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Lattice modules over semi-local Noether lattices

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§ 1. Introduction. For Noetherian lattice modules, the concept of the a-adic pseudometric has been introduced and studied in [2] and [3]. Recently the natural completion of a local Noether lattice was related to the completeness of a local ring in its natural topology ([1]). The purpose of this paper is to establish some properties of Noetherian lattice modules over semi-local Noether lattices and their completions.

The basic concepts are introduced in § 2, and some preliminary results are obtained. Let L be a multiplicative lattice and let M be a Noetherian L-module. In § 3 an interesting property concerning certain sequences in M is established (Theorem 3.2). If L is a Noether lattice and m is the Jacobson radical of L, then it is shown (Corollary 3.4) that the m-adic pseudometric on M is a metric ([2], § 3). § 4 contains some results on dimensions. If L is semilocal, it is shown in § 5 that $[mA, A]$ is finite dimensional, for all A in M (Theorem 5.1), L^* is a Noether lattice, and M^* is a Noetherian L^*-module (Theorem 5.9), where L^* and M^* are the m-adic completions of L and M, respectively ([2], § 6). In § 6 it is established that L^* is a semi-local Noether lattice whose maximal elements are extensions ([2], § 5) of the maximal elements of L.

§ 2. Preliminary remarks. By a multiplicative lattice we shall mean a complete lattice on which there is defined a commutative, associative, join distributive multiplication such that the unit element of the lattice is an identity for the multiplication. Let L be a multiplicative lattice and let M be a complete lattice. We shall denote elements of L by $a, b, c, \ldots$ with the exception that the null element and unit element of L will be denoted by 0 and I, respectively. We shall denote elements of M by $A, B, C, \ldots$, with the exception that the null element and unit element of M will be denoted by 0_M and M, respectively. When no confusion is possible, 0 will also be used in place of 0_M. Recall that M is an L-module ([2], Definition 2.2) in case there is a multiplication between elements of L and M, denoted by aA for a in L and A in $\mathfrak{M}$, which satisfies: (i) $(ab)A = a(bA)$, (ii) $(\bigvee_\alpha a_\alpha)(\bigvee_\beta B_\beta) = \bigvee_{\alpha,\beta} a_\alpha B_\beta$; (iii) $IA = A$; and (iv) $0A = 0$; for all a, a_α, b in L and for all A, B_β in M.

Let M be an L-module. For a, b in L and for A, B in M, (i) $a:b$ denotes the largest c in L such that $cb \leqslant a$; (ii) $A:b$ denotes the largest C in M such that $bC \leqslant A$; and, (iii) $A:B$ denotes the largest c in L such that $cB \leqslant A$. An element A in M is said to be meet principal in case $(b \wedge (B:A))A = bA \wedge B$, for all b in L and for all B in M; A is said to be join principal in case $b \vee (B:A) = (bA \vee B):A$, for all b in L and for all B in M; and, A is said to be principal in case A is both meet and join principal. If each element of M is the join (finite or infinite) of principal elements, M is called principally generated. M is said to be Noetherian if M satisfies the ascending chain condition, is modular, and is principally generated. If M is a Noetherian L-module, L is called a Noether lattice. For other general properties and definitions concerning Noetherian lattice modules, the reader is referred to [2].

In the special case where M and L are both modular, we can prove the following characterization of principal elements which will be useful later.

LEMMA 2.1. *Let M be an L-module and let A be an element of M. If M and L are modular, then A is a principal element of M if and only if*

$$C \wedge A = (C:A)A \tag{2.1}$$

and

$$bA:A = b \vee (0:A) \tag{2.2}$$

for all b in L and for all C in M.

Proof. Assume M and L are modular. Suppose that A is principal. Then clearly A satisfies (2.1) and (2.2). Conversely, assume that A satisfies (2.1) and (2.2). Then, for b in L and C in M, we have

$$\begin{aligned} b \wedge (C:A)\big) A &= \big((b \wedge (C:A)) \vee (0:A)\big)A = \big((C:A) \wedge (b \vee (0:A))\big)A \\ &= \big((C:A) \wedge (bA:A)\big) A = \big((C \wedge bA):A\big) A = C \wedge bA \wedge A = C \wedge bA \end{aligned}$$

by the modularity of L. Also,

$$\begin{aligned} (C \vee bA):A &= \big((C \vee bA) \wedge A\big):A = \big((C \wedge A) \vee bA\big):A = \big(((C:A) \vee b)A\big):A \\ &= \big((C:A) \vee b\big) \vee (0:A) = (C:A) \vee b \end{aligned}$$

by the modularity of M. It follows that A is both meet and join principal, and hence principal. q.e.d.

We will also need the following result.

LEMMA 2.2. *Let M be an L-module. Let a be a principal element of L and let A be a principal element of M. Then aA is a principal element of M.*

Proof. Let b be an element of L and let B be an element of M. Then, since a and A are principal, it follows that

$$\begin{aligned} \big(b \wedge (B:(aA))\big)(aA) &= \big(b \wedge ((B:A):a)\big)(aA) = \Big(\big(b \wedge ((B:A):a)\big)a\Big)A \\ &= \big(ba \wedge (B:A)\big)A = b(aA) \wedge B \end{aligned}$$

by ([2], (2.16)). And also,

$$\begin{aligned} \big(b(aA) \vee B\big):(aA) &= \big((b(aA) \vee B):A\big):a = \big(ba \vee (B:A)\big)a \\ &= b \vee \big((B:A):a\big) = b \vee \big(B:(aA)\big) \end{aligned}$$

by ([2], (2.15) and (2.16)). Hence aA is meet and join principal, and thus principal.

In later parts of this paper we will need to use a generalization of quotient lattices. This construction is developed in Remark 2.3 below. If A, B are elements of a lattice K with $A \leqslant B$, then the set $\{D \in K \mid A \leqslant D \leqslant B\}$ is a sublattice of K which will be denoted by $[A, B]$. If K is a complete lattice with unit element U, then for arbitrary A in K, we will also write K/A in place of $[A, U]$.

Remark 2.3. Let M be an L-module, let A, B be elements of M with $A \leqslant B$, and let a be an element of L such that $aC \leqslant A$, for all C in $[A, B]$. Then $[a, I]$ is "naturally" a multiplicative lattice and $[A, B]$ is "naturally" an $[a, I]$-module.

Proof. For b, c in $[a, I]$, define $b \circ c = bc \vee a$. For C in $[A, B]$ and b in $[a, I]$, define $b \circ C = bC \vee A$. Since M and L are both complete lattices, it follows immediately that $[A, B]$ and $[a, I]$ are complete lattices. It is easily verified that the above definitions of multiplication make $[a, I]$ into a multiplicative lattice and $[A, B]$ into an $[a, I]$-module. The computations will be omitted. q.e.d.

Remark 2.4. Let M be a Noetherian L-module, let A, B be elements of M with $A \leqslant B$, and let a be an element of L such that $aC \leqslant A$, for all C in $[A, B]$. Then, with respect to the "natural" multiplications given in Remark 2.3, $[A, B]$ becomes a Noetherian $[a, I]$-module.

Proof. This is a straight forward computation. The details will be omitted. The reader is referred to ([2], Remarks 2.8 and 2.9). q.e.d.

LEMMA 2.5. *Let M be an L-module, let A be a principal element of M, let a, b be elements of L such that $a \leqslant b$, let L be modular, and let $(0:A) \wedge b \leqslant a$. Then the map $\varphi\colon [a, b] \to [aA, bA]$ defined by $\varphi(x) = xA$ is a lattice isomorphism of $[a, b]$ onto $[aA, bA]$.*

Proof. Let x, y be elements of $[a, b]$ and assume $\varphi(x) = \varphi(y)$. Then $xA = yA$. Hence $xA:A = yA:A$. Consequently, since A is principal,

$$\begin{aligned} x &= x \vee \big(b \wedge (0:A)\big) = b \wedge \big(x \vee (0:A)\big) = b \wedge (xA:A) \\ &= b \wedge (yA:A) = b \wedge \big(y \vee (0:A)\big) = y \vee \big(b \wedge (0:A)\big) = y \end{aligned}$$

by the modularity of L. It follows that φ is one-to-one. To see that φ is onto, let B be an element of $[aA, bA]$. Since

$$a \leqslant aA:A \leqslant a\vee(0:A) \leqslant B:A \leqslant bA:A = b\vee(0:A),$$

we have that

$$a = a\wedge b \leqslant b\wedge(B:A) = b\wedge\big(b\vee(0:A)\big) = b.$$

Thus $b\wedge(B:A)$ is an element of $[a, b]$. Applying φ we obtain

$$\varphi\big(b\wedge(B:A)\big) = \big(b\wedge(B:A)\big)A = bA\wedge B = B,$$

since A is principal, and consequently φ is onto. Since φ is clearly order preserving, we have that φ is a lattice isomorphism of $[a, b]$ onto $[aA, bA]$. q.e.d.

§ 3. A preliminary theorem.

Definition 3.1. Let L be a multiplicative lattice and let M be a Noetherian L-module. For a in L and A in M, let $T(a, A)$ be the collection of all sequences $\langle B_i\rangle$, $i = 1, 2, \ldots$, of elements of M satisfying

$$a^iA \geqslant B_i \geqslant B_{i+1} \geqslant aB_i, \tag{3.1}$$

for all integers $i \geqslant 1$. For $\langle C_i\rangle$ and $\langle B_i\rangle$ in $T(a, A)$, define

(3.2) $\langle C_i\rangle \leqslant \langle B_i\rangle$ if and only if $C_i \leqslant B_i$, for all integers $i \geqslant 1$

$$\langle C_i\rangle \vee \langle B_i\rangle = \langle C_i \vee B_i\rangle \tag{3.3}$$

$$\langle C_i\rangle \wedge \langle B_i\rangle = \langle C_i \wedge B_i\rangle. \tag{3.4}$$

It is easily seen that $T(a, A)$ forms a complete, modular lattice under the relation $\leqslant$ with the resulting join and meet being given by (3.3) and (3.4). The resulting lattice will be denoted by $R(a, A)$.

Theorem 3.2. *Let L be a multiplicative lattice, let M be a Noetherian L-module, let a be an element of L, let A be an element of M, and let $\langle B_i\rangle$, $i = 1, 2, \ldots$, be an element of $R(a, A)$. Then there exists a natural number n such that $B_{m+i} = a^iB_m$, for all integers $m \geqslant n$ and for all integers $i \geqslant 0$.*

Proof. Let $F(a, A)$ be the collection of all sequences $\langle B_i\rangle$ in $R(a, A)$ for which the theorem fails. Assume that $F(a, A) \neq \emptyset$. We shall show that $F(a, A)$ has maximal elements.

Let $\mathfrak{C}$ be a chain in $F(a, A)$. For each C in $\mathfrak{C}$, let C_i be the ith coordinate of C. For each natural number i, set $S_i = \bigvee\{C_i \mid C \in \mathfrak{C}\}$.

Suppose $\langle S_i\rangle$ is not an element of $F(a, A)$. Then there exists a natural number n such that $S_{k+i} = a^iS_k$, for all integers $k \geqslant n$ and for all integers $i \geqslant 0$. Since M satisfies the ascending chain condition, for each i, $1 \leqslant i \leqslant n$, there exists an element in $\mathfrak{C}$ with ith coordinate S_i. Select one and call it $B(i)$.

Set $\langle B_i\rangle = \max\{B(1), \ldots, B(n)\}$. Thus $\langle B_i\rangle$ is in $F(a, A)$, and also $B_i = S_i$, for $1 \leqslant i \leqslant n$. In particular $B_n = S_n$. Consequently,

$$S_{n+i} = a^iS_n = a^iB_n \leqslant B_{n+i} \leqslant S_{n+i},$$

for all integers $i \geqslant 0$. It follows that $B_{n+i} = S_{n+i}$, for all integers $i \geqslant 0$. Thus, $B_i = S_i$, for all integers $i \geqslant 1$, and consequently $\langle S_i\rangle$ is in $F(a, A)$, which is a contradiction to the assumption that $\langle S_i\rangle$ is not an element of $F(a, A)$. Thus $\mathfrak{C}$ has an upper bound and hence $F(a, A)$ has maximal elements by Zorn's Lemma.

Let $\langle F_i\rangle$ be a maximal element of $F(a, A)$. By definition, we know $F_1 \leqslant aA$. Also, if $F_1 = aA$, then $F_{i+1} = a^iF_1$, for all integers $i \geqslant 0$, and hence $\langle F_i\rangle$ would not be in $F(a, A)$. Hence $F_1 \neq aA$. Thus, there exists a principal element E of M such that $E \leqslant aA$ and $E \nleqslant F_1$. It follows that $F_1 < F_1\vee E \leqslant aA$.

Now, define $\langle D_i\rangle$ by $D_i = F_i\vee a^{i-1}E$, for all integers $i \geqslant 1$. Observe that $F_1 < D_1 = F_1\vee E \leqslant aA$ and that $\langle D_i\rangle$ is an element of $R(a, A)$. Hence $\langle F_i\rangle < \langle D_i\rangle$, and $\langle D_i\rangle$ is not in $F(a, A)$. Consequently, there is a natural $n > 1$ such that $D_{k+i} = a^iD_k$, for all integers $k \geqslant n$ and for all integers $i \geqslant 0$. Hence

$$F_{k+i}\vee a^{k+i-1}E = D_{k+i} = a^iD_k = a^i(F_k\vee a^{k-1}E) = a^iF_k\vee a^{k+i-1}E,$$

for all integers $k \geqslant n$ and for all integers $i \geqslant 0$. Therefore, since M is modular and E is principal, we obtain

$$\begin{aligned} F_{k+i} &= F_{k+i}\wedge(a^iF_k\vee a^{k+i-1}E) = a^iF_k\vee(F_{k+i}\wedge a^{k+i-1}E) \\ &= a^iF_k\vee\big((F_{k+i}:E)\wedge a^{k+i-1}E\big), \end{aligned} \tag{3.5}$$

for all integers $k \geqslant n$ and for all integers $i \geqslant 0$. Next, for each integer $i \geqslant 1$, set $H_i = (F_{i+1}:E)\wedge a^i$. It follows from (3.5) that

$$F_{k+i} = a^iF_k\vee H_{k+i-1}E \tag{3.6}$$

for all integers $k \geqslant n$ and for all integers $i \geqslant 0$. It is easily verified that $\langle H_i\rangle$ is an element of $R(a, I)$.

Assume for a moment that $M = L$. Then, it is easily seen that $\langle F_i\rangle \leqslant \langle H_i\rangle$ in $R(a, I)$. Furthermore, if $\langle F_i\rangle = \langle H_i\rangle$, then

$$H_i = (F_{i+1}:E)\wedge a^i = F_i, \tag{3.7}$$

for all integers $i \geqslant 1$. Thus, since $E \leqslant aI = a$, we have by (3.6) and (3.7) that

$$F_{k+i+1} = a^{i+1}F_k\vee H_{k+i}E = a^{i+1}F_k\vee F_{k+i}E \leqslant aF_{n+i}, \tag{3.8}$$

for all integers $k \geqslant n$ and for all integers $i \geqslant 0$.

Then, since $a^iF_m \geqslant F_{m+i}$, for all integers $m \geqslant 1$ and for all integers $i \geqslant 0$, it follows from (3.8) that $a^iF_k = F_{k+i}$, for all integers $k \geqslant n$ and for all integers $i \geqslant 0$, which contradicts the fact that $\langle F_i \rangle$ is in $F(a, I)$. Hence $\langle F_i \rangle < \langle H_i \rangle$, and consequently, there exists a natural number $m \geqslant n$ such that

$$H_{k+i} = a^iH_k\,, \tag{3.9}$$

for all integers $k \geqslant m$ and all integers $i \geqslant 0$, by the maximality of $\langle F_i \rangle$. Consequently, since E is principal, we have by (3.6) and (3.9) that

$$\begin{aligned} F_{k+i+1} &= a^{i+1}F_k \vee H_{k+i}E = a^{i+1}F_k \vee H_{(k+i-1)+1} = a^{i+1}F_k \vee aH_{k+i-1} \\ &= a^{i+1}F_k \vee a\Big(\big((F_{k+i}\colon E) \wedge a^{k+i-1}\big)E\Big) \\ &= a^{i+1}F_k \vee a(a^{k+i-1}E \wedge F_{k+i}) \leqslant aF_{k+i}\,, \end{aligned}$$

for all integers $k \geqslant m$ and for all integers $i \geqslant 0$. As above, for (3.8), this implies that $F_{k+i} = a^iF_k$, for all integers $k \geqslant m$ and for all integers $i \geqslant 0$, in contradiction to $\langle F_i \rangle$ being in $F(a, I)$. Hence, when $M = L$, we have $F(a, I) = \emptyset$.

We return now to the general case. Since $\langle H_i \rangle$ is in $R(a, I)$, and since $F(a, I) = \emptyset$, there exists an integer $s \geqslant n$ such that $H_{k+i} = a^iH_k$, for all integers $k \geqslant s$ and for all integers $i \geqslant 0$. Then by (3.6), we have that

$$F_{k+i+1} = a^{i+1}F_k \vee H_{k+i}E = a^{i+1}F_k \vee a^iH_kE \leqslant a^iF_{k+1}\,,$$

for all integers $k \geqslant s$ and for all integers $i \geqslant 0$. This again implies that $a^iF_{k+1} = F_{k+1+i}$, for all integers $k \geqslant s$ and for all integers $i \geqslant 0$. Thus $\langle F_i \rangle$ is not in $F(a, A)$, which is a contradiction. Hence $F(a, A) = \emptyset$ in the general case. q.e.d.

For a Noether lattice L, recall that an element a in L is maximal if $a \neq I$ and if $b > a$ implies $b = I$. Also recall that the Jacobson radical of L is the inf of all such maximal elements of L.

Corollary 3.3. *Let L be a Noether lattice, let M be a Noetherian L-module, let m be the Jacobson radical of L, let B be an element of M, and let a be an element of L such that $a \leqslant m$. Then $\bigwedge_n a^nB = 0$.*

Proof. Let C be a principal element of M such that $C \leqslant \bigwedge_n a^nB$. Then, for all integers $n \geqslant 1$, we have $C = C \wedge a^nB$.

We shall show that $C = 0$. Consider the sequence $\langle C \wedge a^iB \rangle$, $i = 1, 2, \ldots$ Since

$$a^iB \geqslant C \wedge a^iB \geqslant C \wedge a^{i+1}B \geqslant a(C \wedge a^iB)\,,$$

for all integers $i \geqslant 1$, it follows from Theorem 3.2 that there exists a natural number k such that

$$C \wedge a^{k+i}B = a^i(C \wedge a^kB)\,,$$

for all integers $i \geqslant 0$. Hence $C = a^iC$, for all integers $i \geqslant 0$. In particular $C = aC$. Thus, since C is principal, we have

$$I = C\colon C = aC\colon C = a \vee (0\colon C)\,.$$

Since $a \leqslant m \neq I$, it must be that $0\colon C = I$. Consequently, $C = IC = (0\colon C)C = 0$.

Since M is a Noetherian L-module, every element is principally generated. It follows that $\bigwedge_n a^nB = 0$. q.e.d.

Corollary 3.4. *Let L be a Noether lattice, let M be a Noetherian L-module, let m be the Jacobson radical of L, and let a be an element of L such that $a \leqslant m$. Then*

$$A = \bigwedge_n (A \vee a^n\mathfrak{M}), \text{ for all } A \text{ in } M\,, \tag{3.10}$$

and

(3.11) *the a-adic pseudometric on M is a metric.*

Proof. Let A be an element of M. Then $[A, \mathfrak{M}]$ is a Noetherian L-module by Remark 2.4, and $a \leqslant m$. Thus

$$A = \bigwedge_n (a^n \circ \mathfrak{M}) = \bigwedge_n (A \vee a^n\mathfrak{M})$$

by Corollary 3.3. Hence (3.10) has been established. (3.11) follows from (3.10) and ([2], Theorem 3.10). q.e.d.

§ 4. Some results on dimensions. In this section some results are established concerning dimensions of various lattices. These results will be needed later.

Theorem 4.1. *Let L be a Noether lattice and let a be an element of L. Then there exist primes $p_1, \ldots, p_n$ in L such that $p_1p_2 \ldots p_n \leqslant a$.*

Proof. Let $F(L)$ be the collection of all elements in L for which the theorem fails. Suppose $F(L)$ is not empty. Then $F(L)$ has a maximal element b. Clearly b is not prime. Since b is not prime, there exist elements c, d in L such that $cd \leqslant b$, $c \nleqslant b$, and $d \nleqslant b$. Consequently, $c \vee b > b$ and $d \vee b > b$. Thus, since b is maximal in $F(L)$, there exist primes $p_1, \ldots, p_n$, $p'_1, \ldots, p'_m$ in L such that

$$p_1p_2 \ldots p_n \leqslant c \vee b \quad \text{and} \quad p'_1p'_2 \ldots p'_m \leqslant d \vee b.$$

It follows that

$$(p_1p_2 \ldots p_n)(p'_1p'_2 \ldots p'_m) \leqslant (c \vee b)(d \vee b) = cd \vee cb \vee bd \vee bb \leqslant b\,,$$

which is a contradiction to the maximality of b. Hence $F(L)$ is empty. q.e.d.

Lemma 4.2. *Let L be a local Noether lattice with unique maximal element p, and let M be a Noetherian L-module. Then, for each A in M, the lattice $[pA, A]$ is finite dimensional.*

Proof. Let A be an element of M. Since M is Noetherian, there exists principal elements $A_1, \ldots, A_n$ in M such that $A = A_1 \vee \ldots \vee A_n$. Let $S_0 = pA$, and, for each i, $0 \leqslant i \leqslant n-1$, set $S_{i+1} = S_i \vee A_{i+1}$. Since each A_k is principal, for each i, $0 \leqslant i \leqslant n-1$, we obtain

$$(4.1) \qquad [S_i, S_{i+1}] = [S_i, S_i \vee A_{i+1}] \cong [S_i \wedge A_{i+1}, A_{i+1}] = [(S_i : A_{i+1})A_{i+1}, A_{i+1}] \cong [S_i : A_{i+1}, I]$$

by the isomorphism theorems and Lemma 2.5. Since

$$pA_{i+1} \leqslant pA \vee A_1 \vee \ldots \vee A_i = S_i ,$$

it follows that $p \leqslant S_i : A_{i+1}$, for $0 \leqslant i \leqslant n-1$. Hence, the dimension of $[S_i : A_{i+1}, I]$ is either one or zero. Hence $[S_i, S_{i+1}]$ is finite dimensional, $0 \leqslant i \leqslant n-1$, by (4.1). Since $pA = S_0 \leqslant S_1 \leqslant \ldots \leqslant S_n = A$, we have $[pA, A]$ is also finite dimensional. q.e.d.

THEOREM 4.3. *Let L be a Noether lattice. If 0 is a product of maximal elements, then L is finite dimensional.*

Proof. Assume $0 = p_1 p_2 \ldots p_n$ where each p_i is maximal (and hence prime). For each i, $2 \leqslant i \leqslant n$, we know that $[p_i, I]$ is a Noether lattice and that $[p_1 p_2 \ldots p_i, p_1 p_2 \ldots p_{i-1}]$ is a Noetherian $[p_i, I]$—module by Remark 2.4. Thus, since each $[p_i, I]$ is local, we have that each $[p_i \circ (p_1 p_2 \ldots p_{i-1}), p_1 p_2 \ldots p_{i-1}]$ is finite dimensional, $2 \leqslant i \leqslant n$, by Lemma 4.2. Simplifying this expression we obtain $[p_1 p_2 \ldots p_i, p_1 p_2 \ldots p_{i-1}]$ is finite dimensional for each i, $2 \leqslant i \leqslant n$. Since $I > p_1 \geqslant p_1 p_2 \geqslant \ldots \geqslant p_1 p_2 \ldots p_n = 0$, it follows that L is finite dimensional. q.e.d.

COROLLARY 4.4. *Let L be a Noether lattice. If every (proper) prime element of L is maximal, then L is finite dimensional.*

Proof. Assume every (proper) prime element of L is maximal. By Theorem 4.1 there exists prime elements $p_1, \ldots, p_n$ in L such that $p_1 p_2 \ldots p_n \leqslant 0$. Hence $0 = p_1 p_2 \ldots p_n$, where each p_i is prime, and hence maximal by hypothesis. Thus, by Theorem 4.3, L is finite dimensional.

A Noether lattice is said to be *semi-local* if it has only finitely many maximal elements. If L is a semi-local Noether lattice with maximal elements $p_1, \ldots, p_n$, we will say that $(L, p_1, \ldots, p_n)$ is a semi-local Noether lattice.

COROLLARY 4.5. *Let $(L, p_1, \ldots, p_n)$ be a semi-local Noether lattice, and let m be the Jacobson radical of L. Then $[m, I]$ is finite dimensional.*

Proof. Each (proper) prime element of $[m, I]$ is maximal. q.e.d.

§ 5. m-adic completions.

Throughout this section $(L, p_1, \ldots, p_k)$ is a semi-local Noether lattice, M is a Noetherian L-module, and m is the Jacobson radical of L. Since L is semi-local, clearly $m = p_1 \wedge \ldots \wedge p_k$.

By Corollary 3.4, the m-adic pseudometric ([2], § 3) on M and the m-adic pseudometric on L are metrics. Consequently, the m-adic completions of M and L are defined ([2], § 6). Throughout this section, M^* shall denote the m-adic completion of M, and L^* shall denote the m-adic completion of L. It is known that M^* is an L^*-module ([2], § 7). It will be established in this section (Theorem 5.9) that L^* is in fact a Noether lattice and that M^* is a Noetherian L^*-module under the assumptions stated above. We begin with the following result.

THEOREM 5.1. *For each A in M, the quotient $[mA, A]$ is finite dimensional.*

Proof. Let A be an element of M. Since M is principally generated, there exists principal elements $A_1, \ldots, A_n$ in M such that $A = A_1 \vee \ldots \vee A_n$. Set $S_0 = mA$, and, for each i, $0 \leqslant i \leqslant n-1$, set $S_{i+1} = S_i \vee A_{i+1}$. Then, proceed as in the proof of Lemma 4.2 to obtain

$$[S_i, S_{i+1}] \cong [S_i : A_{i+1}, I], \quad 0 \leqslant i \leqslant n-1 .$$

Now, observe that $m \leqslant S_i : A_{i+1}$ and that $[m, I]$ is finite dimensional (Corollary 4.6). The proof is now finished as in Lemma 4.2. q.e.d.

COROLLARY 5.2. *For each A in M, $[m^n A, A]$ is finite dimensional, for each natural number n.*

Proof. Let A be an element of M. Since $m^n A \leqslant m^{n-1} A \leqslant \ldots \leqslant mA \leqslant A$, and since each quotient $[m^i A, m^{i-1} A]$, $1 \leqslant i \leqslant n$, is finite dimensional by Theorem 5.1, the result follows. q.e.d.

COROLLARY 5.3. *For each natural number n, the quotient L/m^n is finite dimensional.*

Proof. L is a Noetherian L-module. q.e.d.

In order to work with "infs" and "residuals" in M^*, it will be necessary to determine representatives of these elements. The following lemma will prove helpful in this respect. It is needed in the proof of Proposition 5.5 and 5.7.

LEMMA 5.4. *Let $\langle A_i \rangle$, $i = 1, 2, \ldots$, be a sequence of elements of M satisfying $A_{i+1} \leqslant A_i \vee m^i \mathfrak{M}$, for all integers $i \geqslant 1$. Then the sequence $\langle A_i \rangle$ is Cauchy.*

Proof. Let $n \geqslant 1$. Since the sequence $\langle A_i \vee m^n \mathfrak{M} \rangle$, $i = 1, 2, \ldots$, is decreasing in i, and since $m^n \mathfrak{M} \leqslant A_i \vee m^n \mathfrak{M}$, for each integer $i \geqslant 1$, it follows from Corollary 5.2 that there exists a natural number q such that

$$A_i \vee m^n \mathfrak{M} = A_j \vee m^n \mathfrak{M} ,$$

for all integers $i, j \geqslant q$. Consequently, $d_m(A_i, A_j) \leqslant 2^{-n}$, for all integers $i, j \geqslant q$. q.e.d.

PROPOSITION 5.5. *Let B, C be elements of M^*. Let $\langle B_i\rangle$ and $\langle C_i\rangle$ be the completely regular representative of B and C, respectively. Then the sequence $\langle B_i\wedge C_i\rangle$ is a representative of $B\wedge C$.*

Proof. Since $\langle B_i\rangle$ and $\langle C_i\rangle$ are decreasing, the sequence $\langle B_i\wedge C_i\rangle$ is decreasing, and hence Cauchy (Lemma 5.4). Let D be the equivalence class determined by $\langle B_i\wedge C_i\rangle$. Since $B_i\wedge C_i\leqslant C_i$, for all integers $i\geqslant 1$, it follows that $D\leqslant C$ ([2], Proposition 5.10). Similarly $D\leqslant B$. Thus $D\leqslant B\wedge C$.

Now, let A be an element of M^* such that $A\leqslant B$ and $A\leqslant C$. Let $\langle A_i\rangle$ be the completely regular representative of A. Then $A_i\leqslant B_i$ and $A_i\leqslant C_i$, for all integers $i\geqslant 1$ ([2], Proposition 5.9). Hence $A_i\leqslant B_i\wedge C_i$, for all integers $i\geqslant 1$. It follows that $A\leqslant D$ ([2], Proposition 5.10). Consequently $D = B\wedge C$. q.e.d.

Before proceeding to representatives of residuals in M^* (Proposition 5.7), we shall establish the following.

PROPOSITION 5.6. *M^* is modular.*

Proof. Let A, B, C be elements of M^* with $A\geqslant B$. Let $\langle A_i\rangle$, $\langle B_i\rangle$, and $\langle C_i\rangle$ be the completely regular representatives of A, B and C, respectively. Since $A\geqslant B$, we know $A_i\geqslant B_i$, for all integers $i\geqslant 1$ ([2], Proposition 5.9). We also know that the sequence $\langle B_i\vee C_i\rangle$ is the completely regular representative of $B\vee C$ ([2], Proposition 5.7). Hence $\langle A_i\wedge(B_i\vee C_i)\rangle$ is a representative of $A\wedge(B\vee C)$ by Proposition 5.5. Since $\langle A_i\wedge C_i\rangle$ is a representative of $A\wedge C$, and since $\langle B_i\rangle$ is a representative of B, we have that $\langle B_i\vee(A_i\wedge C_i)\rangle$ is a representative of $B\vee(A\wedge C)$ ([2], se comments preceding Proposition 5.6). Consequently, since

$$A_i\wedge(B_i\vee C_i) = B_i\vee(A_i\wedge C_i)\,,$$

for all integers $i\geqslant 1$, by the modularity of M, we obtain $A\wedge(B\vee C) = B\vee(A\wedge C)$. q.e.d.

PROPOSITION 5.7. *Let A, B be elements of M^*. Let $\langle A_i\rangle$ and $\langle B_i\rangle$ be the completely regular representatives of A and B, respectively. Then the sequence $\langle A_i\colon B_i\rangle$ is a representative of $A\colon B$.*

Proof. For each integer $i\geqslant 1$, we have

$$\begin{aligned} A_i\colon B_i &= (A_{i+1}\vee m^i\mathfrak{M})\colon(B_{i+1}\vee m^i\mathfrak{M})\\ &= \big((A_{i+1}\vee m^i\mathfrak{M})\colon B_{i+1}\big)\wedge\big((A_{i+1}\vee m^i\mathfrak{M})\colon m^i\mathfrak{M}\big)\\ &= (A_{i+1}\vee m^i\mathfrak{M})\colon B_{i+1}\geqslant A_{i+1}\colon B_{i+1}\,, \end{aligned}$$

since $\langle A_i\rangle$ and $\langle B_i\rangle$ are completely regular. Hence, the sequence $\langle A_i\colon B_i\rangle$ is decreasing, and thus is Cauchy by Lemma 5.4.

Now, let a in L^* denote the equivalence class determined by $\langle A_i\colon B_i\rangle$. Since $(A_i\colon B_i)B_i\leqslant A_i$, for all integers $i\geqslant 1$, it follows that $aB\leqslant A$ ([2], Proposition 5.10 and comments following Definition 5.13). Hence $a\leqslant A\colon B$. Suppose b is an element of L^* such that $bB\leqslant A$. Let $\langle b_i\rangle$ be the completely regular representative of b. Since the sequence $\langle b_iB_i\vee m^i\mathfrak{M}\rangle$ is the completely regular representative of bB ([2], Corollary 5.15), we have $b_iB_i\vee m^i\mathfrak{M}\leqslant A_i$, for all integers $i\geqslant 1$. Consequently,

$$b_iB_i = (b_iB_i\vee m^i\mathfrak{M})\wedge b_iB_i\leqslant A_i\wedge b_iB_i\leqslant A_i\,,$$

for all integers $i\geqslant 1$. So $b_i\leqslant A_i\colon B_i$, for all integers $i\geqslant 1$. Thus $b\leqslant a$, and hence $A\colon B\leqslant a$. Therefore $a = A\colon B$. q.e.d.

In order to establish that M^* is principally generated, we will need to establish a connection between principal elements of M and principal elements of M^*. This relation is provided by the following result.

THEOREM 5.8. *Let $\langle A_i\rangle$ be a Cauchy sequence of principal elements of M. Then the equivalence class determined by $\langle A_i\rangle$ is a principal element in M^* (considered as an L^*-module).*

Proof. Without loss of generality, we may clearly assume that $\langle A_i\rangle$ is a regular Cauchy sequence. Let B in M^* denote the equivalence class determined by $\langle A_i\rangle$, and let $\langle B_i\rangle$ be the completely regular representative of B. Since M^* and L^* are modular, we will use Lemma 2.1 to show that B is principal.

Let C be an element of M^*, and let $\langle C_i\rangle$ be the completely regular representative of C. For each integer $i\geqslant 1$, we have

$$\begin{aligned} C_i\wedge B_i &= C_i\wedge(A_i\vee m^i\mathfrak{M}) = m^i\mathfrak{M}\vee(C_i\wedge A_i) = m^i\mathfrak{M}\vee\big((C_i\colon A_i)A_i\big)\\ &= m^i\mathfrak{M}\vee\big((C_i\colon A_i)(A_i\vee m^i\mathfrak{M})\big)\\ &= m^i\mathfrak{M}\vee\big((C_i\colon A_i)\wedge(C_i\colon m^i\mathfrak{M})\big)(A_i\vee m^i\mathfrak{M})\\ &= m^i\mathfrak{M}\vee\big((C_i\colon(A_i\vee m^i\mathfrak{M}))(A_i\vee m^i\mathfrak{M})\big)\\ &= m^i\mathfrak{M}\vee\big((C_i\colon B_i)B_i\big) \end{aligned}$$

by ([2], Corollary 4.13 and Theorem 4.14). It follows that $C\wedge B = (C\colon B)B$ by Propositions 5.5, 5.7, and ([2], Corollary 4.6). Hence (2.1) of Lemma 2.1 is satisfied.

To see that B satisfies (2.2) of Lemma 2.1, let a be an element of L^* and let $\langle a_i\rangle$ be the completely regular representative of a. Then, for all integers $i\geqslant 1$, we have

$$\begin{aligned} (a_iB_i\vee m^i\mathfrak{M})\colon B_i &= \big(a_i(A_i\vee m^i\mathfrak{M})\vee m^i\mathfrak{M}\big)\colon(A_i\vee m^i\mathfrak{M})\\ &= (a_iA_i\vee m^i\mathfrak{M})\colon(A_i\vee m^i\mathfrak{M})\\ &= \big((a_iA_i\vee m^i\mathfrak{M})\colon A_i\big)\wedge\big((a_iA_i\vee m^i\mathfrak{M})\colon m^i\mathfrak{M}\big)\\ &= (a_iA_i\vee m^i\mathfrak{M})\colon A_i = a_i\vee(m^i\mathfrak{M}\colon A_i)\\ &= a_i\vee\big((m^i\mathfrak{M}\colon A_i)\wedge(m^i\mathfrak{M}\colon m^i\mathfrak{M})\big)\\ &= a_i\vee\big(m^i\mathfrak{M}\colon(A_i\vee m^i\mathfrak{M})\big) = a_i\vee(m^i\mathfrak{M}\colon B_i) \end{aligned}$$

because A_i is principal. Since $\langle (a_i B_i \vee m^i \mathfrak{M}) : B_i \rangle$ is a representative of $aB:B$ by Proposition 5.7 and ([2], Corollary 5.15), and since $\langle a_i \vee (m^i \mathfrak{M} : B_i) \rangle$ is a representative of $a \vee (0:B)$, it follows that $aB:B = a \vee (0:B)$. Thus B is a principal element of M^*. q.e.d.

We are now in a position to establish the main result of this section.

THEOREM 5.9. *L^* is a Noether lattice and M^* is a Noetherian L^*-module.*

Proof. We only need to establish that M^* is principally generated. If, for each pair B, C of elements of M^* such that $B < C$, we can construct a principal element A in M^* satisfying $A \leqslant C$ and $A \nleqslant B$, it will follow from the ascending chain condition in M^* ([2], Theorem 6.3) that M^* is principally generated.

Let B, C be elements of M^* such that $B < C$. Let $\langle B_i \rangle$ and $\langle C_i \rangle$ be the completely regular representatives of B and C, respectively. Since $B < C$, we have $B_i \leqslant C_i$, for all integers $i \geqslant 1$. Also, since $B \neq C$, there exists an integer n such that $B_i < C_i$, for all integers $i \geqslant n$. Then, in particular, $B_n < C_n$. Since M is principally generated, there exist a principal element A'_n in M such that $A'_n \leqslant C_n$ and $A'_n \nleqslant B_n$.

We shall now inductively construct a sequence of element $A'_{n+1}, A'_{n+2}, \ldots$, of M such that, for all integers $j \geqslant 1$,

$$A'_{n+j} \leqslant (A'_{n+j-1} \vee m^{n+j-1}\mathfrak{M}) \wedge C_{n+j}; \tag{5.1}$$

$$A'_{n+j} \nleqslant B_n; \tag{5.2}$$

and

$$A'_{n+j} \text{ is principal}. \tag{5.3}$$

We proceed as follows. Suppose that

$$(A'_n \vee m^n \mathfrak{M}) \wedge C_{n+1} \leqslant B_n. \tag{5.4}$$

Then, since $\langle C_i \rangle$ is a completely regular sequence, we obtain

$$\begin{aligned} A'_n \leqslant (A'_n \vee m^n \mathfrak{M}) \wedge C_n &= (A'_n \vee m^n \mathfrak{M}) \wedge (C_{n+1} \vee m^n \mathfrak{M}) \\ &= \big((A'_n \vee m^n \mathfrak{M}) \wedge C_{n+1}\big) \vee m^n \mathfrak{M} \leqslant B_n \end{aligned}$$

by (5.4) and the modularity of M. This is a contradiction to $A'_n \nleqslant B_n$ above. Hence

$$(A'_n \vee m^n \mathfrak{M}) \wedge C_{n+1} \nleqslant B_n. \tag{5.5}$$

Since M is principally generated, and by (5.5), there exists a principal element A'_{n+1} such that (i) $A'_{n+1} \leqslant (A'_n \vee m^n \mathfrak{M}) \wedge C_{n+1}$, and (ii) $A'_{n+1} \nleqslant B_n$. Thus A'_{n+1} satisfies (5.1), (5.2), and (5.3).

Now, assume that $A'_{n+1}, \ldots, A'_{n+k}$ have been chosen so that, for each j, $1 \leqslant j \leqslant k$, the conditions in (5.1), (5.2), and (5.3) have been satisfied. In a manner similar to the case for $j = 1$, it may be shown that

$$(A'_{n+k} \vee m^{n+k} \mathfrak{M}) \wedge C_{n+k} \nleqslant B_n.$$

Hence, as above, there exists a principal element A'_{n+k+1} such that

$$A'_{n+k+1} \leqslant (A'_{n+k} \vee m^{n+k} \mathfrak{M}) \wedge C_{n+k+1} \tag{5.6}$$

and

$$A'_{n+k+1} \nleqslant B_n. \tag{5.7}$$

Consequently, (5.6), (5.7) imply that A_{n+k+1} satisfies (5.1), (5.2), (5.3).

Next, set $A'_i = A'_n$, for $1 \leqslant i \leqslant n$. Since the sequence $\langle A'_i \rangle$, $i = 1, 2, \ldots$, satisfies condition (5.1), it follows that $\langle A'_i \rangle$ is a Cauchy sequence by Lemma 5.4. Since each A'_i is principal by (5.3), we have that the equivalence class A in M^* determined by $\langle A'_i \rangle$ is a principal element of M^* (considered as an L^*-module). Since $A'_i \leqslant C_i$, for all integers $i \geqslant 1$, we obtain $A \leqslant C$. We now only need to show that $A \nleqslant B$ in order to complete our proof. So, assume $A \leqslant B$. Select a regular subsequence $\langle A''_i \rangle$ from the sequence $\langle A'_i \rangle$ ([2], Lemma 4.11). Then $A''_i \leqslant A''_i \vee m^i \mathfrak{M} \leqslant B_i \leqslant B_n$, for all integers $i \geqslant n$, by ([2], Corollary 4.13). Hence $A'_i \leqslant B_n$, for large i, which is a contradiction to (5.2). Thus $A \nleqslant B$. q.e.d.

§ 6. Contractions. In order to establish our main theorem we will need the following preliminary result.

THEOREM 6.1. *Let $(L, p_1, \ldots, p_n)$ be a semi-local Noether lattice, let M be a Noetherian L-module, let m be the Jacobson radical of L, let M^* be the m-adic completion of M, and let A be an element of M. Then*

$$(C \frown M)M^* = C, \tag{6.1}$$

for all C in M^* such that $(mA)M^* \leqslant C \leqslant AM^*$.

Proof. Let C be an element of M^* such that $(mA)M^* \leqslant C \leqslant AM^*$. Let $\langle C_i \rangle$ be the completely regular representative of C. Since the sequence $\langle C_i \rangle$ is decreasing, the sequence $\langle C_i \wedge A \rangle$ is decreasing, and hence is Cauchy (Lemma 5.4).

We shall now show that the equivalence class determined by the sequence $\langle C_i \wedge A \rangle$ is C. Since

$$\langle C_i \wedge A \rangle \sim \langle (C_i \wedge A) \vee m^i \mathfrak{M} \rangle,$$

and since

$$(C_i \wedge A) \vee m^i \mathfrak{M} = C_i \wedge (A \vee m^i \mathfrak{M}),$$

for all integers $i \geqslant 1$, it is sufficient to show that $\langle C_i \wedge (A \vee m^i \mathfrak{M}) \rangle$ is a representative of C. Since $\langle C_i \rangle$ is the completely regular representative of C and since $\langle A \vee m^i \mathfrak{M} \rangle$ is the completely regular representative of AM^* ([2], Remark 5.2), it follows that the sequence $\langle C_i \wedge (A \vee m^i \mathfrak{M}) \rangle$ is a representative of $C \wedge AM^*$ (Proposition 5.5). But $C \wedge AM^* = C$ since $C \leqslant AM^*$.

Since we know that $(D \frown M)M^* \leqslant D$, for all D in M^* ([2], Proposition 7.5), in order to prove the theorem, it is sufficient to show that

$(C \cap M)M^* \geqslant C$. This shall now be established. Since $(mA)M^* \leqslant C \leqslant AM^*$, we have that

(6.2) $$mA \leqslant mA \vee m^i\mathfrak{M} \leqslant C_i \leqslant A \vee m^i\mathfrak{M}\,,$$

for all integers $i \geqslant 1$. Hence, from (6.2), we obtain

(6.3) $$mA = mA \wedge A \leqslant C_i \wedge A \leqslant (A \vee m^i\mathfrak{M}) \wedge A = A\,,$$

for all integers $i \geqslant 1$. Since $[mA, A]$ is finite dimensional (Theorem 5.1), it follows from (6.3) that there exists a natural number k such that

(6.4) $$C_i \wedge A = C_k \wedge A\,,$$

for all integers $i \geqslant k$. Since $\langle C_i \wedge A\rangle$ is a representative of C, we obtain from (6.4) that

(6.5) $$(C_k \wedge A)M^* = C\,.$$

Since

$$C_i \geqslant C_i \wedge A = C_k \wedge A\,,$$

for all $i \geqslant k$, we have that

(6.6) $$\bigwedge_i C_i \geqslant \bigwedge_i (C_i \wedge A) = C_k \wedge A\,.$$

Consequently,

$$(C \cap M)M^* = (\bigwedge_i C_i)M^* \geqslant \big(\bigwedge_i (C_i \wedge A)\big)M^* = (C_k \wedge A)M^* = C$$

by (6.5), (6.6), and ([2], Corollary 5.11). q.e.d.

We are now in a position to determine some structural properties of L^*.

THEOREM 6.2. *Let $(L, p_1, \ldots, p_n)$ be a semi-local Noether lattice, let m be the Jacobson radical of L, let L^* be the m-adic completion of L, and let m^* be the Jacobson radical of L^*. Then,*

(6.7) *L^* is a semi-local Noether lattice with maximal elements* $$p_1L^*, \ldots, p_nL^*;$$

(6.8) $$m = m^* \cap L\,;$$

and

(6.9) $$(p_1 \wedge \ldots \wedge p_n)L^* = mL^* = m^* = p_1L^* \wedge \ldots \wedge p_nL^*.$$

Proof. First, we show that each p_kL^*, $1 \leqslant k \leqslant n$, is maximal in L^*. Choose k such that $1 \leqslant k \leqslant n$. Since the extension map is one-to-one, $p_kL^* \neq IL^*$. Let b be an element of L^* such that $p_kL^* \leqslant b \leqslant IL^*$ and $b \neq IL^*$. Let $\langle b_i\rangle$ be the completely regular representative of b. Since $\langle p_k \vee m^i\rangle$, $i = 1, 2, \ldots$, is the completely regular representative of p_kL^*, we have

$$p_k \leqslant p_k \vee m^i \leqslant b_i \leqslant I\,,$$

for all integers $i \geqslant 1$. Since $b \neq IL^*$, there exists a natural number N such that $b_i < I$, for all integers $i \geqslant N$. It follows that $p_k = b_i$, for all integers $i \geqslant N$. Hence $p_kL^* = b$, and p_kL^* is maximal.

Next, suppose b is an element of L^* such that $b \neq IL^*$ and b is maximal. Let $\langle b_i\rangle$ be the completely regular representative of b. Since $b \neq IL^*$, there exists a natural number N such that $b_i < I$, for all integers $i \geqslant N$. Consequently, there exists a natural number k, $1 \leqslant k \leqslant n$, such that $b_i \leqslant p_k$, for infinitely many integers $i \geqslant N$. It follows that $b \leqslant p_kL^*$. Hence $b = p_kL^*$, since p_kL^* is maximal. Thus (6.7) has been established.

Now, since $m = p_1 \wedge \ldots \wedge p_n \leqslant p_k$, for each integer k, $1 \leqslant k \leqslant n$, we have that $mL^* \leqslant p_kL^*$, for each integer k, $1 \leqslant k \leqslant n$. Thus

$$mL^* \leqslant p_1L^* \wedge \ldots \wedge p_nL^* = m^* \leqslant p_kL^*,$$

for each integer k, $1 \leqslant k \leqslant n$. Consequently,

$$m = mL^* \cap L \leqslant m^* \cap L \leqslant p_kL^* \cap L = p_k\,,$$

for each integer k, $1 \leqslant k \leqslant n$, ([2], Propositions 7.2 and 7.6). Thus $m \leqslant m^* \cap L \leqslant \bigwedge_{i=1}^{n} p_i = m$. Hence $m^* \cap L = m$. Thus (6.8) has been established.

Since $p_1L^* \wedge \ldots \wedge p_nL^* = m^*$ by (6.7), and since $p_1 \wedge \ldots \wedge p_n = m$, in order to establish (6.9), we need only show that $m^* = mL^*$. From (6.8) we obtain $mL^* = (m^* \cap L)L^* = m^*$ by Theorem 6.1. q.e.d

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