Non-archimedian metrizability of topological groups

by

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A *non-archimedian right* (*left*) *invariant metric* on a group G is a real valued function $d(x, y)$ of pairs of elements $x, y \in G$ such that

(i) $d(x, y) = 0$ if and only if $x = y$,

(ii) $d(x, y) = d(y, x)$,

(iii) $d(x, y) \leqslant \max\{d(x, z), d(z, y)\}$, $z \in G$,

(iv) $d(xa, ya) = d(x, y)$ for any $a \in G$ $(d(ax, ay) = d(x, y)$ for any $a \in G)$.

The metric is said to be *two sided invariant* if it is both right and left invariant. A topological group G is said to be *non-archimedian metrizable* if its topology is induced by a non-archimedian right (or left) invariant metric.

The main object of this note is to prove a criterion (Theorem 1) for non-archimedian metrizability of a topological group analogous to the classical Birkhoff-Kakutani criterion ([1], [4]) for ordinary metrizability. As supplementary results we prove (Theorem 2) that the quotient of a non-archimedian metrizable group by a closed normal subgroup is again non-archimedian metrizable and that (Theorem 3) a two sided invariant non-archimedian metric group can be imbedded in a complete non-archimedian metric group.

THEOREM 1. *For a topological group G to be non-archimedian metrizable it is necessary and sufficient that there exists a countable base of neighbourhoods at the identity e of G consisting of subgroups of G.*

Proof. Let G be a non-archimedian metrizable topological group and d a right invariant non-archimedian metric on G which induces the given topology on G. Then $U_n = [x|\ d(x, e) < 1/n]$, $n = 1, 2, \ldots$, form a base of neighbourhoods at e for G. If $x \in U_n$, the right invariance of d implies $d(e, x^{-1}) < 1/n$ and hence $x^{-1} \in U_n$. Also

$$d(xy, e) = d(x, y^{-1}) \leqslant \max\{d(x, e), d(e, y^{-1})\} < 1/n \quad \text{if} \quad x, y \in U_n .$$

Thus U_n, $n = 1, 2, \ldots$, are subgroups, i.e. the condition is necessary.

To prove the sufficiency, let us suppose that G has a countable base of neighbourhoods $\{H_n\}$ of subgroups of G. We assume without loss of generality, that $H_{n+1} \subset H_n$ (see [5], p. 50). Now, $\Phi_n(x, y)$, defined for every $n = 1, 2, \ldots$, by

$$\Phi_n(x, y) = \begin{cases} 0 & \text{if} \quad xy^{-1} \in H_n, \\ 1 & \text{if} \quad xy^{-1} \notin H_n, \end{cases}$$

is a non-archimedean pseudo-metric. By a familiar argument (see [7], p. 473) it follows that

$$d(x, y) = \sup_n \frac{1}{2^n} \cdot \frac{\Phi_n(x, y)}{1+\Phi_n(x, y)}$$

is a non-archimedean metric on G. The essential part of this argument can be outlined as follows. If

$$\Phi_n'(x, y) = \frac{\Phi_n(x, y)}{1+\Phi_n(x, y)},$$

then

$$\max\{\Phi_n'(x, z), \Phi_n'(z, y)\} \geqslant \frac{\max\{\Phi_n(x, z), \Phi_n(z, y)\}}{1+\max\{\Phi_n(x, z), \Phi_n(z, y)\}} \geqslant \Phi_n'(x, y)$$

and hence each of the $\Phi_n'(x, y)$ is a non-archimedean pseudo-metric on G. Consequently d satisfies condition (iii) of the definition of a non-archimedean metric. The other conditions are easily verified and d is a right invariant metric.

It remains to show that the topology defined by the metric d is the same as the topology of G. To this end, we observe that

$$U_n = [x|\ d(x, e) < 1/2^{n+1}], \quad n = 1, 2, \ldots,$$

form a base of neighbourhoods at e for the metric topology. Further

$$x \in U_n \Rightarrow d(x, e) < \frac{1}{2^{n+1}} \Rightarrow \frac{1}{2^n} \cdot \frac{\Phi_n(x, e)}{1+\Phi_n(x, e)} < \frac{1}{2^{n+1}}$$

$$\Rightarrow \Phi_n(x, e) < 1 \Rightarrow \Phi_n(x, e) = 0 \Rightarrow x \in H_n, \quad \text{i.e.} \quad U_n \subset H_n.$$

If $x \in H_{n+1}$, then $\Phi_{n+1}(x, e) = 0$. As $H_{n+1} \subset H_n \subset \ldots \subset H_1$,

$$\Phi_{n+1}(x, e) = \Phi_n(x, e) = \ldots = \Phi_1(x, e) = 0.$$

We note that

$$\frac{1}{2^k} \cdot \frac{\Phi_k(x, e)}{1+\Phi_k(x, e)} \leqslant \frac{1}{2^k} \leqslant \frac{1}{2^{n+2}} \quad \text{if} \quad k \geqslant n+2,$$

$$0 = \frac{1}{2^k} \cdot \frac{\Phi_k(x, e)}{1+\Phi_k(x, e)} < \frac{1}{2^{n+2}} \quad \text{if} \quad k < n+2.$$

Hence

$$d(x, e) = \sup_k \frac{1}{2^k} \cdot \frac{\Phi_k(x, e)}{1+\Phi_k(x, e)} \leqslant \frac{1}{2^{n+2}} < \frac{1}{2^{n+1}}.$$

If $x \notin H_{n+1}$, but $x \in H_n$, then $\Phi_k(x, e) = 1$ for $k \geqslant n+1$, so that

$$\frac{1}{2^k} \cdot \frac{\Phi_k(x, e)}{1+\Phi_k(x, e)} = \frac{1}{2^{k+1}} \leqslant \frac{1}{2^{n+2}} \quad \text{for} \quad k \geqslant n+1,$$

$$0 = \frac{1}{2^k} \cdot \frac{\Phi_k(x, e)}{1+\Phi_k(x, e)} < \frac{1}{2^{n+2}} \quad \text{for} \quad k < n+1.$$

Thus

$$d(x, e) = \sup_k \frac{1}{2^k} \cdot \frac{\Phi_k(x, e)}{1+\Phi_k(x, e)} \leqslant \frac{1}{2^{n+2}} < \frac{1}{2^{n+1}}.$$

Hence

$$x \in H_n \Rightarrow d(x, e) < \frac{1}{2^{n+1}} \Rightarrow x \in U_n, \quad \text{i.e.} \quad H_n \subset U_n.$$

As we have already shown that $U_n \subset H_n$, it follows that $H_n = U_n$. This completes the proof of the sufficiency of the condition of the theorem. Theorem 1 is thus proved.

Remark. The metric is two sided invariant if and only if there exists a countable base of neighbourhoods consisting of normal subgroups.

THEOREM 2. *If G is a non-archimedean metrizable topological group and H a closed normal subgroup of G, then G/H is also non-archimedean metrizable.*

Proof. Let d be the metric on G which gives its topology. For $xH, yH \in G/H$ define

$$d^*(xH, yH) = \inf[d(a, b)|\ a \in xH, b \in yH].$$

Then an easy argument (cf. [8], p. 36) shows that d^* is a non-archimedean metric. If d is right (left) invariant, d^* is right (left) invariant. Also

$$\left[xH|\ d^*(xH, H) < \frac{1}{2^n}\right] = \left[xH|\ d(x, e) < \frac{1}{2^n}\right] \quad \text{for} \quad n = 1, 2, \ldots$$

For,

$$d(x, e) < \frac{1}{2^n} \Rightarrow d^*(xH, H) \leqslant d(x, e) < \frac{1}{2^n}$$

and, conversely,

$$d^*(xH, H) < \frac{1}{2^n} \Rightarrow \inf[d(xa, b)|\ a, b \in H] < \frac{1}{2^n}.$$

Hence, corresponding to n there exist $a, b \in H$ such that $d(xa, b) < \dfrac{1}{2^n}$;

$$d(xa, b) < \frac{1}{2^n} \Rightarrow d(xab^{-1}, e) < \frac{1}{2^n}$$

$$\Rightarrow xH\ (= xab^{-1}H) \in \left[xH|\ d(x, e) < \frac{1}{2^n}\right].$$

Thus the topology of the metric d^* on G/H and the quotient topology of G/H coincide. This completes the proof of Theorem 2.

The proof of Theorem 3 below illustrates the use of Theorem 1. Theorem 3 can be proved alternatively by introducing a non-archimedian metric in the set of Cauchy sequences in G (cf. [6], p. 485).

THEOREM 3. *If G is a two sided invariant non-archimedian metric group, then there exists a non-archimedian complete metric group $\hat{G}$ such that G is a dense subgroup of $\hat{G}$.*

Proof. G being a two sided invariant non-archimedian metric group (consequently a metric group, in the usual sense), it can be imbedded as a dense subgroup of a complete metric group $\hat{G}$ ([6], p. 485, (1.4)). Since the non-archimedian metric on G is two sided invariant, there exists a countable base of neighbourhoods of normal subgroups at the identity e of G (see Remark following Theorem 1). The closures in $\hat{G}$ of these subgroups, which are also normal in $\hat{G}$ ([3], p. 46, 5.37 (c)), constitute a base of neighbourhoods ([2], p. 30, Proposition 7) (1) at e for $\hat{G}$. Hence, by Theorem 1, $\hat{G}$ is also non-archimedian metrizable. Further $\hat{G}$ is complete with respect to this non-archimedian metric (see [5], p. 212, Exercise $Q(d)$). The proof of Theorem 3 is now complete.

(1) It is sufficient to take a base at e for G, instead of all neighbourhoods at e, for the validity of the proposition referred to.

References

[1] G. Birkhoff, *A note on topological groups*, Compositio Math. 3 (1936), pp. 427–430.
[2] N. Bourbaki, *Topologie générale*, Chapter 3, Actualités Sci. Ind., No. 916–1143, Paris 1951.
[3] E. Hewitt and K. A. Ross, *Abstract harmonic analysis*, Vol. I, Berlin 1963.
[4] S. Kakutani, *Über die Metrization der topologischen Gruppen*, Proc. Imp. Acad. Japan 12 (1936), pp. 82–84.
[5] J. L. Kelley, *General topology*, New York 1955.
[6] V. L. Klee, *Invariant metrics in groups*, Proc. Amer. Math. Soc. 3 (1952), pp. 484–487.
[7] A. F. Monna, *Remarques sur les metriques non-archimediennes I*, Nederl. Akad. Wetensch. Proc., ser A, 53 (1950), pp. 470–481.
[8] D. Montgomery and L. Zippin, *Topological transformation groups*, New York 1955.

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A three-dimensional spheroidal space which is not a sphere

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1. Introduction. In [1], we described an upper semicontinuous decomposition of E^3 into straight arcs and singletons such that the associated decomposition space E^3/G is topologically distinct from E^3. In this note, we study local properties of the decomposition space.

We shall show that E^3/G is *locally peripherally spherical*, i.e., each point of E^3/G has arbitrarily small neighborhoods bounded by 2-spheres. In fact, each point of E^3/G has arbitrarily small closed neighborhoods which are compact absolute retracts and have 2-spheres as their topological boundaries. In particular, each point of the space has arbitrarily small compact simply connected neighborhoods.

We shall also use the decomposition of [1] to settle a question of Borsuk's concerning spheroidal spaces. A metric space X is a *spheroidal space* if and only if for each point p of X and each neighborhood U of p, there is a neighborhood V of p such that $V \subset U$ and $X-V$ is a compact absolute retract. It is known that each spheroidal space of dimensions 0, 1, and 2 is a sphere [3]. In [3], Borsuk describes an example (due to Ganea) of a spheroidal space of dimension 4 not a sphere. Borsuk [3] raises the following question: *Does there exist a 3-dimensional spheroidal space which is not a sphere?* We give an affirmative answer to this question. Regard S^3 as the one-point compactification $E^3 \cup \{\infty\}$ of E^3. Let G^* denote the upper semicontinuous decomposition of S^3 consisting of all the elements of G, together with $\{\infty\}$. Then associated decomposition space, S^3/G^*, is a 3-dimensional spheroidal space which is not a sphere. In fact, S^3/G^* has the following property: Each point of S^3 has arbitrarily small open neighborhoods V such that the closure of V is a compact absolute retract, the complement of V is a compact absolute retract, and the boundary of V is a 2-sphere.

Throughout this note, we retain the notation of [1]. G denotes the decomposition of E^3 described in [1], E^3/G denotes the associated