A representation theorem for distributive quasi-lattices*

by

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1. Introduction. In [4], J. Płonka introduces the notion of a distributive quasi-lattice (DQL). Roughly speaking, they are algebras which satisfy the usual axioms for a distributive lattice (considered as an algebra with two binary operations) except that the absorption laws are deleted. The purpose of this paper is to obtain a representation theorem for distributive quasi-lattices which is analogous to the representation of distributive lattices as rings of sets. Indeed, in the case when a DQL is a distributive lattice, our representation coincides with the classical one.

2. Definitions and lemmas. A *distributive quasi-lattice* (DQL) is an algebra $\mathfrak{D} = (D, +, \cdot)$ with axioms:

(1) $$x+x = x; \quad xx = x;$$

(2) $$x+y = y+x; \quad xy = yx;$$

(3) $$(x+y)+z = x+(y+z); \quad (xy)z = x(yz);$$

(4) $$x(y+z) = xy+xz; \quad x+yz = (x+y)(x+z).$$

For a finite non-empty subset $S = \{x_1, \ldots, x_n\}$ of a DQL, $\Pi(S)$ will denote $x_1 \cdot \ldots \cdot x_n$ and we adopt the convention that $x \cdot \Pi(\emptyset) = x$. The following properties of distributive quasi-lattices were proved in [4]:

(5) $$\Pi(S)\,\Sigma(S) = \Pi(S) \quad (S \neq \emptyset);$$

(6) $$\Pi(S)+\Sigma(S) = \Sigma(S) \quad (S \neq \emptyset);$$

(7) For finite non-empty subsets $S_1, \ldots, S_n$

$$\sum_{i=1}^{n} \Pi(S_i) = \Big(\sum_{j=1}^{n} \Pi(S_{i_j})\Big) + \Pi\Big(\bigcup_{i=1}^{n} S_i\Big)$$

where $S_{i_1}, \ldots, S_{i_n}$ are the minimal members of $\{S_1, \ldots, S_n\}$ under inclusion.

In order to simplify notation, define the relation: $x \leq y$ iff $xy = x$. It is evident that $(D, \leq)$ is a meet semi-lattice where $x \wedge y = xy$. Although $x+y$ may not be an upper bound for x and y in $(D, \leq)$, $x+y$ is $\leq$ any

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upper bound for x and y. Dually, define $x \leqslant y$ iff $x+y=y$. $(D, \leqq)$ and $(D, \leqslant)$ are isomorphic under $x \to x$ iff $\mathfrak{D}$ is a distributive lattice.

THEOREM 1. *There is a one-to-one correspondence between the class of distributive quasi-lattices and the class of relational systems of the form* $(D, \leqq, \leqslant)$, *where* $(D, \leqq, \wedge)$ *is a meet semi-lattice*, $(D, \leqslant, \vee)$ *is a join semi-lattice and*

(8) $$x\wedge(y\vee z) = (x\wedge y)\vee(x\wedge z);$$

(9) $$x\vee(y\wedge z) = (x\vee y)\wedge(x\vee z).$$

The main result of this paper (Theorem 4) is possible because of the variant of the "prime filter theorem" which appears below.

DEFINITION 2. A *filter* F in a DQL $\mathfrak{D} = (D, +, \cdot)$ is a non-empty subset of D which satisfies

(10) $$x \leqq y, \quad x \in F \Rightarrow y \in F;$$

(11) $$x, y \in F \Rightarrow xy \in F.$$

If $F \neq D$ and $x+y \in F \Rightarrow x \in F$ or $y \in F$, then F is a *prime filter*.

A *prime ideal* is defined dually (that is; $+$, $\cdot$ and $\leqq$ are replaced with $\cdot$, $+$, $\leqslant$ respectively). If $\emptyset \neq S \subseteq D$, then the smallest filter, $[S]$, containing S is, as usual, $\{y \in D \mid y \geqq x_1 \cdot \ldots \cdot x_n$ where $x_i \in S\}$ so that for $x \in D$, $[x] = \{y \mid y \geqq x\}$.

LEMMA 3. *If* $\mathfrak{D}$ *is a* DQL *and* $x \nleqq y$ *then there is a prime filter* F *which contains* x *and not* y.

Proof. By using Zorn's lemma, it can be verified that the family $\mathfrak{F}$ of filters that contain x and not y has a maximal member F. To show F is prime, let $u+v \in F$, $u \notin F$, $v \notin F$. Then $F \subset [F \cup \{u\}]$ implies $y \in [F \cup \{u\}]$, so $y \geqq f_1 u$ for some $f_1 \in F$, similarly $y \geqq f_2 v$ for some $f_2 \in F$. We have already noted that this implies that $y \geqq f_1 u + f_2 v$. Now by (6)

$$(f_1u+f_2v)f_1f_2(u+v) = f_1f_2u+f_1f_2uv+f_1f_2v$$
$$= f_1f_2(u+uv+v) = f_1f_2(u+v).$$

So $f_1f_2(u+v) \leqq f_1u+f_2v \leqq y$, which means $y \in F$, a contradiction.

3. Representation theorem.

THEOREM 4. *An algebra* $\mathfrak{D}$ *is a* DQL *if and only if it is of the form* $(X_{\mathfrak{D}}, +, \cdot)$ *where* $X_{\mathfrak{D}}$ *and* Y *are families of sets closed under finite intersections and unions respectively*, $\theta\colon X_{\mathfrak{D}} \to Y$ *is a one-to-one correspondence satisfying*

(12) $$A \cap \theta^{-1}\big(\theta(B) \cup \theta(C)\big) = \theta^{-1}\big(\theta(A\cap B) \cup \theta(A \cap C)\big) \quad \text{for} \quad A, B, C \in X_{\mathfrak{D}};$$

(13) $$R \cup \theta\big(\theta^{-1}(S) \cap \theta^{-1}(T)\big) = \theta\big(\theta^{-1}(R\cup S) \cap \theta^{-1}(R \cup T)\big) \quad \text{for} \quad R, S, T \in Y$$

and for $A, B \in X_{\mathfrak{D}}$:

$$A+B = \theta^{-1}\big(\theta(A) \cup \theta(B)\big), \quad A \cdot B = A \cap B.$$

Proof. It is immediate that the algebra described in the theorem is a DQL. Now suppose $\mathfrak{D} = (D, +, \cdot)$ is a DQL. Let $\mathfrak{F}$ be the set of prime filters and $\mathfrak{J}$ the prime ideals in $\mathfrak{D}$. For each $x \in D$, let

$$x^* = \{F \in \mathfrak{F} \mid x \in F\} \quad \text{and} \quad x^+ = \{I \in \mathfrak{J} \mid x \notin I\};$$

also

$$X_{\mathfrak{D}} = \{x^* \mid x \in D\} \quad \text{and} \quad Y = \{x^+ \mid x \in D\}.$$

By using Lemma 3, and its dual, $\theta\colon x^* \to x^+$ is seen to be a one-to-one correspondence between $X_{\mathfrak{D}}$ and Y. Clearly

(14) $$x^* \cap y^* = (xy)^* \quad \text{and} \quad x^+ \cup y^+ = (x+y)^+.$$

For $x^*, y^*, z^* \in X_{\mathfrak{D}}$:

$$x^* \cap \theta^{-1}\big(\theta(y^*) \cup \theta(z^*)\big)$$
$$= x^* \cap \theta^{-1}(y^+ \cup z^+) = x^* \cap \theta^{-1}\big((y+z)^+\big)$$
$$= x^* \cap (y+z)^* = \big(x(y+z)\big)^* = (xy+xz)^* = \theta^{-1}\big((xy+xz)^+\big)$$
$$= \theta^{-1}\big((xy)^+ \cup (xz)^+\big) = \theta^{-1}\Big(\theta\big((xy)^*\big) \cup \theta\big((xy)^*\big)\Big)$$
$$= \theta^{-1}\big(\theta(x^* \cap y^*) \cup \theta(x^* \cap z^*)\big).$$

Similarly for (13). Finally the mapping $x \to x^*$ is an isomorphism of $\mathfrak{D}$ onto $(X_{\mathfrak{D}}, +, \cdot)$.

COROLLARY 5. *In Theorem* 4, $\mathfrak{D} \cong (X_{\mathfrak{D}}, +, \cdot)$ *where* $x^*+y^* = (x+y)^*$ *and* $x^* \cdot y^* = (xy)^* = x^* \cap y^*$.

COROLLARY 6 (Birkhoff, Stone). *Every distributive lattice* $\mathfrak{D}$ *is isomorphic with a ring of sets.*

Proof. For each $x \in D$, where $\mathfrak{D}$ is a distributive lattice, $x^* = x^+$ so $X_{\mathfrak{D}} = Y$ and $\theta = 1_{X_{\mathfrak{D}}}$. Hence $\mathfrak{D}$ is isomorphic with the ring $X_{\mathfrak{D}}$.

4. Independent sets.

In order to show that prime filters in a DQL can sometimes be explicitly determined, we will consider the case of a DQL generated by an independent set J, ([4], § 3). First observe that in any finite DQL, F is a prime filter if and only if $F = [x]$ where x is *subjoin irreducible* (SJI); that is

(15) $x \leqq y+z \Rightarrow x \leqq y$ or $x \leqq z$ and x is not the least element of $(D, \leqq)$.

Since we want to consider the most general case, we assume throughout the section that $\mathfrak{D} = (D, +, \cdot)$ is a DQL generated by a set J satisfying:

For S, T finite non-empty subsets of J

(16) $$\Pi(S)\Pi(T)+\Sigma(T)=\Sigma(T)\Rightarrow S\cap T\neq\emptyset;$$

(17) $$\Pi(S)+\Pi(S)\Sigma(T)=\Pi(S)\Sigma(T)\Rightarrow S\cap T\neq\emptyset$$

(see [4], Theorem 6).

LEMMA 7. *Suppose* $\emptyset\neq A\subseteq B\subseteq J$ *and* $\emptyset\neq S_i\subseteq J$ *for* $i=1,\dots,n$. *Then* $\Pi(A)+\Pi(B)\leq\sum_{i=1}^{n}\Pi(S_i)$ if and only if

(18) $$\bigcup_{i=1}^{n} S_i\subseteq B$$

and

(19) $$S_{i_0}\subseteq A\quad \textit{for some}\quad i_0\in\{1,\dots,n\}.$$

Proof. Suppose $\Pi(A)+\Pi(B)\leq\sum_{i=1}^{n}\Pi(S_i)$. Let $s_i\in S_i$ for $i=1,\dots,n$. Then

$$\Pi(A)+\Pi(B)\leq\sum_{i=1}^{n}\Pi(S_i)=\Pi\{u_1+\dots+u_n|\ u_i\in S_i\}\leq s_1+\dots+s_n.$$

Let $x_1,\dots,x_m$ be the distinct members of $(s_1,\dots,s_n)$ so

(20) $$\Pi(A)+\Pi(B)=\sum_{i=1}^{m}\Pi(A)x_i+\sum_{i=1}^{m}\Pi(B)x_i.$$

Multiply (20) by $\Pi(B)\cdot x_1\cdot\dots\cdot x_{p-1}x_{p+1}\cdot\dots\cdot x_m$ to obtain

(21) $$\Pi(B)x_1\cdot\dots\cdot x_{p-1}x_{p+1}\cdot\dots\cdot x_m$$
$$=\Pi(B)\cdot x_1\cdot\dots\cdot x_{p-1}x_{p+1}\cdot\dots\cdot x_m+\Pi(B)\cdot x_1\cdot\dots\cdot x_m.$$

Let $T=B\cup\{x_1,\dots,x_{p-1},x_{p+1},\dots,x_m\}$, and add $\Sigma(T)$ to (21). By (6) we obtain: $\Sigma(T)=\Sigma(T)+\Pi(T)x_p$. This means $x_p\in T$ so $\{s_1,\dots,s_n\}\subseteq B$ and hence (18) is proved.

If (19) is false, then there is a non-empty subset $S'\subseteq B-A$ such that $\Pi(A)+\Pi(B)\leq\Sigma(S')$. Hence,

$$\begin{aligned}\Pi(A)+\Pi(A)\big(\Pi(B-A)+\Sigma(B-A)\big)&=\Pi(A)+\Pi(B)+\Pi(A)\Sigma(B-A)\\&=\Pi(A)\Sigma(S')+\Pi(B)\Sigma(S')+\Pi(A)\Sigma(B-A)\\&=\Pi(A)\big(\Sigma(S')+\Pi(B-A)\Sigma(S')+\Sigma(B-A)\big).\end{aligned}$$

But $S'\subseteq B-A$ so $\Pi(A)+\Pi(A)\Sigma(B-A)=\Pi(A)\Sigma(B-A)$, which by independence implies that $A\cap(B-A)\neq\emptyset$.

Conversely, without loss of generality, let us assume that $S_1\subseteq A$. Then

$$\begin{aligned}\big(\Pi(A)+\Pi(B)\big)\sum_{i=1}^{n}\Pi(S_i)&=\Pi(A-S_1)\Big(\Pi(S_1)+\Pi\big(B-(A-S_1)\big)\Big)\Big(\Pi(S_1)+\sum_{i=2}^{n}\Pi(S_i)\Big)\\&=\Pi(A-S_1)\Big(\Pi(S_1)+\sum_{i=2}^{n}\Pi\big(B-(A-S_1)\big)\Pi(S_i)\Big)\\&=\Pi(A-S_1)\Pi(S_1)+\sum_{i=2}^{n}\Pi\big([A-S_1]\cup[B-(A-S_1)]\cup S_i\big)\\&=\Pi(A)+\Pi(B)\end{aligned}$$

so

$$\Pi(A)+\Pi(B)\leq\sum_{i=1}^{n}\Pi(S_i).$$

THEOREM 8. *An element* $a\in\mathfrak{D}$ *is* SJI *if and only if* $a=\Pi(S)+\Pi(T)$ *where* $\emptyset\neq S\subseteq T\subseteq J$, *and* $S\neq J$. *Moreover the representation is unique.*

Proof. Suppose $a=\Pi(S)+\Pi(T)$, $\emptyset\neq S\subseteq T\subseteq J$, $S\neq J$ and $a\leq b+c$ where $b=\sum_{i=1}^{n}\Pi(S_i)$, $c=\sum_{j=1}^{m}\Pi(T_j)$; S_i, T_i are non-empty subsets of J. Then Lemma 7 implies, without loss of generality, that $S_1\subseteq A$ and $\bigcup_{i=1}^{n}S_i\subseteq B$. So by the converse of Lemma 7, $a\leq b$. Also if $s\in J-S$ then $a\not\leq s$.

Conversely, suppose $a=\sum_{i=1}^{n}\Pi(S_i)$ is SJI, where $\emptyset\neq S_i\subseteq J$, and that for no t is it true that $S_t\subseteq\bigcap_{i=1}^{n}S_i$. Then for each $i\in\{1,\dots,n\}$ there exists $x_i\in S_i-S_{i_0}$ for some $i_0\in\{1,\dots,n\}$. Since $a\leq x_1+\dots+x_n$ and a is SJI, $a\leq x_p$ for some p. Hence

(22) $$\sum_{i=1}^{n}\Pi(S_i)=\sum_{i=1}^{n}x_p\Pi(S_i).$$

Let $S=(\bigcup_{i=1}^{n}S_i)-\{x_p\}$. Then

$$\Pi(S_i)\Pi(S)=\begin{cases}x_p\Pi(S) & \text{if}\quad x_p\in S_i;\\ \Pi(S) & \text{if}\quad x_p\notin S_i.\end{cases}$$

Since $x_p \in S_p - S_{p_0}$, the product of (22) with $\Pi(S)$ yields:

$$x_p \Pi(S) + \Pi(S) = x_p \Pi(S) .$$

But (17) implies the contradiction $x_p \in S$. We now invoke (7) to obtain

$$a = \Pi(S_t) + \Pi(\bigcup_{i=1}^{n} S_i) .$$

If $S_t = J$ then $a \leqq b$ for all $b \in \mathfrak{D}$.

The uniqueness follows from Lemma 7.

THEOREM 9. *Let $\mathfrak{D}$ be a DQL generated by a finite independent set J. Then $\mathfrak{D}$ has the form $(Z, +, \cdot)$ where Z is the family of all finite intersections of $\{A' \mid \emptyset \neq A \subseteq J\}$ where*

$$(\bigcap_{i=1}^{n} A_i') + (\bigcap_{j=1}^{m} B_j') = \bigcap_{\substack{1 \leqslant i \leqslant n \\ 1 \leqslant j \leqslant m}} (A_i \cup B_j)' ;$$

$$(\bigcap_{i=1}^{n} A_i') \cdot (\bigcap_{j=n+1}^{m} A_j') = \bigcap_{i=1}^{m} A_i'$$

and $A' = \{\Pi(S) + \Pi(T) \mid S \subseteq T \subseteq J,\ S \neq J,\ A \cap S \neq \emptyset,\ A \subseteq T\}$.

Proof. For each $d \in D$, let $d^- = \{a \in D \mid a \leqq d,\ a \text{ is SJI}\}$ and $Z = \{d^- \mid d \in D\}$. Then since $\mathfrak{D}$ is finite, $d^* \to d^-$ is an isomorphism of $(X_{\mathfrak{D}}, +, \cdot)$ onto $(Z, +, \cdot)$ where $(X_{\mathfrak{D}}, +, \cdot)$ is as in Corollary 5,

$$d^- + e^- = (d+e)^- \quad \text{and} \quad d^- \cdot e^- = (d \cdot e)^- = d^- \cap e^- .$$

Set $A' = (\Sigma(A))^-$. Since J generates $\mathfrak{D}$, $d \in D$ has the form

$$d = \prod_{i=1}^{n} \Sigma(S_i) , \quad \emptyset \neq S_i \subseteq J \quad \text{so} \quad d^- = \bigcap_{i=1}^{n} (\Sigma(S_i))^- = \bigcap_{i=1}^{n} S_i' .$$

Next

$$(\bigcap_{i=1}^{n} A_i') + (\bigcap_{j=1}^{m} B_j') = \bigcap_{i=1}^{n} (\Sigma(A_i))^- + \bigcap_{j=1}^{m} (\Sigma(B_j))^-$$

$$= \Big(\prod_{i=1}^{n} (\Sigma(A_i))\Big)^- + \Big(\prod_{j=1}^{m} (\Sigma(B_j))\Big)^-$$

$$= \Big(\prod_{i=1}^{n} (\Sigma(A_i)) + \prod_{j=1}^{m} (\Sigma(B_j))\Big)^-$$

$$= \Big(\prod_{\substack{1 \leqslant i \leqslant n \\ 1 \leqslant j \leqslant m}} (\Sigma(A_i) + \Sigma(B_j))\Big)^-$$

$$= \bigcap_{\substack{1 \leqslant i \leqslant n \\ 1 \leqslant j \leqslant m}} (\Sigma(A_i \cup B_j))^- = \bigcap_{\substack{1 \leqslant i \leqslant n \\ 1 \leqslant j \leqslant m}} (A_i \cup B_j)' ;$$

$$(\bigcap_{i=1}^{n} A_i') \cdot (\bigcap_{i=n+1}^{m} (A_i')) = \Big(\prod_{i=1}^{n} \Sigma(A_i)\Big)^- \cdot \Big(\prod_{i=n+1}^{m} \Sigma(A_i)\Big)^-$$

$$= \Big(\prod_{i=1}^{m} \Sigma(A_i)\Big)^- = \bigcap_{i=1}^{m} A_i' .$$

It remains to show that if $\emptyset \neq A \subseteq J$ then

$$(\Sigma(A))^- = \{\Pi(S) + \Pi(T) \mid S \subseteq T \subseteq J,\ S \neq J,\ A \cap S \neq \emptyset,\ A \subseteq T\}.$$

Now $x \in (\Sigma(A))^-$ iff $x \leqq \Sigma(A)$ and x is SJI. But x is SJI iff $x = \Pi(S) + \Pi(T)$ where $S \subseteq T \subseteq J$, $S \neq J$ so by Lemma 7, $x \leqq \Sigma(A)$ iff $A \subseteq T$ and $A \cap S \neq \emptyset$. This completes the proof.

EXAMPLE. Let $J = \{a, b\}$ be an independent set that generates $(D, +, \cdot)$. Then

$$\{a\}' = \{a, a+ab\} ;$$

$$\{b\}' = \{b, b+ab\} ;$$

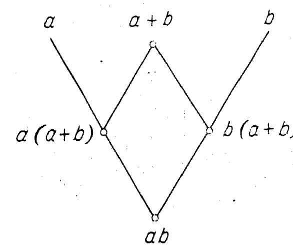
$$\{a, b\}' = \{a+ab, b+ab\} ;$$

$$\{a\}' \cap \{b\}' = \emptyset ;$$

$$\{a\}' \cap \{a, b\}' = \{a+ab\} ;$$

$$\{b\}' \cap \{a, b\}' = \{b+ab\} .$$

It follows from Theorem 9 that the diagram for $(D, \leqq)$ is:



Corollary 5 shows that every DQL $\mathfrak{D}$ determines a unique semi-lattice $X_{\mathfrak{D}}$ which is isomorphic—as a DQL—with $\mathfrak{D}$. We close with the following question: To which meet semi-lattices M do there correspond distributive quasi-lattices $\mathfrak{D}$ such that $X_{\mathfrak{D}}$ is isomorphic to M—as a semi-lattice?

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Characterizing the 3-cell by its metric

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The purpose of this article is to prove that certain elementary properties enjoyed by the Euclidean metric on the unit cube or ball in 3-space yield some rather simple characterizations of the topological 3-cell.

Introduction. In a metric space (X, d), m is a *midpoint* of x and y provided $d(x, m) = d(m, y) = \frac{1}{2}d(x, y)$. We say that a space (or its metric) is *convex* if each pair of points has at least one midpoint, *strongly convex* (SC) if each pair has exactly one midpoint, and *without ramifications* (WR) if no midpoint of x and y is also a midpoint of x' an y, unless $x = x'$. A metric space which is simultaneously SC and WR will be called SC-WR.

Following are the paper's main results (these have already been announced in [10] without detailed proofs).

THEOREM A. *Each 3-dimensional compact* SC-WR *metric space is homeomorphic to the 3-cell.*

THEOREM B. *Each compact 3-manifold (with boundary) having a* SC *metric is homeomorphic to the 3-cell.*

THEOREM C. *Any crumpled cube having a* SC *metric is homeomorphic to the 3-cell. (A crumpled cube is defined to be the closure of the bounded complementary domain of some 2-sphere in E^3.)*

Theorem A generalizes the analogous result in 2-dimensions proved by Lelek and Nitka [6]. The content of Theorem B is that if "fake cubes" (i.e. homotopy 3-cells which are not real cells) exist, then they fail to have a metric with unique midpoints. Although the higher dimensional versions of these theorems seem at present to be unanswered, the following related result is true at least for $n > 5$: Each compact n-manifold having a SC-WR metric is topologically an n-cell [see 10]. Section I recalls some general properties of convex metric spaces, Section II contains the proof of Theorem A, while Theorems B and C are proved simultaneously in Section III.

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