Thus the topology of the metric d^* on G/H and the quotient topology of G/H coincide. This completes the proof of Theorem 2.

The proof of Theorem 3 below illustrates the use of Theorem 1. Theorem 3 can be proved alternatively by introducing a non-archimedian metric in the set of Cauchy sequences in G (cf. [6], p. 485).

THEOREM 3. *If G is a two sided invariant non-archimedian metric group, then there exists a non-archimedian complete metric group $\hat{G}$ such that G is a dense subgroup of $\hat{G}$.*

Proof. G being a two sided invariant non-archimedian metric group (consequently a metric group, in the usual sense), it can be imbedded as a dense subgroup of a complete metric group $\hat{G}$ ([6], p. 485, (1.4)). Since the non-archimedian metric on G is two sided invariant, there exists a countable base of neighbourhoods of normal subgroups at the identity e of G (see Remark following Theorem 1). The closures in $\hat{G}$ of these subgroups, which are also normal in $\hat{G}$ ([3], p. 46, 5.37 (c)), constitute a base of neighbourhoods ([2], p. 30, Proposition 7) (1) at e for $\hat{G}$. Hence, by Theorem 1, $\hat{G}$ is also non-archimedian metrizable. Further $\hat{G}$ is complete with respect to this non-archimedian metric (see [5], p. 212, Exercise $Q(d)$). The proof of Theorem 3 is now complete.

(1) It is sufficient to take a base at e for G, instead of all neighbourhoods at e, for the validity of the proposition referred to.

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A three-dimensional spheroidal space which is not a sphere

by

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1. Introduction. In [1], we described an upper semicontinuous decomposition of E^3 into straight arcs and singletons such that the associated decomposition space E^3/G is topologically distinct from E^3. In this note, we study local properties of the decomposition space.

We shall show that E^3/G is *locally peripherally spherical*, i.e., each point of E^3/G has arbitrarily small neighborhoods bounded by 2-spheres. In fact, each point of E^3/G has arbitrarily small closed neighborhoods which are compact absolute retracts and have 2-spheres as their topological boundaries. In particular, each point of the space has arbitrarily small compact simply connected neighborhoods.

We shall also use the decomposition of [1] to settle a question of Borsuk's concerning spheroidal spaces. A metric space X is a *spheroidal space* if and only if for each point p of X and each neighborhood U of p, there is a neighborhood V of p such that $V \subset U$ and $X - V$ is a compact absolute retract. It is known that each spheroidal space of dimensions 0, 1, and 2 is a sphere [3]. In [3], Borsuk describes an example (due to Ganea) of a spheroidal space of dimension 4 not a sphere. Borsuk [3] raises the following question: *Does there exist a 3-dimensional spheroidal space which is not a sphere?* We give an affirmative answer to this question. Regard S^3 as the one-point compactification $E^3 \cup \{\infty\}$ of E^3. Let G^* denote the upper semicontinuous decomposition of S^3 consisting of all the elements of G, together with $\{\infty\}$. Then associated decomposition space, S^3/G^*, is a 3-dimensional spheroidal space which is not a sphere. In fact, S^3/G^* has the following property: Each point of S^3 has arbitrarily small open neighborhoods V such that the closure of V is a compact absolute retract, the complement of V is a compact absolute retract, and the boundary of V is a 2-sphere.

Throughout this note, we retain the notation of [1]. G denotes the decomposition of E^3 described in [1], E^3/G denotes the associated decomposition space, and Pr denotes the projection map from E^3 onto E^3/G. A and B are horizontal planes in E^3 as described in section 4 of [1].

2. Local properties of E^3/G.

THEOREM 1. *Each point of E^3/G has arbitrarily small (closed) neighborhoods which are compact absolute retracts and have a 2-sphere as their topological boundary in E^3/G.*

Proof. Suppose $g \in E^3/G$ and W is an open set in E^3/G containing g. Suppose g is a non-degenerate element of G. There is an index α such that $g \subset \mathrm{Int}\, T_\alpha$ and $T_\alpha \subset \mathrm{Pr}^{-1}[W]$.

We may assume the construction of G carried out so that each component of $(A \cup B) \cap T_\alpha$ is a disc and each such disc intersects Γ_α. Suppose m is the integer such that α is a stage m index. If $j = 1, 2, \ldots,$ or n_{m-1}, let $D_{\alpha 1j}$ denote the component of $T_\alpha \cap A$ intersecting $\langle p_{\alpha 1} q_{\alpha j}\rangle$, and let $E_{\alpha 1j}$ denote the component of $T_\alpha \cap B$ intersecting $\langle p_{\alpha 1} q_{\alpha j}\rangle$. If $j = 1, 2, \ldots,$ or n_{m-1}, let $D_{\alpha 2j}$ and $E_{\alpha 2j}$ denote the components of $T_\alpha \cap A$ and $T_\alpha \cap B$, respectively, intersecting $\langle p_{\alpha 2} q_{\alpha j}\rangle$.

Now there exist integers k and l such that g intersects both $D_{\alpha kl}$ and $E_{\alpha kl}$, but no other D or E. Let L_α denote the closure of the component of $T_\alpha - \bigcup \{D_{\alpha ij} \cup E_{\alpha ij}\colon\ i = 1 \text{ or } 2,\ j = 1, 2, \ldots, \text{ or } n_{m-1}, \text{ and } (i,j) \neq (k,l)\}$ containing g. Let Σ_α denote the boundary of L_α. Then L_α is a polyhedral 3-cell and Σ_α is a polyhedral 2-sphere. Note that if $i = 1$ or 2, and $j = 1, 2, \ldots,$ or n_{m-1}, then (1) $D_{\alpha ij} \subset \Sigma_\alpha$ if and only if both $i = k$ and $j \neq l$, and (2) $E_{\alpha ij} \subset \Sigma_\alpha$ if and only if both $i \neq k$ and $j = l$. Clearly, g does not intersect Σ_α.

It is easily seen that if $g' \in G$, then (1) $g' \cap L_\alpha$, if non-empty, is an arc, and (2) g' does not intersect Σ_α in more than one point. Hence $\mathrm{Pr}[\Sigma_\alpha]$ is a 2-sphere, and it follows from [3], p. 131 that $\mathrm{Pr}[L_\alpha]$ is a compact absolute retract. (Since each point of E^3/G has arbitrarily small neighborhoods bounded by 2-spheres, E^3/G is finite-dimensional. Thus $\mathrm{Pr}[L_\alpha]$ is finite-dimensional.)

Let Σ_α^* denote $\bigcup \{g'\colon\ g' \in G \text{ and } g' \text{ intersects } \Sigma_\alpha\}$. Since G is upper semicontinuous and Σ_α is closed, Σ_α^* is closed; clearly, $\mathrm{Pr}[\Sigma_\alpha^*] = \mathrm{Pr}[\Sigma_\alpha]$. Now g is disjoint from Σ_α^*, and $(\mathrm{Int}\, \Sigma_\alpha) - \Sigma_\alpha^*$ is an open set V such that $g \subset V$, $V \subset \mathrm{Int}\, \Sigma_\alpha$, and V is a union of elements of G. The boundary, in E^3 of V is contained in Σ_α^*, and in fact, if g' is an element of G lying in Σ_α^*, g' contains a limit point of V. Thus $\mathrm{Pr}[V]$ is open in E^3/G, $g \in \mathrm{Pr}[V]$, and $\mathrm{Cl}\,\mathrm{Pr}[V] \subset W$. It is easily seen that the topological boundary, in E^3/G, of $\mathrm{Pr}[L_\alpha]$ is $\mathrm{Pr}[\Sigma_\alpha^*]$, or $\mathrm{Pr}[\Sigma_\alpha]$. Hence $\mathrm{Pr}[L_\alpha]$ is a compact absolute retract which is a closed neighborhood of g in E^3/G lying in W and bounded by a 2-sphere.

If g is a singleton, then since H_G is closed, g has a neighborhood V in E^3 such that $\mathrm{Cl}\,V$ is a 3-cell missing H_G and lying in $\mathrm{Pr}^{-1}[W]$. It follows that $\mathrm{Pr}[\mathrm{Cl}\,V]$ is a compact absolute retract which is a closed neighborhood of g in E^3/G lying in W and bounded by a 2-sphere. This establishes Theorem 1.

COROLLARY 1. *E^3/G is locally peripherally spherical.*

COROLLARY 2. *Each point of E^3/G has arbitrarily small compact, connected, locally connected, and simply connected neighborhoods.*

A space X is *strongly locally simply connected* if and only if each point of X has arbitrarily small simply connected *open* neighborhoods. We conjecture that E^3/G is not strongly locally simply connected.

It is not difficult to show that the following holds: Suppose $g \in G$ and W is an open set in E^3 containing g. Then there is a 3-cell C such that $g \subset \mathrm{Int}\, C$, $C \subset W$, and C is an union of elements of G.

3. Spheroidal spaces.

Let $E^3 \cup \{\infty\}$ be the one-point compactification of E^3; $E^3 \cup \{\infty\}$ is homeomorphic to the 3-sphere S^3, and we shall identify the two spaces. Let G^* denote the decomposition of S^3 consisting of all the elements of G, together with $\{\infty\}$. Then G^* is an upper semicontinuous decomposition of S^3 into arcs and singletons. Let S^3/G^* denote the associated decomposition space, and let Pr^* denote the projection map from S^3 onto S^3/G^*.

THEOREM 2. *S^3/G^* is a 3-dimensional spheroidal space which is not a sphere.*

Proof. By a simple modification of the argument given in the proof of Theorem 1, we may establish the following: If $g \in G^*$ and U is a neighborhood of g, there is a 2-sphere Σ in U missing g and such that if V is the component of $S^3 - \Sigma$ containing g, then (1) $V \subset U$ and (2) $\mathrm{Pr}^*[S^* - V]$ is a compact absolute retract. Hence S^3/G^* is a spheroidal space. In fact, Σ may be selected in the argument above so that $\mathrm{Pr}^*[\Sigma \cup V]$ is a compact absolute retract and that $\mathrm{Pr}^*[\Sigma]$ is the topological boundary, in S^3/G^*, of both $\mathrm{Pr}^*[\Sigma \cup V]$ and $\mathrm{Pr}^*[S^3 - V]$.

Since each point of S^3/G^* has arbitrarily small neighborhoods bounded by 2-spheres, S^3/G^* has dimension at most 3. Since S^3/G^* contains a 3-cell (about ∞), S^3/G^* has dimension 3.

If S^3/G^* were homeomorphic to S^3, it would follow that E^3/G is homeomorphic to E^3. This would contradict the results of [1]. Thus S^3/G^* is not a sphere.

In fact, S^3/G is not a 3-manifold. If it were a 3-manifold, then by Corollary 1 of [2], S^3/G would be homeomorphic to S^3.

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Lattice modules over semi-local Noether lattices

by

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§ 1. Introduction. For Noetherian lattice modules, the concept of the a-adic pseudometric has been introduced and studied in [2] and [3]. Recently the natural completion of a local Noether lattice was related to the completeness of a local ring in its natural topology ([1]). The purpose of this paper is to establish some properties of Noetherian lattice modules over semi-local Noether lattices and their completions.

The basic concepts are introduced in § 2, and some preliminary results are obtained. Let L be a multiplicative lattice and let M be a Noetherian L-module. In § 3 an interesting property concerning certain sequences in M is established (Theorem 3.2). If L is a Noether lattice and m is the Jacobson radical of L, then it is shown (Corollary 3.4) that the m-adic pseudometric on M is a metric ([2], § 3). § 4 contains some results on dimensions. If L is semilocal, it is shown in § 5 that $[mA, A]$ is finite dimensional, for all A in M (Theorem 5.1), L^* is a Noether lattice, and M^* is a Noetherian L^*-module (Theorem 5.9), where L^* and M^* are the m-adic completions of L and M, respectively ([2], § 6). In § 6 it is established that L^* is a semi-local Noether lattice whose maximal elements are extensions ([2], § 5) of the maximal elements of L.

§ 2. Preliminary remarks. By a multiplicative lattice we shall mean a complete lattice on which there is defined a commutative, associative, join distributive multiplication such that the unit element of the lattice is an identity for the multiplication. Let L be a multiplicative lattice and let M be a complete lattice. We shall denote elements of L by $a, b, c, \ldots$ with the exception that the null element and unit element of L will be denoted by 0 and I, respectively. We shall denote elements of M by $A, B, C, \ldots$, with the exception that the null element and unit element of M will be denoted by 0_M and M, respectively. When no confusion is possible, 0 will also be used in place of 0_M. Recall that M is an L-module ([2], Definition 2.2) in case there is a multiplication between elements of L and M, denoted by aA for a in L and A in $\mathfrak{M}$, which satisfies: (i) $(ab)A = a(bA)$, (ii) $(\bigvee_\alpha a_\alpha)(\bigvee_\beta B_\beta) = \bigvee_{\alpha,\beta} a_\alpha B_\beta$; (iii) $IA = A$; and (iv) $0A = 0$; for all a, a_α, b in L and for all A, B_β in M.