References

[1] G. Birkhoff, *Lattice theory*, 3rd ed., Amer. Math. Soc. Colloq. Publ. 25, Providence, R. I, 1967.
[2] E. Marczewski, *Concerning independence in lattices*, Colloq. Math. 10 (1963), pp. 21–23.
[3] — *Independence and homomorphisms in abstract algebras*, Fund. Math. 50 (1961), pp. 45–61.
[4] J. Płonka, *On distributive quasi-lattices*, Fund. Math. 60 (1967), pp. 191–200.
[5] M. H. Stone, *Topological representations of distributive lattices and Browerian logics*, Časapis Pěst. Mat. Fiz. 67 (1937), pp. 1–25.

UNIVERSITY OF MISSOURI
St. Louis, Missouri

Reçu par la Rédaction le 24. 6. 1969

Characterizing the 3-cell by its metric

by

Dale Rolfsen* (Princeton, N. J.)

The purpose of this article is to prove that certain elementary properties enjoyed by the Euclidean metric on the unit cube or ball in 3-space yield some rather simple characterizations of the topological 3-cell.

Introduction. In a metric space (X, d), m is a *midpoint* of x and y provided $d(x, m) = d(m, y) = \frac{1}{2}d(x, y)$. We say that a space (or its metric) is *convex* if each pair of points has at least one midpoint, *strongly convex* (SC) if each pair has exactly one midpoint, and *without ramifications* (WR) if no midpoint of x and y is also a midpoint of x' an y, unless $x = x'$. A metric space which is simultaneously SC and WR will be called SC-WR.

Following are the paper's main results (these have already been announced in [10] without detailed proofs).

Theorem A. *Each 3-dimensional compact* SC-WR *metric space is homeomorphic to the 3-cell.*

Theorem B. *Each compact 3-manifold (with boundary) having a* SC *metric is homeomorphic to the 3-cell.*

Theorem C. *Any crumpled cube having a* SC *metric is homeomorphic to the 3-cell. (A crumpled cube is defined to be the closure of the bounded complementary domain of some 2-sphere in E^3.)*

Theorem A generalizes the analogous result in 2-dimensions proved by Lelek and Nitka [6]. The content of Theorem B is that if "fake cubes" (i.e. homotopy 3-cells which are not real cells) exist, then they fail to have a metric with unique midpoints. Although the higher dimensional versions of these theorems seem at present to be unanswered, the following related result is true at least for $n > 5$: Each compact n-manifold having a SC-WR metric is topologically an n-cell [see 10]. Section I recalls some general properties of convex metric spaces, Section II contains the proof of Theorem A, while Theorems B and C are proved simultaneously in Section III.

---

* Supported in part by National Science Foundation grant GP-7952X, and based on the author's doctoral dissertation written under Joseph Martin at the University of Wisconsin.

I. Segments and deformations. In a complete convex metric space, each pair of points x, y determines at least one arc (called a *segment*) between x and y which is isometric with a closed real line interval [8]. If the metric is SC, this segment is uniquely determined and for each t, $0 \leqslant t \leqslant 1$, there is a unique point z such that $d(x, z) = t \cdot d(x, y)$ and $d(z, y) = (1-t) \cdot d(x, y)$. We denote z by $P(x, y, t)$, and the interval $[0, 1]$ by I.

(1) *If X is a compact* SC *metric space, the function P defined above is a continuous function mapping $X \times X \times I$ onto X.*

Proof. Only the continuity is in question. Suppose $x_i \to x$, $y_i \to y$ and $t_i \to t$, where $x_i, x, y_i, y \in X$ and $t_i, t \in I$. By compactness of X, the sequence $p_i = P(x_i, y_i, t_i)$ has a subsequence $\{p_{n_i}\}$ which converges, say to p. But since $d(x_{n_i}, p_{n_i}) = t_{n_i} \cdot d(x_{n_i}, y_{n_i})$, we let $i \to \infty$ and see that $d(x, p) = td(x, y)$. Similarly $d(p, y) = (1-t)\,d(x, y)$, so p is, by definition, the point $P(x, y, t)$. Since every convergent subsequence must converge to this point, we have $P(x_i, y_i, t_i) \to P(x, y, t)$, proving (1). ■

Thus for a fixed $x_0 \in X$, $H_t(x) = P(x, x_0, t)$ defines a homotopy of X in X such that (i) $H_0 =$ identity, (ii) $H_1(X) = x_0$, and (iii) $H_t(x)$ lies on the segment from x to x_0. We shall say that H "shrinks" X (or any subset of X) along segments toward x_0. It may readily be seen that a compact convex metric space is SC-WR if and only if the union of any two segments which meet in at least two points is an arc. Thus in the SC-WR case H_t is a homeomorphism for $t < 1$.

(2) *Each compact* SC *metric space X is contractible and locally contractible (so if $\dim X < \infty$ it is an absolute retract). If X is also* WR, *then the dimension of each open set is* $\dim X$.

Proof. Choosing any $x_0 \in X$, we may shrink X to x_0 as described above. Also any subset A of X may be shrunk in X to any point of A. In fact, since the segments are defined by the metric, one sees that this shrinking of A takes place in a set of diameter $\leqslant 2 \cdot \operatorname{diam} A$, so X is locally contractible.

Finally if X is SC-WR and U is an open set we choose $x_0 \in U$ and shrink X toward x_0 until the image of X is inside U but before it becomes a point. This image is a homeomorphic copy of X, so $\dim U = \dim X$. ■

By compactness each segment in X extends to a segment which is maximal (with respect to inclusion).

(3) *If e is an endpoint of a maximal segment in the compact* SC *metric space X, then $X - e$ is a contractible space.*

Proof. Let x_0 be the other endpoint of such a maximal segment. Then in shrinking X to x_0 along segments, the image of $X - e$ always lies in $X - e$. ■

II. Proof of Theorem A. In order to begin the construction proving Theorem A, it is necessary to locate "interior" points to the space. This is the purpose of the following lemma.

Lemma A. *If X is a compact* SC-WR *metric space and $0 < \dim X < \infty$, then there exists an open subset U of X such that $X - x$ fails to be contractible (in itself) whenever $x \in U$. Further, we can find such an open set U which is dense in X.*

Proof. We use the notion of Vietoris homology cycles with compact carriers and adopt the definitions, notation, and propositions outlined in [2]. The key to the proof is Alexandroff's theorem [1] that the dimension of a compact metric space is $> k$ if and only if some k-dimensional essential infinite cycle is homologous to 0 in the space. So if $\dim X = n$, there is an $n-1$ dimensional infinite cycle γ in X and a compact subset B of X which carries γ such that $\gamma \sim 0$ in X, but $\gamma \nsim 0$ in B. Then one may construct a „membrane" A for γ spanned on B, i.e. A is compact, $B \subset A \subset X$, and $\gamma \sim 0$ on A but $\gamma \nsim 0$ on $A - a$ when a is any point of $A - B$. Now we choose any point $a \in A - B$ and a number $\varepsilon > 0$ such that $d(a, B) > 2\varepsilon$, and let U be the ε-neighborhood in X of the point a.

If $x \in U$ we show that $X - x$ is not contractible by assuming otherwise. Let V be the ε-neighborhood of x, so V intersects A but not B. It follows from the general theory that $D = \operatorname{bd}(V) \cap A$ contains an $n-1$ dimensional infinite cycle γ' such that $\gamma' \nsim 0$ in D. However, since X is a SC-WR metric space, the set C obtained by taking the union of all segments from D to x, is homeomorphic to the cone on D with vertex at x. So $\gamma' \sim 0$ in C but $\gamma' \nsim 0$ in $C - x$. Now if $X - x$ were contractible, there would be a compact set $E \subset X - x$ such that $\gamma' \sim 0$ in E. But since $\gamma' \sim 0$ in each of C and E, but $\gamma' \nsim 0$ in $C \cap E$ (because $C \cap E \subset C - x$), the Phragmen–Brouwer theorem guarantees the existence of an n-dimensional infinite cycle in $C \cup E$ which is not homologous to 0 in $C \cup E$, hence essential. But X is contractible, so this n-cycle is homologous to 0 in X. By Alexandroff's theorem, we obtain the contradiction that $\dim X > n$.

To verify the last part of the lemma, note that all the above construction of A, B, and U could have been confined to stay inside any preassigned open set, since the dimension of any open set is also n. Then we just take the union of the U's so obtained. ■

We assume in the rest of Section II *that (X, d) is a metric space satisfying the hypothesis of Theorem* A (i.e. compact, 3-dimensional and SC-WR). By Lemma A there exist $p \in X$ and $\varepsilon > 0$ such that the set $N = \{x \in X\colon d(x, p) < \varepsilon\}$ contains no points with contractible complements. Thus by (3), no maximal segment ends in N. If $x, y \in X$ and $x \neq y$, let xy and $xy\rangle$ denote, respectively, the (unique) segment from x to y and the (unique by the WR property) largest segment which contains y and ends at x. Let $S = \{x \in X\colon d(x, p) = \varepsilon\}$, and let B the set of endpoints (other than p) of segments $px\rangle$, $x \in X - p$. Observe that each point of B is an endpoint of a maximal segment. The plan is to prove first that S is a 2-sphere, then that B is compact and (hence) also a 2-sphere. Then we may conclude that X is a 3-cell.

(4) *S is compact and 2-dimensional and $\overline{N}$ is homeomorphic to $C(S)$, the cone on S, with vertex at p and segments as cone elements.*

Proof. S is a closed subset of the compact space X, and any segment from S to p hits S just once, so we can embed the cone on S into $\overline{N}$ via segments running from S to p. Since no maximal segment ends in N, every point of $\overline{N}$ lies on some segment from S to p. Hence the embedding is surjective. Since $\dim(\overline{N}) = 3$ by (2), we see also that $\dim S = 2$. ∎

(5) *There exists a retraction r of $X - p$ onto S such that for each x in $X - p$, $r(x) \in px\rangle$.*

Proof. We shrink $X - p$ toward p along segments until the image lies in $N - p$ and then push the cone-less-vertex $N - p$ along segments to the base S. ∎

It follows from (5) and the fact that $X - p$ is an absolute neighborhood retract (ANR) that:

(6) *S is an ANR and a 1-1 continuous image of B.*

(7) *If $s \in S$, then $S - s$ is contractible (in itself).*

Proof. Note first that the segment $sp\rangle$ intersects S in exactly one other point s'. Now, if $x \in S - s$, the segment xs' cannot meet $ps\rangle$, for if it did the metric would not be WR. So we may shrink $S - s$ along segments toward s' in $X - ps\rangle$ by the map $H_t(x) = P(x, s', t)$, $x \in S - s$, $t \in I$. Then $rH_t\colon S - s \to S - s$ is a contraction. ∎

(8) *There is a homeomorphism of S onto itself without fixed points and having period 2.*

Proof. We just use the correspondence $s \to s'$ described in the proof of (7). Note that s' is the only point on S having distance 2ε from s. For if s'' were another, then p would be a midpoint of s and s'' as well as s and s', contradicting the WR property. Now if s_n converges in S to s, then $d(s_n, s_n') = 2\varepsilon$ and any limit point l of s_n' must satisfy $d(s, l) = 2\varepsilon$ and $l \in S$, so $l = s'$. This establishes continuity, and bijectiveness and periodicity are obvious. ∎

(9) *S is connected and not separated by any finite set.*

Proof. Let $A = \{s_1, \ldots, s_n\}$ be a finite subset of S and choose any $x, y \in S - A$. We shall connect x and y by a path in $S - A$. First choose $\delta > 0$ small enough to insure that if M is the closed δ-neighborhood of y, then any point of M may be connected to y by a segment missing every segment $ps_i\rangle$. Now, for $i \leqslant n$, let M_i be the set of all $m \in M$ such that mx intersects $ps_i\rangle$. If not empty, M_i is compact and since $mx \cap ps_i\rangle$ can only be one point, there is a continuous function $f\colon M_i \to ps_i\rangle$ defined by $f(m) = mx \cap ps_i\rangle$. Now if $z \in ps_i\rangle$, $f^{-1}(z)$ is a subset of $xz\rangle$, hence at most one-dimensional. Since the range of f, and each point's preimage have dimension at most 1, a standard dimension theorem [4] shows that $\dim M_i \leqslant 2$. Thus $\dim(\bigcup M_i) \leqslant 2$, while $\dim M = 3$ by (2), so there is a point $q \in M - \bigcup M_i$. This means the segment qx misses each of the $ps_i\rangle$, as does the segment qy. Now if $r\colon X - p \to S$ is the retraction of (5) we find that $r(qx \cup qy)$ misses A and is a path from x to y. ∎

(10) *S is a 2-sphere.*

To prove this we appeal to the following algebraic characterization due to McCord [7] and based on the Kline sphere characterization:

A Hausdorff space Y is a 2-sphere if $H_2(Y) \neq 0$ but $H_i(Y - y) = 0$ for $i = 0, 1, 2$ and each $y \in Y$. (Here H_* is reduced singular homology with integral coefficients.)

The condition $H_i(S - s) = 0$ for $s \in S$ and $i = 0, 1, 2$ is certainly true by (7). We get at $H_2(S)$ by looking at other dimensions. $H_0(S) = 0$ since S is connected and $H_i(S) = 0$ for $i > 2$ since $\dim S = 2$. Further, we see $H_1(S) = 0$ by examining the following portion of the Mayer–Vietoris sequence of the triad $(S, S - x, S - y)$, where x and y are distinct points of S; it is exact because $S - x$ and $S - y$ are open:

$$\ldots \to H_1(S - x) \oplus H_1(S - y) \to H_1(S) \to H_0(S - \{x, y\}) \to \ldots$$

The groups on the left are trivial by (7) and the group on the right is trivial by (9), so $H_1(S)$ must be trivial. If $H_2(S)$ were also trivial, we could conclude by a theorem of Lefschetz [5] that the acyclic compact ANR space S had the fixed-point property. Since this contradicts (8), $H_2(S)$ must be nontrivial and hence S is a 2-sphere. ∎

Since S is a 1-1 continuous image of B, we will establish that B is a 2-sphere as soon as we prove it is compact. $\overline{N}$ is a 3-cell, being the cone on S, and by shrinking X toward p until it lies in N, we obtain an embedding of X in E^3.

(11) *Under any embedding of X in E^3, B is its boundary and is therefore compact.*

Proof. Suppose $x \in B$. Then, by (3), $X - x$ is contractible, and we conclude that x is a boundary point of X. Conversely, suppose now that $x \notin B$. Let q be the endpoint (other than p) of $px\rangle$, so that x is interior to the segment pq. It is now possible to shrink X along segments toward q far enough to obtain an embedding $h\colon X \to X$ such that $h(p) = x$. Since $p \subset N$, we have $x \in h(N) \subset X$. Since $h(N)$ is an open 3-cell, it follows that x is interior to X, and (11) is proved. ∎

The proof of Theorem A is concluded by observing that X is homeomorphic to the cone on the 2-sphere B.

III. Proof of Theorems B **and** C. *In the remainder of the paper* (M, d) *will denote a compact* SC (*but not necessarily* WR) *metric space satisfying the hypothesis of either Theorem* B *or* C. Thus M ambiguously denotes either a 3-manifold (hence by (2) a homotopy cell) or a crumpled cube. We wish to show that M is a 3-cell. Certain properties of M are apparent under either assumption: $\mathrm{Bd}(M)$ is a 2-sphere and by an argument as in (11) any maximal segment ends in the boundary. The abbreviations "Bd" and "Int" will have the usual meanings of boundary and interior for manifolds. For a crumpled cube they will be used with reference to some (hence any) embedding in 3-space.

LEMMA B. *Suppose that* (i) K *is either a crumpled cube or homotopy cell,* (ii) J *is a simple closed curve in* $\mathrm{Bd}(K)$, *and* (iii) α *is an arc in* $K-J$ *whose endpoints* a *and* b *are in different components of* $\mathrm{Bd}(K)-J$. *Then* J *is not contractible in* $M-\alpha$.

Proof. Suppose first that K is a crumpled cube $K\subset E^3$. There is an arc β from a to b with $\mathrm{Int}(\beta)\subset E^3-K$. Since $\alpha\cup\beta$ is a simple closed curve which pierces once a disk (part of $\mathrm{Bd}(K)$) bounded by J, the curves J and $\alpha\cup\beta$ are linked in E^3. Thus J is not contractible in $K-\alpha \subset E^3-(\alpha\cup\beta)$.

Now suppose that K is a homotopy cell, which we may assume to be triangulated. It is sufficient to assume that α is polygonal and $\alpha\cap\mathrm{Bd}(K) = \{a, b\}$, and prove that J represents a nontrivial element of the singular homology group $H_1(K-\alpha)$. Letting T be a tubular neighborhood of α in K, there is an exact sequence:

$$\ldots\to H_2(K-\alpha, T-\alpha)\to H_1(T-\alpha)\to H_1(K-\alpha)\to H_1(K-\alpha, T-\alpha)\to\ldots$$

By excision and contractibility of K and T we have:

$$H_i(K-\alpha, T-\alpha)\simeq H_i(K, T)\simeq 0\,.$$

Thus the map $H_1(T-\alpha)\to H_1(K-\alpha)$ induced by inclusion is an isomorphism. Since J separates a from b in $\mathrm{Bd}(K)$ and the infinite cyclic group $H_1(T-\alpha)$ is generated by a small cycle around a in $\mathrm{Bd}(K)$, we see that J is homologous in $K-\alpha$ to a generator of $H_1(T-\alpha)$. Therefore J represents a generator of $H_1(K-\alpha)$ and the lemma is proved.

A construction. It is convenient to enlarge the original space M to give it a constant "radius". Choose a fixed point $*\in\mathrm{Int}(M)$ and a fixed $r>\sup\{d(*, b): b\in\mathrm{Bd}(M)\}$. Let $A\subset\mathrm{Bd}(M)\times[0, \infty)$ be the set of all pairs (b, t) such that $0\leqslant t\leqslant r-d(*, b)$. The map $(b, t)\to\big(b, t(r-d(*, b))^{-1}\big)$ shows that actually $A\cong\mathrm{Bd}(M)\times I\cong S^2\times I$. Although $M\cup A$ has no obvious segment structure, we may easily define a "radial" structure.

(By $M\cup A$ we mean disjoint union with each $b\in\mathrm{Bd}(M)$ identified with $(b, 0)\in A$.) For a point $x\in M\cup A$, define the *ray* $*x$ as follows: if $x\in M$, $*x$ is just the segment from $*$ to x; if $x\in A$ (say $x=(b, t)$), $*x$ is the union of the segment $*b$ in M and the arc $b\times[0, t]$ in A. The natural parametrizations of segments and intervals $[0, t]$ induce a parametrization of rays, and rays clearly vary continuously with their endpoints so deformations may be defined as with segments. The notion of segment length extends to rays: the length of $*x$, $x=(b, t)\in A$, is just $d(*, b)+t$. All rays extend to maximal rays and maximal rays all have length r and end in $Q=\{(b, t)\in A\colon\ t=r-d(*, b)\}$, which is a 2-sphere. We may think of $M\cup A$ as a metric space, although it is not required that this metric correspond to length on rays.

Finally define a map $f\colon (M\cup A)-*\to(0, r]$ by $f(x)=$ the length of the ray $*x$. That is:

$$f(x)=\begin{cases} d(*, x)\,, & x\in M\,,\\ d(*, b)+t\,, & x=(b, t)\in A\,.\end{cases}$$

Let us show that f is really a fibration over $(0, r]$ with 2-spheres as fibres.

(12) *For each* $t\in(0, r]$, $f^{-1}(t)$ *is a 2-sphere.*

Proof. The set $f^{-1}(t)$ consists of those x such that $*x$ has length t. Thus for *fixed* t, there is a continuous function $g_t\colon Q\to f^{-1}(t)$ defined by $g_t(q)=*q\cap f^{-1}(t)$. Since rays extend to maximal rays, g_t is a surjection. We now show that if $x\in f^{-1}(t)$, then (i) $g_t^{-1}(x)$ is connected, and (ii) $Q-g_t^{-1}(x)$ is connected. Clearly $g_t^{-1}(x)$ is just the set of $q\in Q$ such that $x\in *q$, thus (i) and (ii) hold when $x\in A$. So assume $x\in\mathrm{Int}(M)$. Suppose for (i) that $g_t^{-1}(x)$ is *not* connected. Then there exist points $a, b\in g_t^{-1}(x)$ and a simple closed curve $J\in Q-g_t^{-1}(x)$ separating a from b in Q. Let a', b', J' be the respective projections of a, b, J onto $\mathrm{Bd}(M)$, via rays. Since $*a'$ and $*b'$ meet at x, and J' contracts in M along rays missing x, there is an arc $\alpha\subset *a'\cup *b'\subset M$ from a' to b' such that J' contracts in $M-\alpha$. This contradicts Lemma B, so $g_t^{-1}(x)$ must be connected. To prove (ii) suppose $g_t^{-1}(x)$ separates Q; say y and z are in different components of $Q-g_t^{-1}(x)$. Let G, y', z' be the projections of $g_t^{-1}(x)$, y and z onto $\mathrm{Bd}(M)$. Now $*y'$ and $*z'$ miss x while G contracts to x along rays, all of which must miss $*y'\cup *z'$. By continuity of g_t and local contractibility, we see that some neighborhood U of G in $\mathrm{Bd}(M)$ also contracts to x missing $*y'\cup *z'$. Then U contains a simple closed curve separating y' from z' and we proceed as before to contradict Lemma B and establish that $g_t^{-1}(x)$ cannot separate S. In summary, we have a surjective map $g_t\colon Q\to f^{-1}(t)$ such that the preimage of each point is connected and fails to separate Q. Since $Q\cong S^2$, a theorem of Moore [9] guarantees that $f^{-1}(t)\cong S^2$. ■

(13) *f is a 0-regular map.*

Proof. In other words we must show that (i) f is an open map, and (ii) if $t \in (0, r]$, $x \in f^{-1}(t)$ and $\varepsilon > 0$ are given, then there exists $\delta > 0$ such that whenever $s \in (0, r]$, $f^{-1}(s) \cap N_\delta(x)$ lies in one component of $f^{-1}(s) \cap N_\varepsilon(x)$. ($N_\varepsilon = \varepsilon$-neigborhood.) Part (i) is clear; to prove (ii) assume such t, x, ε are given. Choose $V \subset f^{-1}(t)$ to be a closed, connected neighborhood of x in the sphere $f^{-1}(t)$ and such that $V \subset N_\varepsilon(x)$. Let T be the union of all rays which intersect V. Now there exists a neighborhood W of t in $(0, r]$ such that $f^{-1}(W) \cap T \subset N_\varepsilon(x)$. Note that $x \in V \subset f^{-1}(W) \cap T$, and continuity of rays implies that $f^{-1}(W) \cap T$ is a neighborhood of x in X. So we choose $\delta > 0$ such that $N_\delta(x) \subset f^{-1}(W) \cap T$. To see that δ is properly chosen, suppose $f^{-1}(s) \cap N_\delta(x) \neq \emptyset$ for some $s \in (0, r]$. Then $s \in W$ and we have:

$$f^{-1}(s) \cap N_\delta(x) \subset f^{-1}(s) \cap T \subset f^{-1}(s) \cap N_\varepsilon(x) .$$

But $f^{-1}(s) \cap T$ is identical with $g_s g_t^{-1}(V)$, which is connected since V is connected and g_t is a monotone map. Thus $f^{-1}(s) \cap N_\delta(x)$ lies in a single component of $f^{-1}(s) \cap N_\varepsilon(x)$, proving that f is 0-regular. ∎

We now appeal to Theorem 7 of Dyer and Hamstrom [3] to conclude from (12) and (13) that f is a fiber map. In fact, there is a homeomorphism h: $(M \cup A) - * \to (0, r] \times S^2$ such that the following diagram commutes, where π is projection onto the first coordinate:

$$\begin{tikzcd} (M \cup A) - * \arrow[rr, "h"] \arrow[dr, "f"'] & & (0, r] \times S^2 \arrow[dl, "\pi"] \\ & (0, r] & \end{tikzcd}$$

Taking the one-point compactifications, this provides a homeomorphism H: $M \cup A \to B_r$, where B_r is the closed ball in E^3 of radius r, such that $f(x) = \|H(x)\|$, where $\|\cdot\|$ is the Euclidean norm. Since M is therefore homeomorphic to a starlike neighborhood of $0 \in E^3$, bounded by a sphere, M is a 3-cell.

References

[1] P. S. Alexandroff, *Dimensiontheorie. Ein Beitrag zur Geometrie der abgeschlossenen Mengen*, Math. Ann. 106 (1932), pp. 161–238.

[2] R. H. Bing and K. Borsuk, *Some remarks concerning topologically homogeneous spaces*, Annals of Math. 81 (1965), pp. 100–111.

[3] E. Dyer and M.-E. Hamstrom, *Completely regular mappings*, Fund. Math. 45 (1958), pp. 103–118.

[4] W. Hurewicz and H. Wallman, *Dimension theory*, Princeton 1948.

[5] S. Lefschetz, *Topics in topology*, Annals of Math. Studies, v. 10 (1942).

[6] A. Lelek and W. Nitka, *On convex metric spaces I*, Fund. Math. 49 (1961), pp. 183–204.

[7] M. McCord, *Spaces with acyclic point complements*, Proc. Amer. Math. Soc. 17 (1966), pp. 886–890.

[8] K. Menger, *Untersuchungen über allgemeine Metrik*, Math. Ann. 100 (1928), pp. 75–163.

[9] R. L. Moore, *Concerning upper semicontinuous collections of continua*, Trans. Amer. Math. Soc. 27 (1925), pp. 416–428.

[10] D. Rolfsen, *Strongly convex metrics in cells*, Bull. Amer. Math. Soc. 74 (1968), pp. 171–175.

THE INSTITUTE FOR ADVANCED STUDY
Princeton, New Jersey USA

Reçu par la Rédaction le 4. 4. 1968