M such that $f(K)\cap \mathrm{Bd}(M)=f(K_a)$. Let μ be a continuous, non-negative, real function on M which is positive on $f(K)$.*

There is a positive, continuous, real function ν on K such that if f_0 and f_1 are homeomorphisms of K onto tame sets in M where $f_e(K)\cap \mathrm{Bd}(M)=f_e(K_a)$ $(e=0,1)$ and for each point x of K, $\varrho\big(f(x),f_e(x)\big)<\nu(x)$ $(e=0,1)$, then there is an isotopy H_t $(0\leqslant t\leqslant 1)$ of M onto itself so that $H_1f_0=f_1$ and the track of each point x of M under H_t has diameter no greater than $\mu(x)$.

Proof. Since M can be triangulated [1, 2, 18] we might as well assume that it is a pwl manifold.

Let μ_1 be a continuous, non-negative, real function on M which is positive on $f(K)$ so that $(M,\ \mu/3,\ \mu_1,\ 3)$ has Property R.

Substitute $(M\to M,\ K\to K,\ K_a\to K_a,\ \emptyset\to L,\ f\to f,\ \mu_1\to\mu)$ in Theorem 8.1 to get an associated positive, continuous function ν_1 on K. For the function ν take $\nu_1/2$.

Let f_0 and f_1 be homeomorphisms of K onto tame subsets of M such that $f_e(K)\cap \mathrm{Bd}(M)=f_e(K_a)$ $(e=0,1)$ and for each point x of K, $\varrho\big(f(x),f_e(x)\big)<\nu(x)$ $(e=0,1)$.

By combining [1, 18] with [15] we find isotopies H^e_t $(0\leqslant t\leqslant 1;\ e=0,1)$ of M onto itself such that $H^e_1f_e$ is a locally pwl homeomorphism of K into M, for each point x of K, $\varrho\big(H^e_1f_e(x),f(x)\big)<\nu_1(x)$, and the track of each $x\in M$ under H^e_t has diameter no greater than $\mu_1(x)$. Since $f_e(K)$ $(e=0,1)$ is tame, $H^e_1f_e(K)$ is a closed subset of M and $H^e_1f_e$ is a pwl homeomorphism by ([6], Prop. 2.1).

Theorem 8.1 provides an isotopy H^2_t $(0\leqslant t\leqslant 1)$ of M onto itself such that $H^2_1H^0_1f_0=H^1_1f_1$ and the track of each $x\in M$ under H^2_t has diameter no greater than $\mu_1(x)$.

Define the promised isotopy H_t by the rule $H_t=H^0_{3t}$ $(0\leqslant t\leqslant 1/3)$, $H_t=H^2_{3(t-1/3)}H_{1/3}(1/3\leqslant t\leqslant 2/3)$, and $H_t=H^1_{1-3(t-2/3)}(H^1_1)^{-1}H_{2/3}(2/3\leqslant t\leqslant 1)$.

We have $H_1f_0=(H^1_1)^{-1}H^2_1H^0_1f_0=f_1$. The track of each $x\in M$ under H_t has diameter no greater than $\mu_1(x)+\mu_1(H^0_1(x))+\mu_1\big((H^1_1)^{-1}H_{2/3}(x)\big)\leqslant\mu(x)$ since $(M,\ \mu/3,\ \mu_1,\ 3)$ has Property R.

9. Applications I-Improving isotopies of polyhedra.

THEOREM 9.1. *Suppose M is a (pwl) 3-manifold with boundary, K is a finite polyhedron with no local cut points, and K_a is a subpolyhedron of K which has no point components.*

Suppose h_t $(0\leqslant t\leqslant 1)$ is an isotopy of K into M such that $h_t(K)\cap \mathrm{Bd}(M)=h_t(K_a)$ $(0\leqslant t\leqslant 1)$ and both h_0 and h_1 take K onto tame sets (both h_0 and h_1 are pwl).

Suppose $\varepsilon>0$.

Then there is a (pwl) isotopy H_t $(0\leqslant t\leqslant 1)$ of M onto itself such that $H_1h_0=h_1$ and for each t $(0\leqslant t\leqslant 1)$, $d(h_t,H_th_0)<\varepsilon$.

Proof. We prove only the topological version of the theorem.

For each t let $\delta(t)$ be a positive number such that the constant function $\delta(t)$ on K is subject to the restrictions on ν in Theorem 8.2 for the substitution $(M\to M,\ K\to K,\ K_a\to K_a,\ h_t\to f,\ \varepsilon/3\to\mu)$. Let $N(t)$ be a neighborhood of t in $[0,1]$ such that if $s\in N(t)$ then $d(h_s,h_t)<\delta(t)/3$.

From the compactness of $[0,1]$ there are numbers $t_1,\ldots,t_m$ such that $\bigcup N(t_i)=[0,1]$. Let δ be a positive number less than each $\delta(t_i)/3$. Let n be a positive integer such that if $|t-s|<1/n$ then $d(h_t,h_s)<\delta$.

Use [2] to find for each i $(1\leqslant i<n)$ a homeomorphism f_i of K onto a tame set in M such that $f_i(K)\cap \mathrm{Bd}(M)=f_i(K_a)$ and $d(f_i,h_{i/n})<\delta$. Set $f_0=h_0$ and $f_n=h_1$. Note that for each i $(0\leqslant i<n)$ there is a t_j such that $i/n\in N(t_j)$; thus $d(h_{i/n},h_{t_j})<\delta(t_j)/3$ and $d(h_{(i+1)/n},h_{t_j})<d(h_{i/n},h_{(i+1)/n})+d(h_{i/n},h_{t_j})<2\delta(t_j)/3$ so $d(f_i,h_{t_j})<\delta(t_j)$ and $d(f_{i+1},h_{t_j})<\delta(t_j)$.

Use Theorem 8.2 to find for each i $(0\leqslant i<n)$ an $\varepsilon/3$-isotopy H^i_t $(0\leqslant t\leqslant 1)$ of M onto itself so that $H^i_1f_i=f_{i+1}$.

Define H_t by the rule $H_0=I$ and $H_t=H^i_{n(t-i/n)}H_{i/n}(i/n\leqslant t\leqslant(i+1)/n$, $0\leqslant i<n)$. We have $H_1h_0=h_1$. For each t there is an i so that $t\in[i/n,(i+1)/n]$; thus $d(H_th_0,h_t)\leqslant d(H_th_0,H_{i/n}h_0)+d(H_{i/n}h_0,h_{i/n})+d(h_{i/n},h_t)<\varepsilon/3+d(f_i,h_{i/n})+\delta<\varepsilon/3+\delta+\delta<\varepsilon$.

10. Applications II-On pseudo isotopies.

LEMMA 10.1. *Suppose M is a 3-manifold with boundary, K is a polyhedron with no local cut points, K_a is a subpolyhedron of K with no point components and f is a homeomorphism of K into M such that $f(K)\cap \mathrm{Bd}(M)=f(K_a)$.*

Suppose $f(K)$ is locally tame modulo $f(K_a)$.

Then $f(K)$ is locally tame.

Proof. Let x be a point of K_a. We show that $f(K)$ is locally tame at x.

Choose a tame 3-cell C in M such that $C\cap \mathrm{Bd}(M)=D$ a disk whose interior contains $f(x)$, and construct a homeomorphism h of C onto a tame 3-cell B in E^3. Set $E=h(D)$. Let P be a finite subpolyhedron of K and P_a a subpolyhedron of P such that

(1) P contains a neighborhood of x in K,

(2) P has no local cut points,

(3) P_a has no point components,

(4) $f(P)\subset C$, and

(5) $f(P)\cap \mathrm{Bd}(C)=f(P_a)\subset \mathrm{Int}(D)$.

Now $hf(P-P_a)$ is locally tame, and $f(P)$ contains a neighborhood of $f(x)$ in $f(K)$.

As in the proof of Theorem 8.1 define a polyhedral disk Q which contains P_a as a subpolyhedron and which intersects P in exactly P_a, and extend hf to a homeomorphism f' of $R = P \cup Q$ onto $hf(P) \cup E$. Let T be a triangulation of R in which Q underlies a full subcomplex, and let T_Q denote the subcomplex of T which Q underlies. Let R_i denote the i-skeleton of T and T' the first barycentric subdivision of T. It is apparent that $f'(R)$ is locally tame modulo $f'(P_a \cap R_1)$.

Let Δ be a 2-simplex of T. We show that $f'(\Delta)$ is tame. If Δ fails to intersect Q then $f'(\Delta)$ is tame by the hypothesis of the lemma. If $\Delta \subset Q$ then $f'(\Delta)$ is tame by the two dimensional Schoenflies theorem. The fullness condition on T shows that if $\mathrm{Int}(\Delta)$ misses Q but $\Delta \cap Q$ is non-empty then $\Delta \cap Q$ is either a 1-simplex σ or a vertex v. Suppose $\Delta \cap Q = \sigma$. Then $f'(\Delta)$ is locally tame modulo the tame arc $f'(\sigma)$ so $f'(\Delta)$ is tame [10, 1, 18]. Suppose $\Delta \cap Q = v$. Find an arc A which underlies a subcomplex of $\mathrm{lk}(v, T)$ so that $\mathrm{lk}(v, T) \cap \Delta \subset A$ and $A \cap Q$ is a single point which is an endpoint of A. But then $f'(\Delta)$ is contained in the disk $f'(v * A)$ which is tame as in the previous case.

Each 1-simplex of T is the face of some 2-simplex so we have established that $f'(R_1)$ is locally tame modulo $f'(R_0 \cap P_a)$. Now we show that $f'(R_1)$ is locally tame at each of these isolated points. Let v be a vertex of T in P_a. Observe that $\mathrm{lk}(v, T_Q)$ is a simple closed curve. Let G be a subcomplex of the 1-skeleton of $\mathrm{lk}(v, T)$ so that $\mathrm{lk}(v, T_Q)$ is a subcomplex of G, each vertex of $\mathrm{lk}(v, T)$ is in G, and there are no simple closed curves in G other than $\mathrm{lk}(v, T_Q)$. The graph G is obtained by adding to $\mathrm{lk}(v, T_Q)$ the sum over a finite collection of mutually exclusive trees in $\mathrm{lk}(v, T)$ each of which intersects $\mathrm{lk}(v, T_Q)$ in a single vertex. Let G' denote the subcomplex $\mathrm{lk}(v, T') \cap (v * G)$ of $\mathrm{lk}(v, T')$.

Consider now a polyhedral disk $F = u * J$ where u is a point and J is a simple closed curve. Define a pwl map φ of J onto G' so that there is a finite collection of arcs $\{A_i\}$ which fill up J and have mutually exclusive interiors where for each A_i, φ takes A_i homeomorphically onto some 1-simplex of G', for no 1-simplex σ of $\mathrm{lk}(v, T')$ in Q are two A_i's mapped onto σ, for no 1-simplex σ of G' does $\varphi^{-1}(\sigma)$ contain more than two A_i's, and for each pair i, j $(i \neq j)$ such that $A_i \cup A_j$ is an arc, $f'(\varphi(A_i \cup A_j))$ pierces no disk in $f'(v * G)$. Extend φ to take F onto $v * G'$ by taking the join of the maps of J onto G' and u onto v.

By making a series of cuts along tame arcs like the cuts illustrated in Figure 10.1 we convert the map $f'\varphi\colon F \to B$ into an embedding g of F into B so that $g(u) = f'\varphi(u)$, $g(F)$ is locally tame modulo $g(u)$, $g^{-1}(\mathrm{Bd}(B)) = (f'\varphi)^{-1}(\mathrm{Bd}(B))$, and $g(F)$ contains $f'(N(v, T') \cap R_1)$. It is possible to make these cuts because of the piercing condition mentioned in the preceding paragraph.

Let σ be a 1-simplex of T' in $R_1 \cap Q$ which contains v. The disk $g(F)$ is locally tame modulo the tame arc $f'(\sigma)$ so $g(F)$ is tame by the argument used earlier in this proof. Thus $f'(R_1)$ is locally tame at $f'(v)$.

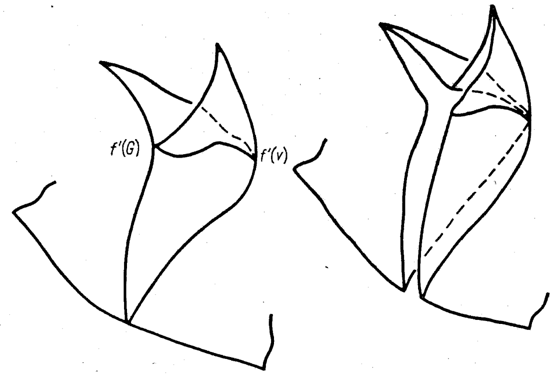

Fig. 10.1

From [1, 18], $f'(R_1)$ is tame, and hence from [8], $f'(R)$ is tame. This shows that $f'(P)$ is tame in B and thus that $f(K)$ is locally tame at the point $f(x)$. Since x is an arbitrary point of K_a, $f(K)$ is locally tame.

Now we give the promised extension of the Keldyš theorem. See [13, 14].

THEOREM 10.1. *Suppose K is a polyhedron with no local cut points, K_a is a subpolyhedron of K with no point components, M is a 3-manifold with boundary, and f is a homeomorphism of K onto a closed subset of M such that $f(K) \cap \mathrm{Bd}(M) = f(K_a)$.*

Let μ be a continuous, non-negative, real function on M which is positive on the set of wild points of $f(K) \cap \mathrm{Int}(M)$ and zero on $\mathrm{Bd}(M)$.

There is a tame embedding g of K into M, there is a pseudo isotopy H_t $(0 \leqslant t \leqslant 1)$ of M onto itself, and there is a 0-dimensional F_σ-subset Y of $f(K - K_a)$ which is contained in the set of wild points of $f(K - K_a)$ such that

1. *$H_1 g = f$,*
2. *H_1 takes $M - H_1^{-1}(Y)$ homeomorphically onto $M - Y$, and*
3. *the track of each point x of M under H_t has diameter no greater than $\mu(x)$.*

Proof. We assume μ has been cut down if necessary so that for each $t > 0$, $\mu^{-1}([t, \infty))$ is compact. Let T be a triangulation of $K - K_a$ with 2-skeleton T_2 and 2-simplexes $\Delta_1, \ldots, \Delta_i, \ldots$ For each integer n let W_n denote the set $f(\bigcup_{i \leqslant n} \Delta_i)$. Set $W = \bigcup W_n$.

Although the lemma is stated for embeddings of finite polyhedra, the proof of Lemma 5.3 of [5] applies equally well to embeddings of in-finite polyhedra and shows that there is a sequence of subsets $X_1, \ldots, X_j, \ldots$ of W such that each $X_j \cap W_n$ is a tame universal curve in W_n which is normally situated with respect to the curvilinear triangulation induced on W_n by T_2, the diameter of each component of $f(\Delta_i) - X_j$ is less than $1/2^{i+j}$, and if $1 \leqslant k < j$ then for all integers n, $X_k \cap W_n \subset I(X_j \cap W_n, W_n)$.

Choose open sets O_{jk} whose closures are contained in $\mathrm{Int}(M) \cap \mu^{-1}(0, \infty))$ so that for each j, the O_{jk}'s have mutually exclusive closures where each $\mathrm{Cl}(O_{jk}) \cap W = \mathrm{Cl}(O_{jk} \cap W)$ is the closure of some component of $W - X_j$, the diameter of every O_{jk} is less than twice the diameter of $\mathrm{Cl}(O_{jk}) \cap W$, every $\mathrm{Cl}(O_{jk})$ $(j > 1)$ is contained in some O_{j-1m}, every $\mathrm{Cl}(O_{jk})$ contains some wild point of W, and every wild point of W is contained in $X_j \cup (\bigcup_k \mathrm{Cl}(O_{jk}))$.

For each $j > 0$ define a continuous, non-negative, real function μ_j on M so that $\mu_j^{-1}((0, \infty)) = \bigcup_k O_{jk}$ and so that if $O_{jk} \subset O_{1m}$ then the maximum value of μ_j on $\mathrm{Cl}(O_{jk})$ is less than $1/2^j$ times the minimum value of μ on $\mathrm{Cl}(O_{1m})$. For each O_{jk} let μ_{jk} denote the restriction of μ_j to O_{jk}.

Define polyhedra $P_{jk} = f^{-1}(f(K) \cap O_{jk})$. For each O_{jk} substitute $(O_{jk} \to M,\ P_{jk} \to K,\ \emptyset \to K_a,\ f \to f,\ \mu_{jk} \to \mu)$ in Theorem 8.2 to find an associated positive, continuous function ν_{jk} on P_{jk}. We assume that the ν_{jk}'s are cut down if necessary so that if $O_{jk} \subset O_{j-1m}$ then for each point x of P_{jk}, $\nu_{jk}(x) < \nu_{j-1m}(x)$.

Use [2] to find for each $j > 0$ a homeomorphism g_j of K into M which agrees with f on $K - \bigcup_k P_{jk}$ so that $g_j(K)$ is locally tame at each point of $g_j(\bigcup_k P_{jk})$ and $x \in P_{jk} \Rightarrow \varrho(f(x), g_j(x)) < \nu_{jk}(x)$. Lemma 4.2 of [5] and Lemma 10.1 here show that each $g_j(K)$ is locally tame. By the assumptions on μ and from the fact that $f(K)$ is a closed subset of M, each $g_j(K)$ is a closed subset of M and is therefore tame [1, 18].

For each $j > 0$, Theorem 8.2 provides an isotopy H_t^j $(0 \leqslant t \leqslant 1)$ of M onto itself such that $H_1^j g_j = g_{j+1}$ and the track of each point x of M under H_t^j has diameter no greater than $\mu_j(x)$.

Set $g = g_1$, and define the promised pseudo isotopy H_t by the rule $H_0 = I$, $H_t = H^j_{2^j(t-(1-1/2^{j-1}))} H_{1-1/2^{j-1}}$ $(1 - 2^{j-1} \leqslant t \leqslant 1 - 1/2^j,\ 1 \leqslant j < \infty)$, and $H_1 = \lim_{t \to 1} H_t$. It is clear that H_t approaches at least a map of M in the limit. Since $\lim_{j \to \infty} g_j = f$ and $H_{1-1/2^j} g_j = g_{j+1}$ for each $j > 0$, $H_1 g = f$. This shows that Condition 1 is satisfied in the conclusion of the theorem.

For each positive integer n let Y_n denote the subset of W which consists of the points x whose pre-images $H_1^{-1}(x)$ have diameters greater than or equal to $1/n$. Each Y_n is closed. Set $Y = \bigcup Y_n$. Since $Y \subset \bigcap (\bigcup_k O_{jk}) \subset \bigcap (W - X_j)$, Y is a 0-dimensional F_σ-subset of the wild points of W. But points x and y are taken by H_1 onto the same point if and only if for each integer j, $H_{1-1/2^j}(x)$ and $H_{1-1/2^j}(y)$ belong to the same O_{jk}. Thus Condition 2 is satisfied.

Consider now a point x of M. If x does not belong to some O_{1k} then H_t is the identity on x. Suppose $x \in O_{1m}$. By the definition of the μ_j's the maximum value of each μ_j over $\mathrm{Cl}(O_{1m})$ is less than $1/2^j$ times the minimum value of μ on $\mathrm{Cl}(O_{1m})$; thus the track of x under H_t has diameter no greater than $\sum 1/2^j \mu(x) = \mu(x)$.

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THE INSTITUTE FOR ADVANCED STUDY
Princeton, New Jersey

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Fiber homotopy type of associated loop spaces

by

F. H. Croom (Lexington, Ky.)

1. Introduction. Let E and B be topological spaces with base points and $\pi\colon E\to B$ a continuous map. This paper gives necessary and sufficient conditions that the fiber structures $(\Omega E, p, \Omega B)$ and $(\Omega B\times\Omega F, q, \Omega B)$ be fiber homotopy equivalent where F is the basic fiber of (E, π, B), ΩE is the space of based loops in E, $p\colon \Omega E\to\Omega B$ is the natural map induced by π and q is the projection on the first factor. From this result it is observed that if (E, π, B) is a Hurewicz fibration with cross section, $(\Omega E, p, \Omega B)$ and $(\Omega B\times\Omega F, q, \Omega B)$ are fiber homotopy equivalent. It follows that the higher loop space $\Omega^n E$ is H-isomorphic to $\Omega^n B\times\Omega^n F$ for $n\geqslant 2$. This naturally implies the known result ([3] p. 152):

$$\pi_n(E)\simeq\pi_n(B)+\pi_n(F)\quad \text{for}\quad n\geqslant 2\,.$$

2. Preliminaries.

DEFINITION. A fiber structure (E, π, B) is a *weak Hurewicz fibration* if there is a *weak lifting function*

$$\lambda\colon\ \Delta=\{(e, a)\in E\times B^I\colon\ p(e)=a(0)\}\to E^I$$

such that λ is continuous,

$$\pi\lambda(e, a)(t)=a(t)\qquad (t\in I)$$

and the map $(e, a)\to\lambda(e, a)(0)$ is fiberwise homotopic to the projection on the first factor.

The following analogue of the Curtis–Hurewicz theorem ([1], [4]) is easily proved:

THEOREM 1. *The fiber structure (E, π, B) is a weak Hurewicz fibration if and only if for each space X, continuous $f\colon X\to E$ and homotopy $\varphi\colon X\times I\to B$ of πf there exists a homotopy $\Phi\colon X\times I\to E$ covering φ such that Φ_0 is fiberwise homotopic to f.*

THEOREM 2. *In order that $(\Omega E, p, \Omega B)$ be fiber homotopy equivalent to $(\Omega B\times\Omega F, q, \Omega B)$, it is necessary and sufficient that $(\Omega E, p, \Omega B)$ be a weak Hurewicz fibration with cross section.*