Lattice modules over semi-local Noether lattices

by

E. W. Johnson (Iowa City, Ia.) and J. A. Johnson (Houston, Tex.)

§ 1. Introduction. For Noetherian lattice modules, the concept of the \( a \)-adic pseudometric has been introduced and studied in [2] and [3]. Recently the natural completion of a local Noether lattice was related to the completeness of a local ring in its natural topology ([1]). The purpose of this paper is to establish some properties of Noetherian lattice modules over semi-local Noether lattices and their completions.

The basic concepts are introduced in § 2, and some preliminary results are obtained. Let \( L \) be a multiplicative lattice and let \( M \) be a Noetherian \( L \)-module. In § 3 an interesting property concerning certain sequences in \( M \) is established (Theorem 3.2). If \( L \) is a Noether lattice and \( m \) is the Jacobson radical of \( L \), then it is shown (Corollary 3.4) that the \( m \)-adic pseudometric on \( M \) is a metric ([2], § 3). § 4 contains some results on dimensions. If \( L \) is semilocally, it is shown in § 5 that \([m, A] \) is finite dimensional, for all \( A \) in \( M \) (Theorem 5.1), \( L^* \) is a Noether lattice, and \( M^* \) is a Noetherian \( L^* \)-module (Theorem 5.9), where \( L^* \) and \( M^* \) are the \( m \)-adic completions of \( L \) and \( M \), respectively ([3], § 6). In § 6 it is established that \( L^* \) is a semi-local Noether lattice whose maximal elements are extensions ([2], § 5) of the maximal elements of \( L \).

§ 2. Preliminary remarks. By a multiplicative lattice we shall mean a complete lattice on which there is defined a commutative, associative, join distributive multiplication such that the null element of the lattice is an identity for the multiplication. Let \( L \) be a multiplicative lattice and let \( M \) be a complete lattice. We shall denote elements of \( L \) by \( a, b, c, \ldots \), with the exception that the null element and unit element of \( L \) will be denoted by \( 0 \) and \( I \), respectively. We shall denote elements of \( M \) by \( A, B, C, \ldots \), with the exception that the null element and unit element of \( M \) will be denoted by \( 0_M \) and \( M \), respectively. When no confusion is possible, \( 0 \) will also be used in place of \( 0_L \). Recall that \( M \) is an \( L \)-module ([2], Definition 2.3) in case there is a multiplication between elements of \( L \) and \( M \), denoted by \( a \cdot a \) for \( a \) in \( L \) and \( A \) in \( M \), which satisfies:

(i) \( (ab) \cdot A = a(b \cdot A) \),
(ii) \( A_0 B_0 \) = \( \bigvee \alpha_0 \bigvee B_0 \),
(iii) \( IA = A \) and
(iv) \( \alpha A = 0 \) for all \( A, B, \alpha \) in \( L \) and for all \( A, B_0 \) in \( M \).
Let \( M \) be an \( L \)-module. For \( a, b \in L \) and for \( A, B \in M \), (i) \( a \triangleleft b \) denotes the largest \( c \in L \) such that \( cb \leq a \); (ii) \( A : B \) denotes the largest \( c \in L \) such that \( cB \leq A \). An element \( A \in M \) is said to be meet principal in case \( (b \land (B : A))A = bA \triangleleft B \) for all \( b \in L \) and for all \( B \in M \); \( A \) is said to be join principal in case \( b \lor (B : A)A = (b \lor (B : A))A \) for all \( b \in L \) and for all \( B \in M \); and, \( A \) is said to be principal in case \( A \) is both meet and join principal.

If each element of \( M \) is the join (finite or infinite) of principal elements, \( M \) is called principally generated. \( M \) is said to be Noetherian if \( M \) satisfies the ascending chain condition, is modular, and is principally generated.

If \( M \) is a Noetherian \( L \)-module, \( L \) is called a Noetherian lattice. For other general properties and definitions concerning Noetherian lattice modules, the reader is referred to [3].

In the special case where \( M \) and \( L \) are both modular, we can prove the following characterization of principal elements which will be useful.

**Lemma 2.1.** Let \( M \) be an \( L \)-module and let \( A \) be an element of \( M \). If \( M \) and \( L \) are modular, then \( A \) is a principal element of \( M \) if and only if

\begin{equation}
C \land A = (C : A)A
\end{equation}

and

\begin{equation}
b \land A = b \lor (0 : A)
\end{equation}

for all \( b \in L \) and for all \( C \in M \).

*Proof.* Assume \( M \) and \( L \) are modular. Suppose that \( A \) is principal. Then clearly \( A \) satisfies (2.1) and (2.2). Conversely, assume that \( A \) satisfies (2.1) and (2.2). Then, for \( b \in L \) and \( C \in M \), we have

\[
b \land (C : A)A = \left\{ b \land (C : A) \lor (0 : A) \right\}A = \left\{ (C : A) \land (b \lor (0 : A)) \right\}A = \left\{ (C : A) \land bA \right\}A = C \land bA
\]

by the modularity of \( L \). Also,

\[
(C \lor bA)A = \left\{ (C \lor bA) \lor (0 : A) \right\}A = \left\{ (C \lor A) \lor bA \right\}A = \left\{ (C \lor A) \lor b \right\}A = (C \lor A) \lor b
\]

by the modularity of \( M \). It follows that \( A \) is both meet and join principal, and hence principal.

We will also need the following result.

**Lemma 2.2.** Let \( M \) be an \( L \)-module. Let \( a \) be a principal element of \( L \) and let \( A \) be a principal element of \( M \). Then \( aA \) is a principal element of \( M \).

*Proof.* Let \( b \in L \) and let \( B \in M \). Then \( b \land A = b \lor (0 : A) \) and \( b \lor A = b \lor (0 : A) \). Therefore, \( aA \) is a principal element of \( M \).

In later parts of this paper we will need to use a generalization of quotient lattices. This construction is developed in Remark 2.3 below. If \( A, B \) are elements of a lattice \( K \) with \( A \leq B \), then the set \( \{ D \in K \mid A \leq D \leq B \} \) is a sublattice of \( K \) which will be denoted by \( [A, B] \). If \( K \) is a complete lattice with unit element \( 1 \), then for arbitrary \( A \in K \), we will also write \( 1 \lor A \) in place of \( [A, 1] \).

**Remark 2.3.** Let \( M \) be an \( L \)-module, let \( A, B \) elements of \( M \) with \( A \leq B \), and let \( a \) be an element of \( L \) such that \( aA \leq A \), for all \( C \) in \( [A, B] \). Then \( aA \) is "naturally" a multiplicative lattice and \( [A, B] \) is "naturally" an \([a, 1]\)-module.

*Proof.* For \( b, c \in [a, 1] \), define \( b \times c = b \lor c \lor a \). For \( C \in [A, B] \) and \( b \in [a, 1] \), define \( b \times C = b \lor C \lor a \). Since \( M \) and \( L \) are both complete lattices, it follows immediately that \( [A, B] \) and \( [a, 1] \) are complete lattices. It is easily verified that the above definition of multiplication makes \([a, 1]\) into a multiplicative lattice and \([A, B] \) into an \([a, 1]\)-module. The computations will be omitted.

**Remark 2.4.** Let \( M \) be a Noetherian \( L \)-module, let \( A, B \in M \), and let \( a \) be an element of \( L \) such that \( aC \leq A \), for all \( C \in [A, B] \). Then, with respect to the "natural" multiplications given in Remark 2.3, \([A, B] \) becomes a Noetherian \([a, 1]\)-module.

*Proof.* This is a straightforward computation. The details will be omitted. The reader is referred to ([2], Remark 2.8).

**Remark 2.5.** Let \( M \) be an \( L \)-module, let \( A \) be a principal element of \( M \), let \( a, b \) be elements of \( L \) such that \( a \leq b \), let \( L \) be modular, and let \( 0 \times (0 : A) \leq a \). Then the map \( \psi : [a, b] \rightarrow [a, bA] \) defined by \( \psi(x) = aA \) is a lattice isomorphism of \([a, b] \) onto \([a, bA] \).

*Proof.* Let \( x, y \in [a, b] \) and assume \( \psi(x) = \psi(y) \). Then \( aA = yA \). Hence \( aA = yA \). Consequently, since \( A \) is principal, \( aA = yA \).

*Proof.* Let \( x, y \in [a, b] \) and assume \( \psi(x) = \psi(y) \). Then \( aA = yA \). Hence \( aA = yA \). Consequently, since \( A \) is principal, \( aA = yA \).

*Proof.* Let \( x, y \in [a, b] \) and assume \( \psi(x) = \psi(y) \). Then \( aA = yA \). Hence \( aA = yA \). Consequently, since \( A \) is principal, \( aA = yA \).
by the modularity of $L$. It follows that $\varphi$ is one-to-one. To see that $\varphi$ is onto, let $B$ be an element of $\{aA, bA\}$. Since

$$a \in aA; A \subseteq a\varphi(0; A) \subseteq B; A \subseteq bA; A = b\varphi(0; A),$$

we have that

$$a = aB \subseteq bB = b\varphi(B; A) = b\varphi(b\varphi(0; A)) = b.$$ 

Thus $b\varphi(B; A)$ is an element of $[a, b]$. Applying $\varphi$ we obtain

$$\varphi(b\varphi(B; A)) = \varphi(b\varphi(B; A))A = bA \cup B = B,$$

since $A$ is principal, and consequently $\varphi$ is onto. Since $\varphi$ is clearly order preserving, we have that $\varphi$ is a lattice isomorphism of $[a, b]$ onto $[aA, bA]$. q.e.d.

§ 3. A preliminary theorem.

**Definition 3.1.** Let $L$ be a multiplicative lattice and let $M$ be a Noetherian $L$-module. For $a$ in $L$ and $A$ in $M$, let $T(a, A)$ be the collection of all sequences $(B_i)$, $i = 1, 2, \ldots$, of elements of $M$ satisfying

$$aA \supseteq B_1 \supseteq B_{i+1} \supseteq aB_i,$$

for all integers $i \geq 1$. For $(C_i)$ and $(B_i)$ in $T(a, A)$, define

$$\langle C_i \rangle \leq \langle B_i \rangle$$

if and only if $C_i \subseteq B_i$, for all integers $i \geq 1$. (3.3)

$$\langle C_i \rangle \cap \langle B_i \rangle = \langle C_i \cap B_i \rangle$$

(3.4)

$$\langle C_i \rangle = \langle B_i \rangle = \langle C_i \cup B_i \rangle.$$ 

It is easily seen that $T(a, A)$ forms a complete, modular lattice under the relation $\leq$ with the resulting join and meet being given by (3.3) and (3.4). The resulting lattice will be denoted by $K(a, A)$.

**Theorem 3.2.** Let $L$ be a multiplicative lattice, and let $M$ be a Noetherian $L$-module, let $aB$ be an element of $L$, let $A$ be an element of $M$, and let $(B_i)$, $i = 1, 2, \ldots$, be an element of $E(a, A)$. Then there exists a natural number $n$ such that $B_n = aB_n$, for all integers $m \geq n$ and for all integers $i \geq 0$.

Proof. Let $E(a, A)$ be the collection of all sequences $(B_i)$ in $K(a, A)$ for which the theorem fails. Assume that $F(a, A) \neq \emptyset$. We shall show that $F(a, A)$ has maximal elements.

Let $C$ be a chain in $E(a, A)$. For each $C$ in $C$, let $C_i$ be the $i$th coordinate of $C$. For each natural number $i$, set $S_i = \bigvee \{C_i; C \in C\}$.

Suppose $(S_i)$ is not an element of $E(a, A)$. Then there exists a natural number $m$ such that $S_{m+1} = a'S_{m+1}$ for all integers $k \geq n$ and for all integers $i \geq 0$. Since $M$ satisfies the ascending chain condition, for each $i$, $1 \leq i \leq n$, there exists an element in $C$ with $i$th coordinate $S_i$. Select one and call it $B_i$.

Set $(B_i) = \max(B(1), \ldots, B(n))$. Thus $(B_i)$ is in $E(a, A)$, and also $B_i = S_i$, for $1 \leq i \leq n$. In particular $B_n = S_n$. Consequently,

$$S_{n+i} = a'S_{n+i} = a'B_n \subseteq S_{n+i},$$

for all integers $i \geq 0$. It follows that $B_{n+i} = S_{n+i}$, for all integers $i \geq 0$. Thus $B_i = S_i$, for all integers $i \geq 1$, and consequently $(S_i)$ is in $E(a, A)$, which is a contradiction to the assumption that $(S_i)$ is not an element of $E(a, A)$. Thus $C$ has an upper bound and hence $E(a, A)$ has maximal elements by Zorn's Lemma.

Let $(C_i)$ be a maximal element of $E(a, A)$. By definition, we know $F_i \leq aA$. Also, if $F_i = aA$, then $F_{i+1} = aF_i$, for all integers $i \geq 0$, and hence $(C_i)$ would not be in $E(a, A)$. Hence $F_i \not= aA$. Thus, there exists a principal element $E$ of $M$ such that $E \leq aA$ and $E \not= F_i$. It follows that $F_i \leq E \leq a$. (3.5)

Now, define $(D_i)$ by $D_i = F_i \cup a'E$, for all integers $i \geq 1$. Observe that $F_i \leq D_i = F_i \cup E \leq aA$ and that $(D_i)$ is an element of $E(a, A)$. Hence $(F_i) \leq (D_i)$, and $(D_i)$ is not in $E(a, A)$. Consequently, there is a natural number $n$ such that $D_{n+i} = a'D_n$, for all integers $k \geq n$ and for all integers $i \geq 0$. Hence

$$F_{n+i} \cup a'E = D_{n+i} = a'D_n = aF_n \cup a'E = aF_n \cup a'E = E,$$

for all integers $k \geq n$ and for all integers $i \geq 0$. Therefore, since $M$ is modular and $E$ is principal, we obtain

$$F_{n+i} = aF_n \cup a'E = aF_n \cup (aE \cup a'E) = aF_n \cup (aE \cup a'E),$$

(3.6)

$$F_{n+i} = aF_n \cup (aE \cup a'E),$$

for all integers $k \geq n$ and for all integers $i \geq 0$. Next, for each integer $i \geq 1$, set $H_i = (F_{i+1} \setminus aA)$. It follows from (3.5) that

$$F_{n+i} = aF_n \cup (aE \cup a'E),$$

for all integers $k \geq n$ and for all integers $i \geq 0$. It is easily verified that $(H_i)$ is an element of $E(a, A)$.

Assume for a moment that $M = L$. Then, it is easily seen that $(E_i)$ is in $E(a, A)$. Furthermore, if $(E_i) = (B_i)$, then

$$H_i = (E_{i+1} \setminus aA) = F_i,$$

for all integers $i \geq 1$. Thus, since $E \leq aA$, we have by (3.6) and (3.7) that

$$F_{n+i} = aF_n \cup (aE \cup a'E) = aF_n \cup (aE \cup a'E),$$

(3.8)

$$F_{n+i} = aF_n \cup (aE \cup a'E),$$

for all integers $k \geq n$ and for all integers $i \geq 0$. 

Then, since \( aP_m \geq P_{m+1} \), for all integers \( m \geq 1 \) and for all integers \( i \geq 0 \), it follows from (3.8) that \( aP_k = P_{k+1} \), for all integers \( k \geq n \) and for all integers \( i \geq 0 \), which contradicts the fact that \( (P_i) \) is in \( P(a, I) \). Hence \( (P_i) \in (H_i) \), and consequently, there exists a natural number \( m \geq n \) such that

\[
H_{k+1} = aH_k,
\]

for all integers \( k \geq m \) and all integers \( i \geq 0 \), by the maximality of \( (P_i) \).

Consequently, since \( E \) is principal, we have by (3.8) and (3.9) that

\[
F_{k+i+1} = a^{i+1}F_k \vee H_{k+i} = a^{i+1}F_k \vee H_{k+i-1} = a^{i+1}F_k \vee aH_{k+i-1}
\]

\[
= a^{i+1}F_k \vee a[(F_{k+i-1} \cap aH_{k+i-1})]
\]

\[
= a^{i+1}F_k \vee a(a^{i+1-1}E \vee F_{k+i}) \leq aE_{k+i},
\]

for all integers \( k \geq m \) and all integers \( i \geq 0 \). As above, for (3.8), this implies that \( F_{k+i} = aF_k \), for all integers \( k \geq m \) and all integers \( i \geq 0 \), in contradiction to \( (P_i) \) being in \( P(a, I) \). Hence, when \( M = L \), we have \( P(a, I) = \emptyset \).

We return now to the general case. Since \( (H_i) \) is in \( P(a, I) \), and since \( P(a, I) = \emptyset \), there exists a natural number \( n \) such that \( H_{k+i} = aH_k \), for all integers \( k \geq n \) and for all integers \( i \geq 0 \). Then by (3.8), we have that

\[
F_{k+i+1} = a^{i+1}F_k \vee H_{k+i} = a^{i+1}F_k \vee aH_{k+i} \leq aE_{k+i},
\]

for all integers \( k \geq n \) and all integers \( i \geq 0 \). This again implies that \( aE_{k+i} = a^iE_k \), for all integers \( k \geq n \) and for all integers \( i \geq 0 \). Thus \( (P_i) \) is not in \( P(a, A) \), which is a contradiction. Hence \( P(a, A) = \emptyset \) in the general case.

For a Noether lattice \( L \), recall that an element \( a \) in \( L \) is maximal if \( a \neq I \) and if \( b \geq a \) implies \( b = I \). Also recall that the Jacobson radical of \( L \) is the inf of all such maximal elements of \( L \).

**Corollary 3.3.** Let \( L \) be a Noether lattice, \( M \) be a Noetherian \( L \)-module, \( m \) be the Jacobson radical of \( L \), \( B \) be a element of \( M \), and \( a \) be an element of \( L \) such that \( a \leq m \). Then \( aB = 0 \).

**Proof.** Let \( C \) be a principal element of \( M \) such that \( C \leq aB \).

Then, for all integers \( n \geq 1 \), we have \( C = C \cup aB \).

We shall show that \( C = 0 \). Consider the sequence \( (C \cup aB, i = 1, 2, \ldots) \). Since

\[
aB \geq C \cup aB \geq C \cup a^{i+1}B \geq a(C \cup aB),
\]

for all integers \( i \geq 1 \), it follows from Theorem 3.2 that there exists a natural number \( k \) such that

\[
C \cup a^{k+1}B = a(C \cup a^kB),
\]

for all integers \( i \geq 0 \). Hence \( C = aC \), for all integers \( i \geq 0 \). In particular \( C = 0 \). Thus, since \( C \) is principal, we have

\[
I = C = aC = 0 \vee (0; C).
\]

Since \( a \leq m \), it must be that \( 0 = I \). Consequently, \( I = 0 \).

Since \( M \) is a Noetherian \( L \)-module, every element is principally generated. It follows that \( \bigwedge_a aB = 0 \).

**Corollary 3.4.** Let \( L \) be a Noether lattice, \( M \) be a Noetherian \( L \)-module, \( m \) be the Jacobson radical of \( L \), and let \( a \) be an element of \( L \) such that \( a \leq m \). Then

\[
A = \bigwedge_a (A \cup aB), \quad \text{for all } A \in M,
\]

and

\[
\text{the } a \text{-adic pseudometric on } M \text{ is a metric}.
\]

**Proof.** Let \( A \) be an element of \( M \). Then \( (A \cup aB) \) is a Noetherian \( L \)-module by Remark 2.4, and \( a \leq m \). Thus

\[
A = \bigwedge_a (A \cup aB) = (A \cup aB)
\]

by Corollary 3.3. Hence (3.10) has been established. (3.11) follows from (3.10) and ([2], Theorem 3.10).

**§ 4. Some results on dimensions.** In this section some results are established concerning dimensions of various lattices. These results will be needed later.

**Theorem 4.1.** Let \( L \) be a Noether lattice and let \( a \) be an element of \( L \). Then there exist primes \( p_1, \ldots, p_n \) in \( L \) such that \( p_1p_2 \cdots p_n \leq a \).

**Proof.** Let \( F(L) \) be the collection of all elements in \( L \) for which the theorem fails. Suppose \( F(L) \) is not empty. Then \( F(L) \) has a maximal element \( b \). Clearly \( b \) is not prime. Since \( b \) is not prime, there exist elements \( a, \beta \) in \( L \) such that \( ab \leq b, \beta b \leq b, \text{ and } \beta \leq b \). Consequently, \( ab \beta > b \) and \( a \beta b > b \). Thus, since \( b \) is maximal in \( F(L) \) there exist primes \( p_1, \ldots, p_n, \beta_1, \ldots, \beta_m \) in \( L \) such that

\[
p_1p_2 \cdots p_n \leq ab \beta \text{ and } p_1p_2 \cdots p_n \leq a \beta b.
\]

It follows that

\[
(p_1p_2 \cdots p_n)(p_1p_2 \cdots p_n) \leq (ab \beta)(ab \beta) = ab \beta b \beta b \leq b,
\]

which is a contradiction to the maximality of \( b \). Hence \( F(L) \) is empty. q.e.d.

**Lemma 4.2.** Let \( L \) be a local Noether lattice with unique maximal element \( p \), and let \( M \) be a Noetherian \( L \)-module. Then, for each \( A \in M \), the lattice \( [p, A] \) is finite dimensional.
Proof. Let $A$ be an element of $M$. Since $M$ is Noetherian, there exists principal elements $A_1, \ldots, A_n$ in $M$ such that $A = A_1 \vee \ldots \vee A_n$. Let $S_i = pA_i$ and, for each $i$, $0 \leq i \leq n-1$, set $S_{i+1} = S_i / A_{i+1}$. Since each $A_i$ is principal, for each $i$, $0 \leq i \leq n-1$, we obtain

\[(S_i, S_{i+1}) = (S_i, S_i / A_{i+1}) \cong (S_i / A_{i+1}, A_{i+1})\]

\[= ([S_i : A_{i+1}]A_{i+1}, A_{i+1}) \cong ([S_i : A_{i+1}], A_i)\]

by the isomorphism theorems and Lemma 2.5. Since

$\exists [S_i : A_{i+1}]$, it follows that $p \leq S_i / A_{i+1}$, for $0 \leq i \leq n-1$. Hence, the dimension of $[S_i : A_{i+1}, I]$ is either one or zero. Hence $[S_i : A_{i+1}, I]$ is finite dimensional, $0 \leq i \leq n-1$, by (4.1). Since $pA_i = S_i \leq S_i \leq \ldots \leq S_n = A$, we have $[pA_i, A]$ is also finite dimensional.

**THEOREM 4.3.** Let $L$ be a Noether lattice. If $0$ is a product of maximal elements, then $L$ is finite dimensional.

Proof. Assume $0 = p_1p_2 \ldots p_n$ where $p_i$ is prime and (hence prime). For each $i$, $2 \leq i \leq n$, we know that $[p_1, I]$ is a Noether lattice and that $[p_1, p_2 \ldots p_n, p_{i+1}]$ is a Noetherian $[p_1, I]$-modul by Remark 2.4. Thus, since each $[p_1, I]$ is local, we have that $[p_1, p_2 \ldots p_n, p_{i+1}]$ is finite dimensional, $0 \leq i \leq n$, by Lemma 4.2. Simplifying this expression we obtain $[p_1, p_2 \ldots p_n, p_{i+1}]$ is finite dimensional, $2 \leq i \leq n$. Since $L > p_1p_2 \ldots p_n$, it follows that $L$ is finite dimensional.

**COROLLARY 4.4.** Let $L$ be a Noether lattice. If every (proper) prime element of $L$ is maximal, then $L$ is finite dimensional.

Proof. Assume every (proper) prime element of $L$ is maximal. By Theorem 4.1 there exists principal elements $p_1, \ldots, p_n$ in $M$ such that $p_1p_2 \ldots p_n < 0$. Hence $0 = p_1p_2 \ldots p_n$, where each $p_i$ is prime, and hence maximal by hypothesis. Thus, by Theorem 4.3, $L$ is finite dimensional.

A Noether lattice is said to be semi-local if it has only finitely many maximal elements. If $L$ is a semi-local Noether lattice with maximal elements $p_1, \ldots, p_n$, we will say that $(L, p_1, \ldots, p_n)$ is a semi-local Noether lattice.

**COROLLARY 4.5.** Let $(L, p_1, \ldots, p_n)$ be a semi-local Noether lattice, and let $m$ be the Jacobson radical of $L$. Then $[m, I]$ is finite dimensional.

Proof. Each (proper) prime element of $[m, I]$ is maximal.

§ 5. $m$-adic completions. Throughout this section $(L, p_1, \ldots, p_n)$ is a semi-local Noether lattice, $M$ is a Noetherian $L$-module, and $m$ is the Jacobson radical of $L$. Since $L$ is semi-local, clearly $m = p_1 \wedge \ldots \wedge p_n$.

By Corollary 3.4, the $m$-adic pseudometric $(2.2, \xi)$ on $M$ and the $m$-adic pseudometric on $L$ are metrics. Consequently, the $m$-adic completions of $M$ and $L$ are defined $(2.2, \xi)$. Throughout this section, $M^\ast$ shall denote the $m$-adic completion of $M$, and $L^\ast$ shall denote the $m$-adic completion of $L$. It is known that $M^\ast$ is an $L^\ast$-module under the assumptions stated above. We begin with the following result.

**THEOREM 5.1.** For each $A$ in $M$, the quotient $[mA, A]$ is finite dimensional.

Proof. Let $A$ be an element of $M$. Since $M$ is principally generated, there exists principal elements $A_1, \ldots, A_n$ in $M$ such that $A = A_1 \vee \ldots \vee A_n$. Set $S_i = mA_i$ and, for each $i$, $0 \leq i \leq n-1$, set $S_{i+1} = S_i / A_{i+1}$. Then, proceed as in the proof of Lemma 4.2 to obtain $[S_i, S_{i+1}] \cong [S_i : A_{i+1}, I]$, $0 \leq i \leq n-1$.

Now observe that $m \leq S_i / A_{i+1}$ and that $[m, I]$ is finite dimensional.

**COROLLARY 5.2.** For each $A$ in $M$, $[mA, A]$ is finite dimensional, for each natural number $m$.

Proof. Let $A$ be an element of $M$. Since $mA \leq m^{a-1}A \leq \ldots \leq mA \leq A$, and since each quotient $[mA, m^{a-1}A]$, $A \leq n$, is finite dimensional by Theorem 5.1, the result follows.

**COROLLARY 5.3.** For each natural number $n$, the quotient $L/m^n$ is finite dimensional.

Proof. $L$ is a Noetherian $L$-module.

In order to work with "inf" and "residuals" in $M$, it will be necessary to determine representatives of these elements. The following lemma will prove helpful in this respect. It is needed in the proof of Proposition 5.3 and 5.7.

**LEMMA 5.4.** Let $A_i$, $i = 1, 2, \ldots, n$, be a sequence of elements of $M$ satisfying $A_i \leq A_i \vee m^nA_i$ for all integers $i > 0$. Then the sequence $\langle A_i \rangle$ is Cauchy.

Proof. Set $n > 1$. Since the sequence $\langle A_i \vee m^nA_i \rangle$, $i = 1, 2, \ldots$, is decreasing in $i$ and, since $mA_i \leq A_i \vee m^nA_i$, for each integer $i > 1$, it follows from Corollary 5.2 that there exists a natural number $g$ such that $A_i \vee m^nA_i \leq A_i \vee m^nA_i$ for all integers $i, j > g$. Consequently, $d_m(A_i, A_j) < 2^{-n}$, for all integers $i, j > g$.

q.e.d.
PROPOSITION 5.5. Let \( B, C \) be elements of \( M^* \). Let \((B_i)\) and \((C_i)\) be the completely regular representatives of \( B \) and \( C \), respectively. Then the sequence \((B_i \cap C_i)\) is a representative of \( B \cap C \).

Proof. Since \((B_i)\) and \((C_i)\) are decreasing, the sequence \((B_i \cap C_i)\) is decreasing, and hence Cauchy (Lemma 5.4). Let \( D \) be the equivalence class determined by \((B_i \cap C_i)\). Since \( B_i \cap C_i \leq C_i \) for all integers \( i \geq 1 \), it follows that \( D \leq C \) (2), Proposition 5.10). Similarly \( D \leq B \). Thus \( D \leq B \cap C \).

Now, let \( A \) be an element of \( M^* \) such that \( A \leq B \) and \( A \leq C \). Let \((A_i)\) be the completely regular representative of \( A \). Then \( A_i \leq B_i \) and \( A_i \leq C_i \), for all integers \( i \geq 1 \) (2), Proposition 5.9). Hence \( A_i \leq B_i \cap C_i \), for all integers \( i \geq 1 \). It follows that \( A \leq D \) (2), Proposition 5.10). Consequently \( D = B \cap C \).

Before proceeding to representatives of residuals in \( M^* \) (Proposition 5.7), we shall establish the following.

PROPOSITION 5.6. \( M^* \) is modular.

Proof. Let \( A, B, C \) be elements of \( M^* \) with \( A \geq B \). Let \((A_i), (B_i)\), and \((C_i)\) be the completely regular representatives of \( A, B \), and \( C \), respectively. Since \( A_i \geq B_i \), we know \( A_i \geq B_i \), for all integers \( i \geq 1 \) (2), Proposition 5.9). We also know that the sequence \((B_i \cup C_i)\) is the completely regular representative of \( B \cup C \) (2), Proposition 5.7). Hence \((A_i \cap (B_i \cup C_i))\) is a representative of \( A \cap (B \cup C) \) by Proposition 5.5. Since \((A_i \cap C_i)\) is a representative of \( A \cap C \), and since \((B_i)\) is a representative of \( B \), we have that \((B_i \cup (A_i \cap C_i))\) is a representative of \( B \cup (A \cap C) \) (2), as comments preceding Proposition 5.6). Consequently, since

\[
A_i \cap (B_i \cup C_i) = B_i \cup (A_i \cap C_i),
\]

for all integers \( i \geq 1 \), by the modularity of \( M \), we obtain \( A \leq B \cup C \).

PROPOSITION 5.7. Let \( A, B \) be elements of \( M^* \). Let \((A_i)\) and \((B_i)\) be the completely regular representatives of \( A \) and \( B \), respectively. Then the sequence \((A_i \cap B_i)\) is a representative of \( A \cap B \).

Proof. For each integer \( i \geq 1 \), we have

\[
A_i \cap B_i = (A_i \cap B_i) \cap (B_i \cap C_i) = (A_i \cap B_i) \cap (B_i \cap C_i) = A_i \cap B_i \geq A_i \cap B_i = A_i \cap B_i,
\]

since \((A_i)\) and \((B_i)\) are completely regular. Hence, the sequence \((A_i \cap B_i)\) is decreasing, and thus is Cauchy by Lemma 5.4.

Now, let \( a \) in \( I^* \) denote the equivalence class determined by \((A_i \cap B_i)\).

Since \((A_i \cap B_i) \leq A_i \), for all integers \( i \geq 1 \), it follows that \( aB \leq A \) (2), Proposition 5.10 and comments following Definition 5.13). Hence \( a \leq A : B \). Suppose \( b \) is an element of \( I^* \) such that \( bB \leq a \). Let \((B_i)\) be the completely regular representative of \( b \). Since the sequence \((B_i \cup C_i)\) is the completely regular representative of \( bB \) (2), Corollary 5.15), we have \( b_i B_i \leq A_i \), for all integers \( i \geq 1 \). Consequently,

\[
b_i B_i = (b_i B_i \cup C_{i+1}) \leq A_i \land B_i \leq A_i \land B_i \leq A_i,
\]

for all integers \( i \geq 1 \). So \( b \leq A : B_i \), for all integers \( i \geq 1 \). Thus \( b \leq a \), and hence \( A : B = a \). Therefore \( a = A : B \).

q.e.d.

In order to establish that \( M^* \) is principally generated, we will need to establish a connection between principal elements of \( M \) and principal elements of \( M^* \). This relation is provided by the following result.

THEOREM 5.8. Let \((A_i)\) be a Cauchy sequence of principal elements of \( M \). Then the equivalence class determined by \((A_i)\) is a principal element in \( M^* \) (considered as an \( I^* \)-module).

Proof. Without loss of generality, we may clearly assume that \((A_i)\) is a regular Cauchy sequence. Let \( B \) in \( M^* \) denote the equivalence class determined by \((A_i)\), and let \((B_i)\) be the completely regular representative of \( B \). Since \( M^* \) and \( I^* \) are modular, we will use Lemma 2.1 to show that \( B \) is principal.

Let \( C \) be an element of \( M^* \), and let \((C_i)\) be the completely regular representative of \( C \). For each integer \( i \geq 1 \), we have

\[
C_i \cap B_i = C_i \cap ((A_i \cup m^i B) \cap (A_i \cup m^i C)) = m^i B \cap (m^i C \cap (A_i \cup m^i C))
\]

\[
= m^i B \cap (m^i C \cap (A_i \cup m^i C)) = m^i B \cap (m^i C \cap (A_i \cup m^i C))
\]

by (2), Corollary 4.13 and Theorem 4.14). It follows that \( C : B = (C : B) : B \) by Propositions 5.5, 5.7, and (2), Corollary 4.6). Hence (2.1) of Lemma 2.3 is satisfied.

To see that \( B \) satisfies (2.2) of Lemma 2.1, let \( a \) be an element of \( I^* \) and let \((A_i)\) be the completely regular representative of \( a \). Then, for all integers \( i \geq 1 \), we have

\[
a_i B_i \leq m^i B = (a_i \land (A_i \cup m^i C)) \land (A_i \cup m^i C) = (a_i \land (A_i \cup m^i C)) \land (A_i \cup m^i C)
\]

\[
= a_i \land (A_i \cup m^i C) = a_i \land (A_i \cup m^i C)
\]

\[
= a_i \land (A_i \cup m^i C) = a_i \land (A_i \cup m^i C)
\]

by (2), Corollary 4.13 and Theorem 4.14). It follows that \( C_i : B_i = (C_i : B_i) : B_i \) by Propositions 5.5, 5.7, and (2), Corollary 4.6). Hence (2.1) of Lemma 2.3 is satisfied.

Fundamenta Mathematicae, T. LVIII
because $A_t$ is principal. Since $\langle (a \lor B_0 \lor \vert m^2 \Bbb R \rangle : B_0 \rangle$ is a representative of $a : B$ by Proposition 5.7 and (2), Corollary 5.15, and since $\langle a \lor (0 : B) \rangle$ is a representative of $a \lor (0 : B_1)$, it follows that $a : B = a \lor (0 : B_1)$. Thus $B$ is a principal element of $M^*$. q.e.d.

We are now in a position to establish the main result of this section.

**Theorem 5.9.** $L$ is a Noether lattice and $M^*$ is a Noetherian $L^*$-module.

**Proof.** We only need to establish that $M^*$ is generically principal. If, for each pair $B$, $C$ of elements of $M^*$ such that $B < C$, we can construct a principal element $A$ in $M^*$ satisfying $A \leq C$ and $a < B$, it will follow from the ascending chain condition in $M^*$ (2), Theorem 6.5) that $M^*$ is principally generated.

Let $B$, $C$ be elements of $M^*$ such that $B < C$. Let $\langle B_0 \rangle$ and $\langle C_0 \rangle$ be the completely regular representatives of $B$ and $C$, respectively. Since $B < C$, we have $B_t \leq C_t$, for all integers $t \geq 0$. Also, since $B \not< C$, there exists an integer $n$ such that $B_t \leq C_t$, for all integers $t \geq n$. Then, in particular, $B_n \leq C_n$. Since $M$ is generically principal, there exist a principal element $A_t$ in $M$ such that $A_n \leq C_n$ and $A_t \leq B_t$.

We shall now inductively construct a sequence of element $A_{t+1}$, $A_{t+2}$, ..., of $M$ such that, for each integer $t \geq 1$,

\begin{equation}
A_t \leq (A_{t+1} \lor (0 : \vert m^2 \Bbb R \rangle) \lor C_{t+1}) \tag{5.1}
\end{equation}

\begin{equation}
A_{t+1} \leq B_{t+1} \tag{5.2}
\end{equation}

and

\begin{equation}
A_{t+1} \text{ is principal.} \tag{5.3}
\end{equation}

We proceed as follows. Suppose that

\begin{equation}
\langle A_t \lor \vert m^2 \Bbb R \rangle \rangle : C_{t+1} \leq B_n \tag{5.4}
\end{equation}

Then, since $\langle C_0 \rangle$ is a completely regular sequence, we obtain

\begin{align*}
A_t \leq (A_t \lor \vert m^2 \Bbb R \rangle \lor C_{t+1} & = (A_t \lor \vert m^2 \Bbb R \rangle \lor C_{t+1} \lor \vert m^2 \Bbb R \rangle \\
& = (A_t \lor \vert m^2 \Bbb R \rangle \lor C_{t+1} \lor \vert m^2 \Bbb R \rangle \leq B_{t+1}
\end{align*}

by (5.4) and the modularity of $M$. This is a contradiction to $A_t \leq B_n$ above. Hence

\begin{equation}
\langle A_t \lor \vert m^2 \Bbb R \rangle \lor C_{t+1} \leq B_{t+1} \tag{5.5}
\end{equation}

Since $M$ is generically principal, and by (5.5), there exists a principal element $A_{t+1}$ such that (i) $A_{t+1} \leq (A_t \lor \vert m^2 \Bbb R \rangle \lor C_{t+1}$, and (ii) $A_{t+1} \leq B_{t+1}$. Thus $A_{t+1}$ satisfies (5.1), (5.2), and (5.3).

Now, assume that $A_{t+1}, \ldots, A_{t+n}$ have been chosen so that, for each $j$, $1 \leq j \leq k$, the conditions in (5.1), (5.2), and (5.3) have been satisfied. In a manner similar to the case for $j = 1$, it may be shown that

\begin{equation}
\langle A_{t+n} \lor \vert m^2 \Bbb R \rangle \lor C_{t+n} \leq B_{t+n} \tag{5.6}
\end{equation}

Hence, as above, there exists a principal element $A_{t+n+1}$ such that

\begin{equation}
A_{t+n+1} \leq (A_{t+n} \lor \vert m^2 \Bbb R \rangle \lor C_{t+n+1} \tag{5.7}
\end{equation}

and

\begin{equation}
A_{t+n+1} \text{ is principal.} \tag{5.8}
\end{equation}

Consequently, (5.6), (5.7) imply that $A_{t+n+1}$ satisfies (5.1), (5.2), (5.3).

Next, set $A_t = A_t$, for $1 \leq t \leq n$. Since the sequence $\langle A_i \rangle, i = 1, 2, \ldots$, satisfies condition (5.1), it follows that $\langle A_t \rangle$ is a Cauchy sequence by Lemma 5.4. Since each $A_t$ is principal by (5.3), we have that the equivalence class $A$ in $M^*$ determined by $\langle A_t \rangle$ is a principal element of $M^*$ (considered as an $L^*$-module). Since $A_t \leq C_t$, for all integers $t \geq 1$, we obtain $A < C$.

We now only need to show that $A \leq B$ in order to complete our proof. So, assume $A < B$. Select a regular subsequence $\langle A'_t \rangle$ from the sequence $\langle A_t \rangle$ (4), Lemma 4.11). Then $A_t \leq A'_t \lor \vert m^2 \Bbb R \rangle \lor B_t \leq B_{t+1}$, for all integers $t \geq n$, by (2), Corollary 4.13). Hence $A_t \leq B_{t+1}$, for large $t$, which is a contradiction to (5.2). Thus $A \leq B$.

q.e.d.

**§ 6. Contractions.** In order to establish our main theorem we will need the following preliminary result.

**Theorem 6.1.** Let $(L_a, p_1, \ldots, p_n)$ be a semi-local Noether lattice, let $M$ be a Noetherian $L$-module, let $b$ be the Jacobson radical of $L$, let $M^*$ be the $m$-adic completion of $M$, and let $A$ be an element of $M$. Then

\begin{equation}
\langle (C \setminus M) \Bbb R \rangle = 0, \tag{6.1}
\end{equation}

for all $C$ in $M^*$ such that $(mA)M^* \leq C \leq AM^*$.

**Proof.** Let $C$ be an element of $M^*$ such that $(mA)M^* \leq C \leq AM^*$. Let $\langle C \rangle$ be the completely regular representative of $C$. Since the sequence $\langle C \rangle$ is decreasing, the sequence $\langle C_t \rangle$ is decreasing, and hence is Cauchy (Lemma 5.4).

We shall now show that the equivalence class determined by the sequence $\langle C_t \rangle$ is $C$. Since $\langle C_t \rangle \sim G_t(A \lor \vert m^2 \Bbb R \rangle)$, and since

\begin{equation}
\langle C_t \rangle \lor \vert m^2 \Bbb R \rangle = G_t(A \lor \vert m^2 \Bbb R \rangle), \tag{6.2}
\end{equation}

for all integers $t \geq 1$, it is sufficient to show that $\langle C_t \rangle \lor \vert m^2 \Bbb R \rangle$ is a representative of $C$. Since $\langle C \rangle$ is the completely regular representative of $C$ and since $\langle A \lor \vert m^2 \Bbb R \rangle$ is the completely regular representative of $AM^*$ (2), Remark 5.2), it follows that the sequence $\langle C_t \rangle \lor \vert m^2 \Bbb R \rangle$ is a representative of $C \lor AM^*$ (Proposition 5.5). But $C \lor AM^* = C$ since $C \leq AM^*$.

Since we know that $(D \setminus M)M^* \leq D$, for all $D$ in $M^*$ (2), Proposition 7.5), in order to prove the theorem, it is sufficient to show that $D \setminus M$ is a $m$-adic completion of $M$.
(C \wedge M)^* \geq C$. This shall now be established. Since $(mA)^* \leq C \leq \lambda M^*$, we have that

(6.2) \quad mA \leq mA \vee m^* \leq A \vee m^* \Rightarrow \lambda A \leq A \wedge m\lambda \theta,

for all integers $\lambda \geq 1$. Hence, from (6.2), we obtain

(6.3) \quad mA = mA \wedge A \leq \lambda A \wedge m\lambda \theta \wedge A = A,

for all integers $\lambda \geq 1$. Since $[mA, A]$ is finite dimensional (Theorem 5.1), it follows from (6.3) that there exists a natural number $k$ such that

(6.4) \quad C \wedge A = C \wedge A,

for all integers $\lambda \geq k$. Since $C \wedge A$ is a representative of $C$, we obtain from (6.4) that

(6.5) \quad (C \wedge A)^* = C.

Since $C \geq C \wedge A = C \wedge A,$

(6.6) \quad \bigwedge_{\lambda \geq k} C \geq \bigwedge_{\lambda \geq k} (C \wedge A) = C \wedge A.

Consequently,

(6.7) \quad (C \wedge M)^* = (\bigwedge_{\lambda \geq k} C)^* \geq \bigwedge_{\lambda \geq k} (C \wedge A)^* = (C \wedge A)^* = C

by (6.5), (6.6), and (23), Corollary 5.11. \quad \text{q.e.d.}

We are now in a position to determine some structural properties of $L^*$. 

**Theorem 6.2.** Let $(L_1, p_1, ..., p_n)$ be a semi-local Noether lattice, let $m$ be the Jacobson radical of $L_1$, let $L^*$ be the $m$-adic completion of $L_1$, and let $m^*$ be the Jacobson radical of $L^*$. Then,

(6.8) \quad L^* is a semi-local Noether lattice with maximal elements

\[ \{p_1L^*, ..., p_nL^* \} \]

and

(6.9) \quad \{p_1 \wedge ... \wedge p_n \} L^* = mL^* = p_1L^* \wedge ... \wedge p_nL^*.

Proof. First, we show that each $p_i L^*$, $1 \leq i \leq n$, is maximal in $L^*$. Choose $k$ such that $1 \leq k \leq n$. Since the extension map is one-to-one, $p_kL^* \neq I^*$. Let $b$ be an element of $L^*$ such that $p_kL^* \leq b \leq I^*$. Let $(b)$ be the completely regular representative of $b$. Since $b \leq I$, $b_i$ is the completely regular representative of $b_i$; we have $p_k \leq p_k \vee m^* \leq b_i \leq I$.

for all integers $i \geq 1$. Since $b \neq I^*$, there exists a natural number $N$ such that $b_i \leq I$, for all integers $i \geq N$. It follows that $p_k \neq b_i$, for all integers $i \geq N$. Hence $p_kL^* = b$, and $p_kL^*$ is maximal.

Next, suppose $b$ is an element of $L^*$ such that $b \neq I^*$ and $b$ is maximal. Let $(b)$ be the completely regular representative of $b$. Since $b \neq I^*$, there exists a natural number $N$ such that $b_i \leq I$, for all integers $i \geq N$. Consequently, there exists a natural number $k$, $1 \leq k \leq n$, such that $b_i \leq p_i$; for infinitely many integers $i \geq N$. It follows that $b \neq p_kL^*$. Hence $b = p_kL^*$, since $p_kL^*$ is maximal. Thus (6.7) has been established.

Now, since $m = p_1 \wedge ... \wedge p_n \leq p_k$, for each integer $k$, $1 \leq k \leq n$, we have that $mL^* \leq p_kL^*$, for each integer $k$, $1 \leq k \leq n$. Thus

\[ mL^* = p_kL^* \wedge ... \wedge p_nL^* = m^* \leq p_kL^*, \]

for each integer $k$, $1 \leq k \leq n$. Consequently,

\[ m = mL^* \wedge L \leq m^* \wedge L \leq p_kL^* \wedge L = p_i \]

for each integer $k$, $1 \leq k \leq n$, (23), Propositions 7.2 and 7.6. Thus

\[ m \leq m^* \wedge L \leq \bigwedge_{i=1}^n p_i = m. \]

Hence $m^* \wedge L = m$. Thus (6.8) has been established.

Since $p_kL^* \wedge ... \wedge p_nL^* = m^*$ by (6.7), and since $p_1 \wedge ... \wedge p_n = m$, in order to establish (6.9), we need only show that $m^* = mL^*$. From (6.8) we obtain $mL^* = (m^* \wedge L)L^* = m^*$ by Theorem 6.1. \quad \text{q.e.d.}

**References**


**University of Houston**

*Houston, Texas*

**University of Iowa**

*Iowa City, Iowa*

*Reçu par la Rédaction le 24 A. 1969*