Thus the topology of the metric $d^*$ on $G/H$ and the quotient topology of $G/H$ coincide. This completes the proof of Theorem 2.

The proof of Theorem 3 below illustrates the use of Theorem 1. Theorem 3 can be proved alternatively by introducing a non-archimedean metric in the set of Cauchy sequences in $G$ (cf. [3], p. 480).

**Theorem 3.** If $G$ is a two sided invariant non-archimedean metric group, then there exists a non-archimedean complete metric group $\hat{G}$ such that $G$ is a dense subgroup of $\hat{G}$.

Proof. $G$ being a two sided invariant non-archimedean metric group (consequently a metric group, in the usual sense), it can be imbedded as a dense subgroup of a complete metric group $\hat{G}$ ([6], p. 485, (1.4)). Since the non-archimedean metric on $G$ is two sided invariant, there exists a countable base of neighbourhoods of normal subgroups at the identity $e$ of $G$ (see Remark following Theorem 1). The closures in $\hat{G}$ of these subgroups, which are also normal in $\hat{G}$ ([3], p. 46, 5.37 (c)), constitute a base of neighbourhoods of $e$ in $\hat{G}$. Hence, by Theorem 1, $\hat{G}$ is also non-archimedean metrizable. Further $\hat{G}$ is complete with respect to this non-archimedean metric (see [5], p. 212, Exercise 5(d)).

The proof of Theorem 3 is now complete.

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(1) It is sufficient to take a base at $e$ for $\hat{G}$, instead of all neighbourhoods at $e$, for the validity of the proposition referred to.

**References**


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A three-dimensional spheroidal space which is not a sphere

by

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**1. Introduction.** In [1], we described an upper semicontinuous decomposition of $E^3$ into straight arcs and singletons such that the associated decomposition space $E^3/G$ is topologically distinct from $E^3$. In this note, we study local properties of the decomposition space.

We shall show that $E^3/G$ is locally peripherally spherical, i.e., each point of $E^3/G$ has arbitrarily small neighborhoods bounded by 2-spheres. In fact, each point of $E^3/G$ has arbitrarily small closed neighborhoods which are compact absolute retracts and have 2-spheres as their topological boundaries. In particular, each point of the space has arbitrarily small compact simply connected neighborhoods.

We shall also use the decomposition of [1] to settle a question of Borsuk's concerning spheroidal spaces. A metric space $X$ is a spheroidal space if and only if for each point $p$ of $X$ and each neighborhood $U$ of $p$, there is a neighborhood $V$ of $p$ such that $V \cup U$ and $X - V$ is a compact absolute retract. It is known that each spheroidal space of dimensions 0, 1, and 2 is a sphere [3]. In [3], Borsuk describes an example (due to Ganea) of a spheroidal space of dimension 4 not a sphere. Borsuk [3] raises the following question: Does there exist a 3-dimensional spheroidal space which is not a sphere? We give an affirmative answer to this question. Regard $S^3$ as the one-point compactification $E^3 + \{\infty\}$ of $E^3$. Let $G$ denote the upper semicontinuous decomposition of $S^3$ consisting of all the elements of $G$, together with $\{\infty\}$. Then associated decomposition space, $S^3/G$, is a 3-dimensional spheroidal space which is not a sphere. In fact, $S^3/G$ has the following property: Each point of $S^3$ has arbitrarily small open neighborhoods $V$ such that the closure of $V$ is a compact absolute retract, the complement of $V$ is a compact absolute retract, and the boundary of $V$ is a 2-sphere.

Throughout this note, we retain the notation of [1], $G$ denotes the decomposition of $E^3$ described in [1], $E^3/G$ denotes the associated decomposition space. 

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A three-dimensional spherical space which is not a sphere

If \( g \) is a singleton, then since \( H_0 \) is closed, \( g \) has a neighborhood \( V \) in \( E^3 \) such that \( C \cup V \) is a 3-cell missing \( H_0 \) and lying in \( P^* [W] \). It follows that \( P^*[C \cup V] \) is a compact absolute retract which is a closed neighborhood of \( g \) in \( E^3 \) lying in \( W \) and bounded by a 2-sphere. This establishes Theorem 1.

**Corollary 1.** \( E^3 \) is locally peripherally spherical.

**Corollary 2.** Each point of \( E^3 \) has arbitrarily small compact, connected, locally connected, and simply connected neighborhoods.

A space \( X \) is strongly locally simply connected if and only if each point of \( X \) has arbitrarily small simply connected open neighborhoods. We conjecture that \( E^3 \) is not strongly locally simply connected.

It is not difficult to show that the following holds: Suppose \( g \in G \) and \( W \) is an open set in \( E^3 \) containing \( g \). Then there is a 3-cell \( U \) such that \( g \in \text{Int } U, U \subseteq W \), and \( U \) is an union of elements of \( G \).

3. **Spheral spaces.** Let \( E^3 \cup \{ \infty \} \) be the one-point compactification of \( E^3 \); \( E^3 \cup \{ \infty \} \) is homeomorphic to the 3-sphere \( S^3 \), and we shall identify the two spaces \( G \) consisting of all the elements of \( G \) together with \( \{ \infty \} \). Then \( G^* \) is an upper semicontinuous decomposition of \( S^3 \) into arcs and singletons. Let \( S^3[G^*] \) denote the associated decomposition space, and let \( P^* \) denote the projection map from \( S^3 \) onto \( S^3[G^*] \).

**Theorem 2.** \( S^3[G^*] \) is a 3-dimensional spheroidal space which is not a sphere.

Proof. By a simple modification of the argument given in the proof of Theorem 1, we may establish the following: If \( g \in G \) and \( U \) is a neighborhood of \( g \), there is a 2-sphere \( S^2 \) in \( U \) missing \( g \) and such that if \( V \) is the component of \( S^2 \cup C \) containing \( g \), then \( \{ 1 \} V \subseteq U \), \( P^*[S^2 \cup V] \) is a compact absolute retract, and that \( P^*[C \cup V] \) is the topological boundary, in \( S^3[G^*] \), of both \( P^*[S^2 \cup V] \) and \( P^*[C \cup V] \).

Since each point of \( S^3[G^*] \) has arbitrarily small neighborhoods bounded by 2-spheres, \( S^3[G^*] \) has dimension at most 3. Since \( S^3[G^*] \) contains a 3-cell (about \( \infty \)), \( S^3[G^*] \) has dimension 3.

If \( S^3[G^*] \) were homeomorphic to \( S^3 \), it would follow that \( E^3[G^*] \) is homeomorphic to \( E^3 \). This would contradict the results of [1]. Thus \( S^3[G^*] \) is not a sphere.

In fact, \( S^3[G^*] \) is not a 3-manifold. If it were a 3-manifold, then by Corollary 1 of [3], \( S^3[G^*] \) would be homeomorphic to \( S^3 \).
Lattice modules over semi-local Noether lattices

by

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§ 1. Introduction. For Noetherian lattice modules, the concept of the $\alpha$-adic pseudometric has been introduced and studied in [2] and [3]. Recently the nature of completion of a local Noether lattice was related to the completeness of a local ring in its natural topology ([1]). The purpose of this paper is to establish some properties of Noetherian lattice modules over semi-local Noether lattices and their completions.

The basic concepts are introduced in § 2, and some preliminary results are obtained. Let $L$ be a multiplicative lattice and let $M$ be a Noetherian $L$-module. In § 3 an interesting property concerning certain sequences in $M$ is established (Theorem 3.2). If $L$ is a Noether lattice and $m$ is the Jacobson radical of $L$, then it is shown (Corollary 3.4) that the $m$-adic pseudometric on $M$ is a metric ([2], § 3). § 4 contains some results on dimensions. If $L$ is semilocal, it is shown in § 5 that $[m^d, A]$ is finite dimensional, for all $A$ in $M$ (Theorem 5.1), $L^*$ is a Noether lattice, and $M^*$ is a Noetherian $L^*$-module (Theorem 5.9), where $L^*$ and $M^*$ are the $m$-adic completions of $L$ and $M$, respectively ([2], § 6). In § 6 it is established that $L^*$ is a semi-local Noether lattice whose maximal elements are extensions ([2], § 5) of the maximal elements of $L$.

§ 2. Preliminary remarks. By a multiplicative lattice we shall mean a complete lattice on which there is defined a commutative, associative, join distributive multiplication such that the unit element of the lattice is an identity for the multiplication. Let $L$ be a multiplicative lattice and let $M$ be a complete lattice. We shall denote elements of $L$ by $a, b, c, \ldots$, with the exception that the null element and unit element of $L$ will be denoted by 0 and 1, respectively. We shall denote elements of $M$ by $A, B, C, \ldots$, with the exception that the null element and unit element of $M$ will be denoted by $0_M$ and $1_M$, respectively. When no confusion is possible, 0 will also be used in place of $0_M$. Recall that $M$ is an $L$-module ([2], Definition 2.3) in case there is a multiplication between elements of $L$ and $M$, denoted by $aA$ for $a$ in $L$ and $A$ in $M$, which satisfies:

(i) $(ab)A = a(bA)$,  
(ii) $(\bigvee a_i)(\bigvee B_j) = \bigvee a_iB_j$;  
(iii) $1A = A$;  
(iv) $0A = 0_M$ for all $a, a_i, b$ in $L$ and for all $A, B_j$ in $M$. 

References