# Tree-likeness of hereditarily equivalent continua

by

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A continuum is a compact, connected, metric space. An hereditarily equivalent continuum is a non-degenerate continuum which is homeomorphic to each of its non-degenerate subcontinua.

In [5], S. Mazurkiewicz raised the question: "Un continu dans l'espace à $m$ dimensions qui est homéomorphe de tout continu qu'il contient, est-il necessairement un arc simple?" Moïse, [6], has answered this question in the negative by showing that the pseudo-arc is an indecomposable, planar, hereditarily equivalent continuum; however, G. W. Henderson, [3], has shown that each decomposable, hereditarily equivalent continuum is an arc. The pseudo-arc and the arc are the only known hereditarily equivalent continua. The author has heard, conversationally, speculation that (especially in light of the proof of D. W. Henderson, [2], that there is a continuum each non-degenerate subcontinuum of which is infinite dimensional) there may be an infinite dimensional, hereditarily equivalent continuum. Lemma 3 of this paper shows that this is not the case.

A tree-like continuum is one which, for every positive number $\varepsilon$, can be $\varepsilon$-mapped onto a finite tree. Each tree-like continuum is one-dimensional. Whyburn has shown, [8], that each planar, hereditarily equivalent continuum is tree-like. In this note it is shown that every hereditarily equivalent continuum is tree-like.

LEMMA 1. *Suppose that $M$ is an hereditarily equivalent continuum and $\varepsilon$ is a positive number. Then there is a homeomorphism $h$ of $M$ onto a proper subcontinuum of $M$ such that, for each point $x$ of $M$, the distance from $x$ to $h(x)$ is less than $\varepsilon$.*

Proof. Let $G$ be an uncountable monotonic collection of non-degenerate subcontinua of $M$ (e.g., for some point $p$, let $G$ be the collection to which $g$ belongs if, and only if, for some positive number $\delta$, $g$ is the component containing $p$ of the closed $\delta$-neighborhood of $p$.) For each element $g$ of $G$, let $h_g$ denote a homeomorphism of $M$ onto $g$. Let $X = \{h_g|\ g \in G\}$. Then $X$ is a subset of $M^M$, the space of all mappings (i.e. continuous transformations) of $M$ into $M$, and, thus, is separable and metric. Since every uncountable subset of $X$ has a limit element, there is an element $k$ of $G$ such that $h_k$ is a limit element of $\{h_g |\ g \in G$ and $g \subset k\}$ and of $\{h_g |\ g \in G$ and $k \subset g\}$, ([7], Theorem 6, p. 3). Thus there is a sequence $\{g_i\}$ of distinct elements of $G$ which are proper subcontinua of $k$ such that the sequence $\{h_{g_i}\}$ converges to $h_k$. Then the sequence $\{h_{g_i} \circ h_k^{-1}\}$ converges to the identity mapping of $k$ onto itself and, for each $i$, $h_{g_i} \circ h_k^{-1}(k)$ is a proper subcontinuum, $g_i$, of $k$. Since $M$ is homeomorphic to $k$, there also exists a sequence $\{f_i\}$ of homeomorphisms, each throwing $M$ onto a proper subcontinuum of $M$ such that the sequence $\{f_i\}$ converges to the identity mapping of $M$ onto $M$. Thus, Lemma 1 is true.

LEMMA 2. *If $M$ is an hereditarily equivalent continuum, every mapping of $M$ into a connected one-dimensional polyhedron is homotopic to a constant.*

Proof. Suppose that $Y$ is connected, one-dimensional polyhedron and there is an essential mapping of $M$ into $Y$. Since the space $Y^M$ of all mappings of $M$ into $Y$ is an ANR ([4], p. 260), if we show that there is an essential mapping of $M$ into $Y$ which is a limit point (in $Y^M$) of the set of inessential mapping of $M$ into $Y$, we will have achieved a contradiction. Let $f$ be a mapping of $M$ onto $Y$ which is essential; there is a subcontinuum $K$ of $M$ such that $f|K$ is essential but, if $K'$ is a proper subcontinuum of $K$, $f|K'$ is inessential. Since $K$ is homeomorphic to $M$, there is an essential mapping $g$ of $M$ into $Y$ such that, if $M'$ is a proper subcontinuum of $M$, $g|M'$ is inessential. Let $\varepsilon > 0$. Since $g$ is uniformly continuous, there is a positive number $\delta$ such that, if $h$ is a mapping of $M$ into $M$ at a distance in $M^M$ less than $\delta$ from the identity mapping of $M$ onto $M$ then $[g|h(M)] \circ h$ is a distance less than $\varepsilon$ from $g$ in $Y^M$. But there is a homeomorphism $h$ of $M$ onto a proper subcontinuum $M'$ of $M$ which moves no point of $M$ a distance greater than $\delta$ and $g|M'$ is inessential. Thus, the distance in $Y^M$ from $g$ to $[g|M'] \circ h$ is less than $\varepsilon$ and is inessential. Thus $g$ is an essential mapping of $M$ into $Y$ which is a limit point in $Y^M$ of the set of inessential mappings of $M$ into $Y$, our contradiction. Hence, every mapping of $M$ into a one-dimensional polyhedron is inessential.

LEMMA 3. *Every hereditarily equivalent continuum is one-dimensional.*

Proof. Suppose $M$ is an hereditarily equivalent continuum and $\dim M > 1$. Then ([4], p. 271) there is a subcontinuum $M'$ of $M$ and an essential mapping $f$ of $M'$ onto a circle. But $M'$ is hereditarily equivalent, a contradiction to Lemma 2.

THEOREM. *Every hereditarily equivalent continuum is tree-like.*

Proof. Every hereditarily equivalent continuum is a one-dimensional continuum each mapping of which onto a one-dimensional polyhedron is inessential and, thus, [1], Theorem 1, is tree-like.



**References**


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