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Non-manifold factors of Euclidean spaces

by

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1. Introduction. The object of this paper is to define a class of factors of Euclidean n-space which contain some non-manifolds (i.e., Theorems 3.6 and 4.3). These factorizations will be general enough to include those given by R. H. Bing [4] and John Hemple [5].

Throughout this paper we will use the following terminology: (i) Any subset of a topological space which is homeomorphic to I^n, where $I=[0,1]$, will be called an n-cell. (ii) An n-manifold will be a paracompact Hausdorff space in which every point has a neighborhood whose closure is an n-cell. (iii) If X is a topological space and $D\subset X$ then by $\operatorname{int} D$ is meant the set $X-\overline{X-D}$, where $\overline{X-D}$ is the closure of $X-D$ in X.

2. Separation Theorems.

LEMMA 2.1. *Let $C_1, C_2, \ldots, C_p$ be disjoint compact subsets of a Hausdorff space X. Let $D_1, D_2, \ldots, D_p$ be (not necessarily disjoint) n-cells such that for each $i=1,2,\ldots,p$, $C_i\subset \operatorname{int} D_i$. Then for any $[a,b]\subset E^1$ and $\varepsilon>0$ there exist disjoint $(n+1)$-cells $E_1, E_2, \ldots, E_p$ contained in $X\times(a-\varepsilon, b+\varepsilon)$ such that for each $i=1,2,\ldots,p$*

(1) $C_i\times[a,b]\subset \operatorname{int} E_i$;

(2) $\Pi_1 E_i = D_i$;

where Π_1 is the projection of $X\times E^1$ onto X.

Proof. Let $f\colon [-\varepsilon, r+\varepsilon]\to[a-\varepsilon, b+\varepsilon]$ be the homeomorphism given by

$$f(x)=\begin{cases} a+x & \text{if } x\in[-\varepsilon,0],\\ \left(\dfrac{b-a}{r}\right)x+a & \text{if } x\in[0,r],\\ b+x-r & \text{if } x\in[r,r+\varepsilon].\end{cases}$$

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Let $k\colon [-\varepsilon, r+\varepsilon]\to[-\varepsilon, 2p-1+\varepsilon]$ be the homeomorphism given by

$$k(t)=\frac{2(p+\varepsilon)-1}{r+2\varepsilon}(t+\varepsilon)-\varepsilon\,.$$

For each $j=1,2,\ldots,p$ let k_j be a homeomorphism of $[-\varepsilon, r+\varepsilon]$ onto $[-\varepsilon, 2p-1+\varepsilon]$ with the properties:

(1) $k_j(-\varepsilon)=-\varepsilon$ and $k_j(t+\varepsilon)=(2p-1+\varepsilon)$,

(2) $k_j(0)=2j-2$,

(3) $k_j(r)=2j-1$.

Let $A=\bigcup_i D_i\subset X$ and note that since A is a compact Hausdorff space it is normal. $\operatorname{Bdy}A=A-\operatorname{int}A$ and C_j for $j=1,\ldots,p$ are closed sets. Thus there exist open sets U_j for $j=1,2,\ldots,p$ satisfying

(1) $U_i\cap U_j=\emptyset$ if $i\neq j$,

(2) $C_i\subset U_i$ for all $i=1,\ldots,p$,

(3) $U_i\subset\operatorname{int}A$.

By the Urysohn Lemma there exists a continuous function g mapping A onto $[0,1]$ such that

(1) $g(\bigcup_i C_i)=1$,

(2) $g(A-\bigcup_i U_i)=0$.

Construct $h\colon A\times[-\varepsilon, r+\varepsilon]\to A\times[-\varepsilon, 2p-1+\varepsilon]$ as follows:

$$h(x,t)=\begin{cases}\big(x, g(x)k_j(t)+(1-g(x))k(t)\big) & \text{for } (x,t)\in\overline{U_j}\times[-\varepsilon, r+\varepsilon]\,,\\ \big(x, k(t)\big) & \text{for } (x,t)\in(A-\bigcup_i U_i)\times[-\varepsilon, r+\varepsilon]\,.\end{cases}$$

For each $j=1,2,\ldots,p$, $h=\mathrm{id}\times k$ on $\operatorname{Bdy}\overline{U_j}\times[-\varepsilon, k+\varepsilon]$, hence h is well-defined. h is continuous since g, k and k_j are all continuous. Suppose $h(x_1,t_1)=h(x_2,t_2)$ then $x_1=x_2=x$.

First, if $x\in U_j$ then

$$g(x)k_j(t_1)+[1-g(x)]k(t_1)=g(x)k_j(t_2)+[1-g(x)]k(t_2)$$

or

$$g(x)[k_j(t_1)-k_j(t_2)]+\big(1-g(x)\big)[k(t_1)-k(t_2)]=0\,.$$

But $g(x)$ and $1-g(x)\geqslant 0$ and both k and k_j are order preserving homeomorphisms, whence $k_j(t_1)-k_j(t_2)$ and $k(t_1)-k(t_2)$ are both positive, zero, or negative together. Therefore $k(t_1)-k(t_2)=0$ and $t_1=t_2$.

Second, if $x\in A-\bigcup_i U_i$ then $k(t_1)=k(t_2)$ and again $t_1=t_2$.

Thus h is one-to-one and continuous hence a homeomorphism. h can be extended to a homeomorphism of $X\times[-\varepsilon, r+\varepsilon]\to X\times[-\varepsilon, 2p-1+\varepsilon]$ by defining $h(x,t)=\big(x,k(t)\big)$ on $X-A$.

For each $j=1,\ldots,p$ let E'_j be the $(n+1)$-cells defined by

$$E'_j=D_j\times[2j-2-\tfrac14, 2j-1+\tfrac14]\,.$$

Now define for each $j=1,2,\ldots,p$

$$E_j=(\mathrm{id}\times f)\big(h^{-1}(E'_j)\big)\,.$$

Clearly $E_j\cap E_i=\emptyset$ if $i\neq j$ and $\Pi_1E_j=D_j$. Moreover

$$E_i\subset\operatorname{int}A\times(a-\varepsilon, b+\varepsilon)\,.$$

If $x\in C_i\times[a,b]$ then

$$(\mathrm{id}\times f)^{-1}(x)\in C_i\times[0,2r-1]$$

and

$$h\cdot(\mathrm{id}\times f)^{-1}(x)\in C_i\times[2i-2, 2i-1]\,.$$

But

$$C_i\times[2i-2,2i-1]\subset\operatorname{int}(D_i\times[2i-2-\tfrac14, 2i-1+\tfrac14])$$

whence

$$(\mathrm{id}\times f)\cdot h^{-1}\cdot h\cdot(\mathrm{id}\times f)^{-1}(x)\in\big(\mathrm{id}\times f\big)h^{-1}(D_i\times[2i-2-\tfrac14, 2i-1+\tfrac14]\big)$$

and $x\in E_i$. Thus the E_i $i=1,2,\ldots,p$ satisfy all the claims of the theorem.

Lemma 2.2. *Suppose $B\neq\emptyset$ is a compact subset of $\operatorname{int}I^n$ and C is a compact subset of I^n disjoint from B. Similarly, suppose $D\neq\emptyset$ is a compact subset of $\operatorname{int}I^m$ and E is a compact subset of I^m disjoint from D. Then there exists an $(n+m)$-cell G with the following properties:*

(1) $B\times D\subset\operatorname{int}G\subset G\subset\operatorname{int}I^n\times\operatorname{int}I^m$,

(2) $G\cap\{(B\times E)\cup(C\times D)\cup(C\times E)\}=\emptyset$.

Proof. Let $T\subset\operatorname{int}I^n$ be an n-cell such that $T\cap(B\cup C)=\emptyset$ and T is the product of its projections. Such an n-cell exists since $C\cap B=\emptyset$ and they are each closed. Similarly let $R\subset\operatorname{int}I^m$ be an m-cell such that $R\cap(D\cup E)=\emptyset$ and R is the product of its projections. Let $\Pi_iT=[t_i,t'_i]$ for each $i=1,2,\ldots,n$ and $\Pi_jR=[r_j,r'_j]$ for each $j=1,2,\ldots,m$ where Π_α is the projection onto the αth coordinate.

Let $\delta_1=\min[\text{distance from } B \text{ to } (\operatorname{Bdy}I^n\cup C), \text{distance from } T \text{ to } (B\cup C\cup\operatorname{Bdy}I^n)]$. Let $\delta_2=\min[\text{distance from } D \text{ to } (\operatorname{Bdy}I^m\cup E), \text{distance from } R \text{ to } (D\cup E\cup\operatorname{Bdy}I^m)]$. Set $\delta=\min(\delta_1,\delta_2)$. Let $k, k_1, k_2, \ldots, k_m$ be homeomorphisms defined as follows:

(1) $k\colon[0,1]\to[0,1]$ such that $k(0)=0$, $k(1)=1$ and $k[\delta/2, 1-\delta/2]=[1/4, 3/4]$.

(2) For each $i=1,2,\ldots,m$ let $k_i\colon[0,1]\to[0,1]$ such that $k_i(0)=0$, $k_i(1)=1$ and $k_i[r_i,r'_i]=[1/4,3/4]$.

Let U_n be an open subset of $(\delta/2, 1-\delta/2)^n$ such that $B \subset U_n$ and $U_n \cap C = \emptyset$. Set $W_n = I^n - [\delta/3, 1-\delta/3]^n$. By the Urysohn Lemma there exists a continuous function $g\colon I^n \to [0,1]$ such that

$$g(B \cup \operatorname{Bdy} I^n) = 1 \quad \text{and} \quad g(I^n - (U_n \cup W_n)) = 0 .$$

Consider the following collection of maps

$$h_i\colon I^n \times I^m \to I^n \times I^m, \qquad i = 1, 2, \ldots, m .$$

For $x \in I^n$ and $(y_1, \ldots, y_m) \in I^m$

$$h_i(x, y_1, \ldots, y_m) = \begin{cases} \big(x, (y_1, \ldots, y_{i-1}, g(x)k(y_i) + (1-g(x))k_i(y_i), y_{i+1}, \ldots, y_m)\big) \\ \qquad \text{for } (x, (y_1, \ldots, y_m)) \in \overline{U_n} \times I^m, \\ \big(x, (y_1, \ldots, y_{i-1}, g(x)y_i + (1-g(x))k_i(y_i), y_{i+1}, \ldots, y_m)\big) \\ \qquad \text{for } (x, (y_1, \ldots, y_m)) \in \overline{W_n} \times I^m, \\ \big(x, (y_1, \ldots, y_{i-1}, k_i(y_i), y_{i+1}, \ldots, y_m)\big) \\ \qquad \text{for } (x, (y_1, \ldots, y_m)) \in (I^n - (W_n \times U_n)) \times I^m . \end{cases}$$

Each h_i is well-defined since $\overline{U_n} \cap \overline{W_n} = \emptyset$, $\Pi_j h_j / \operatorname{Bdy} U_n \times I^m = \Pi_j$ for all $j \neq i$ and

$$\Pi_i h_i / \operatorname{Bdy} U_n \times I^m = k_i \Pi_i ,$$

where again Π_i is the projection onto the ith coordinate axis. And $\Pi_i h_i / \operatorname{Bdy} W_n \times I^m = \Pi_j$ for all $j \neq i$,

$$\Pi_i h_i / \operatorname{Bdy} W_n \times I^m = k_i \Pi_i .$$

Clearly each h_i is continuous and onto $I^n \times I^m$. Suppose for $x, x' \in I$ and $(y_1, \ldots, y_m), (z_1, \ldots, z_m) \in I^m$ we have

$$h_i(x, (y_1, \ldots, y_m)) = h_i(x', (z_1, \ldots, z_m))$$

then $x = x'$ and $y_j = z_j$ for $j \neq i$. Consider the three cases:

(1) $x \in U_n$,

(2) $x \in W_n$, or

(3) $x \in I^n - (U_n \cup W_n)$.

Case (1):

$$g(x)k(y_i) + (1-g(x))k_i(y_i) = g(x)k(z_i) + (1-g(x))k_i(z_i)$$

and

$$g(x)(k(y_i) - k(z_i)) + (1-g(x))(k_i(y_i) - k_i(z_i)) = 0 .$$

Because $0 \leqslant g(x) \leqslant 1$ and k as well as k_1 preserve order it follows that $y_i = z_i$.

Similar arguments show that for Cases (2) and (3), $y_i = z_i$. Thus for each i, h_i is an injection consequently a homeomorphism.

Define $H\colon I^n \times I^m \to I^n \times I^m$ to be the homeomorphism $h_1 \cdot h_2 \cdot \ldots \cdot h_m$. Set

$$J = [\delta/2, 1-\delta/2]^n \times [1/4, 3/4]^m \subset I^n \times I^m .$$

If $(x, y) \in B \times D$ then $x \in U_n$ and $H(x, y) \in J$. Thus $H(B \times D) \subset J$. Let $(x, y) \in (C \times (D \cup E))$ then $x \in I^n - U_n$ and there exists a j such that $\Pi_j(y) \in I - [r_j, r_j']$. If $x \in W_n$ then $H(x, y) \notin J$. If $x \in I^n - (W_n \times U_n)$ then $\Pi_j h_j(x, y) \in I - [1/4, 3/4]$ and $H(x, y) \notin J$. Thus $H(C \times (D \cup E)) \cap J = \emptyset$.

Note that $H/B \times I^m = \mathrm{id} \times k^*$ where $k^* = (K, K, \ldots, K)$ with m factors. Thus it follows that $\Pi_m^* H(B \times D)$ and $\Pi_m^* H(B \times E)$ are disjoint compact subset of $[1/4, 3/4]^m \subset I^m$, where Π_m^* is the projection of $I^n \times I^m$ onto I^m. Also

$$\Pi_m^* H(B \times D) \subset (1/4, 3/4)^m .$$

Let $\gamma = \min(\text{distance from } \Pi_m^* H(B \times D) \text{ to } \operatorname{Bdy}[1/4, 3/4^m, \delta/2)$. Let U_m be an open set in $(1/4 + \gamma/2, 3/4 - \gamma/2)^m$ such that

$$\Pi_m^* H(B \times D) \subset U_m \quad \text{and} \quad U_m \cap \Pi_m^* H(B \times E) = \emptyset .$$

Let $W_m = [1/4, 3/4]^m - [1/4 + \gamma/3, 3/4 - \gamma/3]^m$. There exists a continuous function $f\colon [1/4, 3/4]^m \to [0, 1]$ such that

(1) $f/\Pi_m^*(H(B \times D) \cup \operatorname{Bdy}[1/4, 3/4]^m) = 1$,

(2) $f/[1/4, 3/4]^m - (U_m \cup W_m) = 0$.

Let $\psi, \psi_1, \ldots, \psi_n$ be homeomorphisms defined as follows:

(1) $\psi\colon [\delta/2, 1-\delta/2] \to [\delta/2, 1-\delta/2]$ such that $\psi(\delta/2) = \delta/2$, $\psi(1-\delta/2) = 1-\delta/2$ and $\psi[\delta/2 + \gamma/2, 1-\delta/2-\gamma/2] = [1/4, 3/4]$.

(2) For each $i = 1, 2, \ldots, n$ let $\psi_i\colon [\delta/2, 1-\delta/2] \to [\delta/2, 1-\delta/2]$ such that $\psi_i(\delta/2) = \delta/2$, $\psi_i(1-\delta/2) = 1-\delta/2$ and $\psi_i[t_i, t_i'] = [1/4, 3/4]$. Consider the following collection of maps:

$$\theta_i\colon I^n \times I^m \to I^n \times I^m \qquad i = 1, 2, \ldots, n .$$

For $(x_1, x_2, \ldots, x_n) \in I^n$ and $y \in I^m$

$$\theta_i((x_1, \ldots, x_n), y) = \begin{cases} \mathrm{id} \quad \text{for } ((x_1, \ldots, x_n), y) \in (I^n \times I^m) - \operatorname{int} J, \\ \big((x_1, \ldots, x_{i-1}, f(y)\psi(x_i) + (1-f(y))\psi_i(x_i), x_{i+1}, \ldots, x_n), y\big) \\ \qquad \text{for } ((x_1, \ldots, x_n), y) \in [\delta/2, 1-\delta/2]^n \times U_m, \\ \big((x_1, \ldots, x_{i-1}, f(y)x_i + 1 - f(y)\psi_i(x_i), x_{i+1}, \ldots, x_n), y\big) \\ \qquad \text{for } ((x_1, \ldots, x_n), y) \in [\delta/2, 1-\delta/2]^n \times W_m, \\ \big((x_1, \ldots, x_{i-1}, \psi_i(x_i), x_{i+1}, \ldots, x_n), y\big) \\ \qquad \text{on } [\delta/2, 1-\delta/2]^n \times ([1/4, 3/4^m] - (U_m \cup W_m)) . \end{cases}$$

Each θ_i is well-defined since $\overline{W_m} \cap \overline{U_m} = \emptyset$. By an argument exactly like the one given above for h_i, each θ_i is a homeomorphism. Define $\theta = \theta_1 \circ \theta_2 \circ \ldots \circ \theta_n$.

Set

$$J' = [1/4, 3/4]^n \times [1/4+\gamma/2, 3/4-\gamma/2]^m \subset J .$$

If $(x, y) \in B\times D$ then $\theta H(x, y) \in J'$. Thus $\theta H(B\times D) \subset J'$. If $(x, y) \in \big((C\times D) \cup (C\times E)\big)$ then $H(x, y) \notin J$ hence $\theta\cdot H(x, y) \notin J'$. Suppose $(x, y) \in B\times E$ then $\Pi_m^* H(x, y) \in [1/4, 3/4]^m - U_m$ and there exists a j such that $\Pi_j(x) \notin [t_j, t_j']$. If $\Pi_m^* H(x, y) \in W_m$ then $\theta\cdot H(x, y) \notin J'$ since

$$\Pi_m^* \theta\cdot H(x, y) = \Pi_m^* H(x, y) \quad \text{and} \quad W_m \cap \Pi_m^* J' = \emptyset .$$

If $\Pi_m^* H(x, y) \in \Pi_m^*(J) - W_m - U_m$ then $\Pi_j \Pi_m^* H(x, y) \notin [1/4, 3/4]$ and $\theta\cdot H(x, y) \notin J'$. Therefore $\theta\cdot H\{(B\times E) \cup (C\times D) \cup (C\times E)\}$ is contained in $I^n\times I^m - J'$.

Define $G = H^{-1}\cdot\theta^{-1}(J')$. G is the $(n+m)$-cell contained in $\operatorname{int} I^n \times \operatorname{int} I^m$ satisfying properties 1 and 2 of theorem.

Note that J' defined in the above proof in the product of cells. Thus a proof similar to that of Theorem 2.1 would prove the following theorem.

LEMMA 2.3. *Suppose B_i, $i = 1, 2, \ldots, p$ are disjoint compact subsets of $\operatorname{int} I^n$, one of which is non-empty, and C is a compact subset of I^n disjoint from $B = \bigcup_i B_i$. Similarly suppose D_j, $j = 1, 2, \ldots, q$ are disjoint compact subsets of $\operatorname{int} I^m$, one of which is non-empty, and E is a compact subset of I^m disjoint from $\bigcup_i D_i = D$. Then there exist $(n+m)$-cells G_{ij} $i = 1, 2, \ldots, p$, $j = 1, 2, \ldots, q$ such that*

(1) $G_{ij} \cap G_{rs} = \emptyset$ *if* $i \neq r$ *or* $j \neq s$,

(2) $B_i\times D_j \subset \operatorname{int} G_{ij} \subset G_{ij} \subset \operatorname{int}(I^n\times I^m)$,

(3) $\bigcup_i \bigcup_j G_{ij} \cap \{(C\times D) \cup (B\times E) \cup (C\times E)\} = \emptyset$.

3. A class of factorizations of E^n. In this section we shall define a class of upper semi-continuous decompositions of E^n and prove that the associated decomposition spaces are factors of E^{n+1}. This class contains the decompositions for each of the spaces (a) "dogbone space", (b) "unused example" and (c) "segment space" ([5]).

DEFINITION 3.1. Suppose α is an arc in E^n (i.e. $\alpha = h[0, 1]$ for some homeomorphism $h\colon I\to E^n$) such that $P = \Pi_1/\alpha$ is an injection, where Π_1 is the projection of E^n onto the 1st coordinate. In this case α will be said to have *property* QS.

Let α be an arc with property QS and assume that $\Pi_1 h(1) = b$ and $\Pi_1 h(0) = a$ with $a < b$. Define the continuous function $f\colon E^1\to E^n$ by

$$f(t) = \begin{cases} P^{-1}(a) & \text{for} \quad t \leqslant a , \\ P^{-1}(t) & \text{for} \quad a \leqslant t \leqslant b , \\ P^{-1}(b) & \text{for} \quad b \leqslant t . \end{cases}$$

Define the homeomorphism $k\colon E^1\times E^{n-1}\to E^1\times E^{n-1}$ by $k(t, x) = \big(t, x-f(t)\big)$. For any $\varepsilon > 0$ let

$$C_1 = \{z|\ z \in E^n, \|z-a\| \leqslant \varepsilon\} ,$$

$$C_2 = \{z|\ z \in E^n, \|z-b\| \leqslant \varepsilon\} ,$$

$$C_3 = \{z|\ z \in E^n, a \leqslant \Pi_1 z \leqslant b \text{ and } \|z-\Pi_1 z\| \leqslant \varepsilon\}$$

then $Q_\varepsilon = C_1 \cup C_2 \cup C_3$ is an n-cell containing $\Pi_1(a)$. The n-cell $k^{-1}(Q_\varepsilon)$ will be called an ε-radial neighborhood of α.

Remark 3.1. Note that if α is an arc with property QS then for any $\varepsilon > 0$ the ε-radial neighborhood of α intersects the planes $\Pi_1^{-1}(t) = R_t = \{(t, y)|\ (t, y) \in t\times E^{n-1}\}$ is void, a single point, or an $(n-1)$-cell.

Remark 3.2. Suppose α is an arc which has property QS. Since the homeomorphism used to define radial neighborhood is uniformly continuous, it follows that for any $\varepsilon > 0$ there exists a $\delta > 0$ and a collection of planes $R_i = \Pi^{-1}(t_i)$ with $t_1 = u < t_2 < \ldots < t_p = b$ such that the R_i cut the δ-radial neighborhood of α into $(p+1)$ n-cells C_i; $i = 0, 1, \ldots, p$ and $\operatorname{diam} C_i \leqslant \varepsilon$.

Let $A_1, A_2, \ldots$ be a sequence of compact n-manifolds (not necessarily connected) in E^n satisfying

P1. $A_{i+1} \subset \operatorname{int} A_i$ for all $i = 1, 2, 3, \ldots$

P2. Each component of $A_\infty = \bigcap_i A_i$ is an arc with property QS.

Let Γ_n' be the class of upper semi-continuous decompositions of E^n into arcs A_∞ and points of $E^n - A_\infty$. Further let Γ_n be the class of associated decomposition spaces.

LEMMA 3.1. *Suppose $\varepsilon > 0$ and A_i are as defined above, then there exists a finite collection of n-cells U_i satisfying:*

1. *For each U_i there exists an arc $a_1 \subset A_\infty \cap \operatorname{int} U_i$ such that the distance from x to $\operatorname{Bdy} U_i$ is less than ε for all $x \in a_i$.*

2. *There exists an integer m such that if A is a component of A_m then $A \subset \operatorname{int} U_i$ for some i.*

Proof. For each arc $\alpha \in A_\infty$ let N_α be the $\varepsilon/2$-radial neighborhood of α. For each N_α there exists a neighborhood $V_\alpha \subset N_\alpha$ with the property that if an arc $\beta \subset A_\infty$ intersects V_α non-trivially then $\beta \subset N_\alpha$. The existence of such V_a's follows from the fact that the decomposition of E^n into the arcs of A_∞ and the points of $E^n - A_\infty$ is an upper semi-continuous decomposition. The collection of sets $\{V_a|\ a \subset A_\infty\}$ is an open cover of the compact set A_∞. Thus there is a finite subcollection $V_1, V_2, \ldots, V_p$ which covers A_∞. Let $N_1, N_2, \ldots, N_p$ be the corresponding N_a's. Note that by the choice of the V_a's we have each arc $a \subset A_\infty$ contained in the interior of at least one N_i. For each arc $a \subset A_\infty$ there exists an integer $m(a)$ such that

1. $a \subset A^a_{m(a)} \subset A_{m(a)}$ where $A^a_{m(a)}$ is the component of $A_{m(a)}$ containing a;
2. $A^a_{m(a)} \subset \operatorname{int} N_i$, for some $i = 1, 2, \ldots, p$.

The collection $\{\operatorname{int} A^a_{m(a)}|\ a \subset A_\infty\}$ is an open cover of A_∞. Therefore there is a finite subcover. From this collection of $A^a_{m(a)}$'s there is one with largest subscript $m(a)$. $m = m(a)$ is the desired integer. Each N_i is the $\varepsilon/2$-radial neighborhood of some $a \subset A_\infty$. Therefore the collection $U_i = N_i$ satisfies the claims of the Lemma.

LEMMA 3.2. *Suppose A_i, $i = 1, 2, \ldots$, are defined as above and A is a component of A_r for some r. Given $\varepsilon > 0$ then there exist integers $\gamma(1), \gamma(2), \ldots, \gamma(m+1)$ and sets $K_{ij} \subset A \times E^1$, $i = 1, 2, \ldots, s$; $j = 1, 2, \ldots, m$ which satisfy the following conditions:*

1. *For each i, K_{i0} is an $(n+1)$-cell and K_{ij} is the disjoint union of $(n+1)$-cells K_{ijk}, $k = 1, 2, \ldots, \mu(i, j)$;*
2. *$K_{i0} \cap K_{e0} = \emptyset$ if $i \neq e$;*
3. *$\bigcup_i K_{ij} \subset (A_{\gamma(j)} \cap A) \times [j, 2m+1-j]$, $\bigcup_i K_{ij+1} \subset (\operatorname{int} A_{\gamma(j)} \cap A) \times (j, 2m+1-j)$ for each j;*
4. *For each i K_{i0} can be written as the union of $(n+1)$-cells D_{ie}, $e = 0, 1, \ldots, m$, such that $D_{ie} \cap D_{iv} = \operatorname{Bdy} D_{ie} \cap \operatorname{Bdy} D_{iv}$ is an n-cell if $|e-v| = 1$ and is void if $|e-v| > 1$;*
5. *Diameter of $\Pi^*_n(D_{ie}) < \varepsilon$ for all i, e, where Π^*_n is the projection $E^n \times E^1 \to E^n$;*
6. *$D_{ie} \cap D_{iv} \cap K_{ijk}$ is either void or an n-cell.*

Proof. Let the ε of Lemma 3.1 be the min (ε, distance from $A_\infty \cap A$ to $\operatorname{Bdy} A$) hence there exists a finite set of n-cells K'_{i0}, $i = 1, 2, \ldots, s$ and an integer $\gamma(1)$ satisfying:

a. $K'_{i0} \subset \operatorname{int} A$ for all i;

b. If A' is a component of $A_{\gamma(1)} \cap A$ then $A' \subset \operatorname{int} K'_{i0}$ for some i.

Note that the K'_{i0} may not be disjoint. By Remark 2.2 each n-cell K'_{i0} can be chosen so that there is a finite set of planes R_{ij}, $j = 1, 2, \ldots, m_i$ which cut K'_{i0} into (m_i+1) n-cells D'_{ij} such that

$$D'_{ij} \cap D'_{iv} = \operatorname{Bdy} D'_{ij} \cap \operatorname{Bdy} D'_{iv}$$

is an $(n-1)$-cell if $|j-v| = 1$ and is void if $|j-v| > 1$. Without loss of generality assume $m_i = m$ for all i.

Similarly apply Lemma 3.1 to each component of $A_{\gamma(1)} \cap A$ to obtain an integer $\gamma(2)$ and sets K'_{i1} where K'_{i1} is the union of n-cells K'_{i1k}, $k = 1, 2, \ldots, \mu(i, 1)$, satisfying:

(i) If A^* is a component of $A_{\gamma(2)} \cap A$ then

$$A^* \subset \operatorname{int} K'_{i1k} \subset K_{i1k} \subset \operatorname{int} A' \subset K'_{i0}$$

for some k and some component A' of $A_{\gamma(1)} \cap A$.

(ii) $K_{i1k} \not\subset R_{ij}$ is either void or an $(n-1)$-cell.

Condition (ii) actually follows from the proof of Lemma 3.1. Continue this procedure to obtain the integers $\gamma(3), \gamma(4), \ldots, \gamma(m+1)$ and sets K'_{ij} as well as n-cells K'_{ijk} satisfying conditions analogous to (i) and (ii).

For each i and j define W_{ijl} to be the union of the components of $A_{\gamma(j+1)} \cap A$ which are contained in K'_{ijl} but not in K'_{ijp} for any $p < l$. Note that W_{ijl} are compact and $W_{ijk} \cap W_{ijl} = \emptyset$ if $k \neq l$. Let $\{W_{i01}\}$ and $\{K'_{i0}\}$ be respectively $\{C_i\}$ and $\{D_i\}$ of Lemma 2.1 and let $a - \varepsilon = 0$ and $b + \varepsilon = 2m+1$. Then define $K_{i0} = E_i$ of Lemma 2.1. By the proof of Lemma 2.1 we see that K_{i0} can be written as the union of $(n+1)$-cells D_{il} such that $\Pi^*_n D_{il} = D'_{il}$. Further the D_{il} satisfy condition 4.

In general let $\{W_{ijk}\}$ and $\{K'_{ijk}\}$ be respectively $\{C_{ik}\}$ and $\{D_{ik}\}$ of Lemma 2.1 and let $\varepsilon = 1/2$, $a = j$ and $b = 2m+1-j$. If $K_{ijk} = E_{ik}$ of Lemma 2.1 and $K_{ij} = \bigcup_k E_{ik}$ then conditions 1 through 5 are clearly satisfied and condition 6 follows from (ii) above.

Remark 3.3. Note that if $i \neq r$ and A' is a component of $A_{\gamma(j+1)} \cap A$ contained in K'_{rj} then $K_{ij+1} \cap A' \times E^1 = \emptyset$ since $K'_{ij} \cap K'_{rj} \subset A - A_{\gamma(j+1)}$. Also $K'_{ijp} \cap K'_{ijp} = \emptyset$ if they are not in the same n-cell of K'_{ij-1}.

The proof of the next lemma is based on the following known result.

THEOREM. *Suppose that A is an n-cell which is the union of two n-cells A_1 and A_2 with the properties that $A_1 \cap A_2$ and $\operatorname{Bdy} A_1 \cap \operatorname{Bdy} A_2$ are $(n+1)$-cells and $A_1 \cap A_2 \subset \operatorname{Bdy} A_1 \cap \operatorname{Bdy} A_2$. If $B \subset A$, B is compact and $B \cap \operatorname{Bdy} A \subset A_2$ then there exists a homeomorphism h of A onto A which is fixed on the $\operatorname{Bdy} A$ and such that $h(B) \subset A_2$.*

LEMMA 3.3. *For $\varepsilon \geqslant 0$ and A a component of A_r (where A_i, $i = 1, 2, \ldots$, are defined as above) let $\gamma(f)$, D_{il}, K_{ij}, and K_{ijk} be as in Lemma 3.2. Then there exists a homeomorphism $h\colon E^n \times E^1 \to E^n \times E^1$ such that the following hold:*

1. *$h = \mathrm{id}$ on complement of $\bigcup_i K_{i1}$;*
2. *$h = \mathrm{id}$ on the complement of*

$$\bigcup_i \Big(\big(K_{i1} \cap (D_{i0} \cup D_{i1})\big) \cup \big(K_{i2} \cap (D_{i1} \cap D_{i2})\big) \cup \ldots \cup \big(K_{im} \cap (D_{im} \cup D_{im})\big) \Big);$$

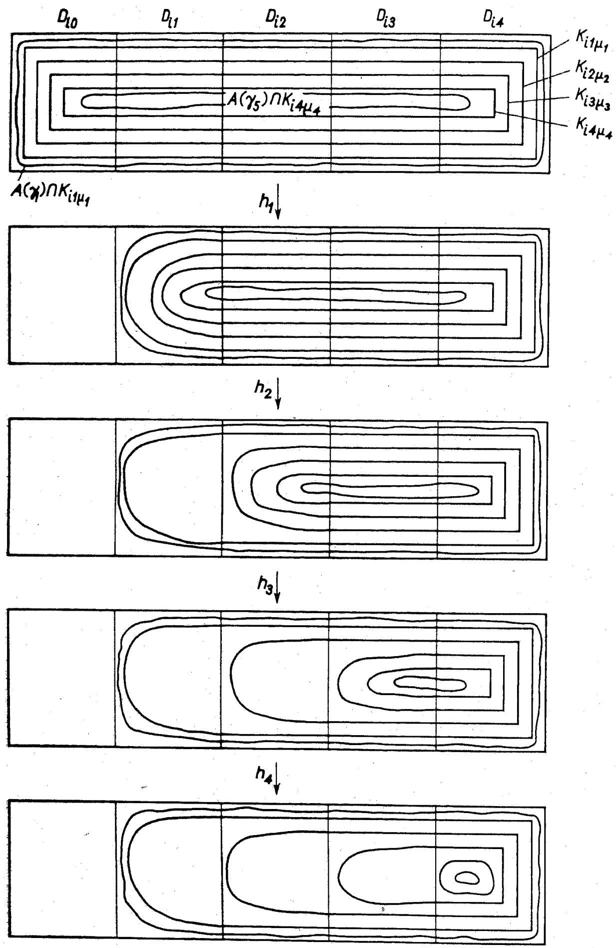

Fig. 1

3. *If* $A' \subset A_{\gamma(j+1)} \cap A$ *and* $K_{ijk} \supset A' \times [j, 2m+1-j]$ *then*

$$h\big((D_{i0} \cup \ldots \cup D_{ij}) \cap A' \times ([j, j+1] \cup [2m-j, 2m+1-j])\big)$$

is contained in $D_{i\psi} \cup D_{i\psi+1}$ *where* $\psi = \min(j, \max\{e \mid K_{iek} \cap D_{ie} \neq \emptyset, K_{iek} \supset A'\})$.

Before reading the proof of Lemma 3.3 it may be helpful to look at Figures 1 and 2. The homeomorphism h will be obtained as the composition of homeomorphisms $h_{m-1} \circ h_{m-2} \circ \ldots \circ h_1$. Figure 1 illustrates how the h_j will be constructed. The shaded region of Figure 2 is that part of $A \times [0, 2m+1]$ which is not moved by h.

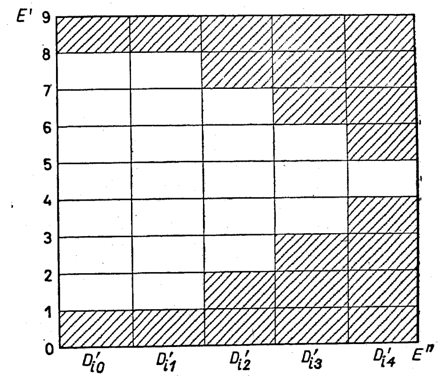

Fig. 2

Proof of Lemma 3.3. Let $h_1: E^n \times E^1 \to E^n \times E^1$ be a homeomorphism defined as follows:

$$h_1 = \mathrm{id} \quad \text{on} \quad E^n \times E' - \bigcup_i \big(K_{i1} \cap (D_{i0} \cup D_{i1})\big).$$

For each i and A' a component of $A_{\gamma(2)}$ with $A' \times [1, 2m] \subset K_{i1k}$, then

a. If $K_{i1k} \cap D_{i0} = \emptyset$ or $K_{i1k} \cap D_{i1} = \emptyset$ then

$$h_1 = \mathrm{id} \text{ on } K_{i1k};$$

b. If $K_{i1k} \cap K_{i0} \cap D_{i1} \neq \emptyset$ then

$h_1 = \mathrm{id}$ on $\mathrm{Bdy}\, K_{i1} \cap (D_{i0} \cup D_{i1})$ and $h_1\big(A' \times [1, 2m] \cap (D_{i0} \cup D_{i1})\big) \subset D_{i1}$.

h_1 as defined exists since $A' \times [1, 2m]$ is compact, $K_{i1k} \cap (D_{i0} \cup D_{i1})$ is the union of two $(n+1)$-cells which intersect in an n-cell in their common boundary and $(A' \times [1, 2m]) \cap \mathrm{Bdy}\big(K_{ijk} \cap (D_{i0} \cup D_{i1})\big)\, q \subset D_{i1}$ and $K_{ijk} \cap K_{ijl} = \emptyset$ if $k \neq l$.

Now proceed inductively to define h_j for $j = 2, 3, \ldots, m-1$. As a notational aid define $L_{ij} = (D_{i0} \cup D_{i1} \cup \ldots \cup D_{ij}) \cap K_{ij}$.
Define $h_j\colon E^n \times E' \to E^n \times E'$ as follows:

a. $h_j = \mathrm{id}$ on $E^n \times E^1 - \big(h_{j-1} \circ h_{j-2} \circ \ldots \circ h_1(\bigcup_i L_{ij})\big)$.

For each i and A' a component of $A_{\gamma(j+1)} \cap A$ with $A' \times [j, 2m+1-j] \subset K_{ijk}$ then

b. If $H \cap D_{ij-1} = \emptyset$ or $H \cap D_{ij} = \emptyset$ then

$$h_j = \mathrm{id} \text{ on } H = \big(h_{j-1} \circ h_{j-2} \circ \ldots \circ h_1(K_{ijk})\big) ;$$

c. If $H \cap D_{ij-1} \cap D_{ij} \neq \emptyset$ then let h_j be such that

$$h_j\Big(h_{i-1} \circ h_{j-2} \circ \ldots \circ h_1\big(A' \times [j, 2m+1-j] \cap (D_{ij-1} \cup D_{ij})\big)\Big)$$

is contained in D_{ij}.

h_j exists since $A' \times [j, 2m+1-j]$ is compact,

$$h_{j-1} \circ \ldots \circ h_1\big(K_{ijk} \cap (D_{ij-1} \cup D_{ij})\big)$$

is the union of two $(n+1)$-cells which intersect in an n-cell in their common boundary and

$$h_{j-1} \circ \ldots \circ h_1(A' \times [j, 2m+1-j]) \cap \mathrm{Bdy}\, h_{j-1} \circ \ldots \circ h_1(K_{ijk}) \cap (D_{ij-1} \cup D_{ij})\big)$$

is contained in D_{ij}.

Set $h = h_{m-1} \circ h_{m-2} \circ \ldots \circ h_1$. Clearly conditions 1 and 2 are satisfied by h. To see that condition 3 is satisfied let

$$x \in (A_{\gamma(j+1)} \cap A) \times ([j, j+1] \cup [2m-j, 2m+1-j]) .$$

There exists some component $A' \subset A_{\gamma(j+1)} \cap A$ such that

$$x \in A' \times ([j, j+1] \cup [2m-j, 2m+1-j])$$

and a unique k_{ijk} containing x. Let $\psi = \min(j, \max\{e \mid K_{iek} \cap D_{ie} \neq \emptyset, D_{iek} \supset A'\})$.

Case 1. If $\psi < j$ then

$$h(x) = h_{m-1} \circ \ldots \circ h_\psi \circ \ldots \circ h_1(x) = h_\psi \circ \ldots \circ h_1(x) \subset D_{i\psi} \cup D_{i\psi+1} .$$

Case 2. If $\psi = j$ then

$$h(x) = h_{m-1} \circ \ldots \circ h_{j+1} \circ h_j \circ \ldots \circ h_1(x) = h_{j+1} \circ h_j \circ \ldots \circ h_1(x)$$

which is a point in $D_{ij} \cup D_{ij+1}$.

LEMMA 3.4. *Suppose $\varepsilon > 0$ and A is a component of A_r (where A_i $i = 1, 2, \ldots$ are defined as above). Then there exists an integer N and a uniformly continuous homeomorphism $h\colon E^n \times E^1 \to E^n \times E^1$ which is the identity on $E^{n+1} - (A \times E^1)$ and such that for each $w \in E^1$*

(1) $\Pi_{n+1}\big(h(A \times w)\big) \subset [w-2m-1, w+2m+1]$,

(2) $\mathrm{diam}\big(\Pi_n^*(A' \times w)\big) < 4\varepsilon$

where A' is a component of $A_n \cap A$, Π_{n+1} is the projection of $E^n \times E^1$ onto E^1, and Π_n^* is the projection onto E^n.

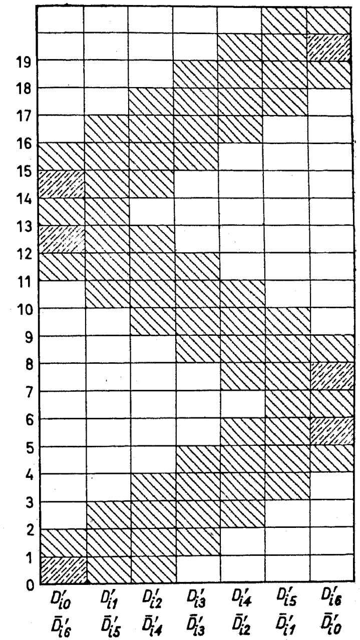

Fig. 3

Figure 3 shows how to apply Lemma 3.3 to prove Lemma 3.4. In Figure 3 only one sequence $K_{i1k}, K_{i2k}, \ldots, K_{im+1k}$ containing a component of $A_N \cap A$, is shown. The $(n+1)$-cells in the figure are shown as if they intersect each of the $(n+1)$-cells $D_{i0}, D_{i1}, \ldots, D_{im}$. This may not be the case; however, an analogous figure is obvious.

Proof. Apply Lemma 2.3 to $A \times E^1$ and integers m and $\gamma(m+1)$ and sets D_{il}, K_{ij} and K_{ijk}. Set $N = \gamma(m+1)$ for $g = 0, \pm 1, \pm 2, \ldots$ let

$$x_g = g(2m+2), \qquad x'_g = x_g + m + 1 ,$$

$$y_g = g(2m+2) + 2m + 1 , \qquad y'_g = y_g + m + 1 .$$

Note that D_{il}, K_{ij}, and $K_{ijk} \subset A\times[x_0, y_0]$ by suitable translations of E' we get sets analogous to D_{il}, K_{il}, and K_{ijk} in $A\times[x_g, y_g]$ for each g. Apply Lemma 3.3 to $A\times[x_g, y_g]$ for each g. Define $\overline{D}_{jl} = D_{im-l}$ and apply Lemma 3.3 to $A\times[x_l', y_l']$ using $\overline{D_{il}}$ in place of D_{il}. Thus there exists a homeomorphism which is uniformly continuous. By the choice of x_g, y_g, x_g' and y_g' and Remark 3.3 there exist integers i and k such that

$$\Pi_n^* h(A'\times w) \subset \Pi_n^*(D_{ik}\cup D_{ik+1}\cup \overline{D}_{im-k-2}\cup \overline{D}_{im-k-3})$$

for each component $A'\subset A\cap A_N$ and $w\in E^1$. Note that i and k depend on A' and w. Diameter $\Pi_n^*(D_{il}) < \varepsilon$ for all i and l. Thus condition (2) is satisfied. For $w\in E^1$ there exist $x_g, y_g, x'_{g+\delta}$ and $y'_{g+\delta}$, where $\delta = 0, -1$ such that $w\in[x_g, y_7]\cap[x'_{g+\delta}, y'_{g+\delta}]$. Thus

$$\Pi_{n+1}(A'\times w)\subset[x_g, y_g]\cup[x'_{g+\delta}, y_{g+\delta}],$$

and condition (1) is satisfied.

THEOREM 3.5. *For each component $A\subset A_r$ (where A_i, $i = 1, 2, \ldots$ are as defined above) and each $\varepsilon > 0$ there exists an integer N and a uniformly continuous homeomorphism $h\colon E^n\times E^1\to E^n\times E^1$ such that*

1. $h = \mathrm{id}$ *in* $E^{n+1} - A\times E^1$;
2. $|\Pi_{n+1}[h(x)] - \Pi_{n+1}(x)| < \varepsilon$;
3. *For each $w\in E^1$ diameter of each component of $A_N\times w$ is less than ε.*

Proof. Let $\varepsilon' = \varepsilon/8$ then by Lemma 3.4 there exists a uniformly continuous homeomorphism h_1 and an integer N satisfying

a. $h_1 = \mathrm{id}$ on $E^{n+1} - A\times E^1$,

b. $|\Pi_{n+1}h_1(x) - \Pi_{n+1}(x)| < 4m+2$ for some positive integer m, and

c. $\operatorname{diam}\Pi_n^*(A'\times w) < 4\varepsilon'$ for all $w\in E^1$ and components A' in $A_N\cap A$.

Let $h_2\colon E^n\times E^1\to E^n\times E^1$ be the homeomorphism given by

$$h_2(x, t) = \left(x, \frac{4m+2}{\varepsilon'}t\right).$$

The homeomorphism $h = h_2^{-1}h_1h_2$ is the desired homeomorphism.

Note that h is isotopic to the identity since the homeomorphisms of Lemmas 3.3 and 3.4 were.

THEOREM 3.6. $X_n\times E^1 = E^{n+1}$ *if* $X_n\in\Gamma_n$, *where* Γ_n *is defined above.*

Theorem follows from Theorem 2.1 and the following theorem which is due to R. H. Bing [4].

THEOREM. *Let $X_n\in\Gamma_n$. Further suppose that for each i and $\varepsilon > 0$ there is an integer N and an isotopy μ of E^{n+1} onto E^{n+1} such that μ_0 is the identity μ_1 is uniformly continuous and*

1. $\mu_t = \mathrm{id}$ *on* $E^{n+1} - (A_j\times E^1)$;
2. $|\Pi_{n+1}\mu_t(x) - \Pi_{n+1}(x)| < \varepsilon$, *where Π_{n+1} is the projection of E^{n+1} onto the $(n+1)$-st coordinate;*
3. *For each $w\in E^1$ the diameter of each component of $\mu_1(A_N\times w)$ is less than ε.*

Then $X_n\times E^1 = E^{n+1}$.

Remark 3.4. Note that there exists a countable collection of compact sets R_i such that

1. $A\times E^1 = \bigcup_i R_i$,
2. $h(R_i)\subset R_i$ for all $i = 1, 2, \ldots$;
3. $h/\mathrm{Bdy}\,R_i = \mathrm{id}$ for all $i = 1, 2, \ldots$,
4. $\operatorname{diam}\Pi_{n+1}(R_i) < \varepsilon/8$, and
5. $\operatorname{diam}h\big(R_i\cap(A_n\times E^1)\big) < \varepsilon/2$,

where h is the homeomorphism of Theorem 3.5.

4. The "Dogbone space" squared is E^6. In [7] K. W. Kwun showed that there exist two non-manifolds whose product is E^n for $n\geqslant 6$. In this section we shall show that the product of any two spaces belonging to Γ_n and Γ_m respectively is E^{n+m}.

Throughout this section let $\{A_i\}$ be as defined above and let $\{B_i\}$ be a collection of m-manifolds in E^m which are analogous to the A_i. That is B_i $(i = 1, 2, \ldots)$ is a collection of compact manifolds in E^m satisfying P1 and P2, where $B_\infty = \bigcap_i B_i$.

LEMMA 4.1. *Given A and B components of A_r and B_s respectively and $\varepsilon > 0$ then there exists an integer $N > \max(r, s)$ and a homeomorphism $h\colon E^n\times E^m\to E^n\times E^m$ such that*

1. $h = \mathrm{id}$ *on* $E^{n+m} - (A\times B)$ *and*
2. $\operatorname{Diam}h(A'\times B') < \varepsilon$ *for each component* $A'\subset A_N\cap A$ *and* $B'\subset B_N\cap B$.

Proof. By Lemma 3.1 there exist integer J and K, a set of n-cells $E_1, \ldots, E_p$, and a set of m-cells $F_1, \ldots, F_q$ such that

1. $E_i\subset\operatorname{int}A$ for each $i = 1, 2, \ldots, p$;
2. $F_j\subset\operatorname{int}B$ for each $j = 1, 2, \ldots, q$;
3. For each component $A'\subset A_J\cap A$ there is at least one i such that $A'\subset\operatorname{int}E_i$;
4. For each component $B'\subset B_K\cap B$ there is at least one j such that $B'\subset\operatorname{int}F_j$.

Let $N = \max(J, K)$ and note that for each component

$$A'\times B'\subset(A_N\times B_N)\cap(A\times B)$$

there exist integers i and j such that $A'\times B'$ is a subset of $\operatorname{int}E_i\times\operatorname{int}F_j$.

By Lemma 2.2 there exists a collection of $(n+m)$-cells $G_1, G_2, \ldots, G_l$ such that

1. For each component $A'\times B'$ of $(A_N\times B_N)\cap(A\times B)$ there exists a unique k such that $A'\times B'\subset \operatorname{int} G_k$ and $A'\times B'\cap G_j=\emptyset$ for all $j\neq k$.

2. $G_k\subset \operatorname{int} E_i\times \operatorname{int} F_j\subset A\times B$ for some i and j.

Note that even though $i\neq j$ it may be the case that $G_i\cap G_j\neq\emptyset$. Nevertheless there exists an $(n+m)$-cell $Q_i\subset G_i$ whose diameter is less than ε and such that $Q_i\cap G_j=\emptyset$ for $i\neq j$. For each component $A'\times B'\subset\{(A_N\times B_N)\cap(A\times B)\}$ there exists an integer i and a homeomorphism $h_i\colon E^{n+m}\to E^{n+m}$ such that

1. $A'\times B'\subset G_i$,
2. $h_i=\mathrm{id}$ on $E^{n+m}-G_i$,
3. $h_i(A'\times B')\subset Q_i$.

Define $h=h_1\circ h_2\circ\ldots\circ h_l$. Even though the G_i's are not disjoint, h_i is the identity on $G_j\cap(A_N\times B_N)$ for $j\neq i$. Thus h satisfies conditions 1 and 2 of the theorem.

Remark 4.1. Since the homeomorphism h of Lemma 4.1 is the identity outside a compact set h is uniformly continuous and isotopic to the identity.

THEOREM 4.2. *Let A_i, $i=1,2,\ldots$; B_j, $j=1,2,\ldots$ be defined as above, then there exists a pseudo-isotopy $H\colon E^{n+m}\times I\to E^{n+m}$ such that*

a. *$H(x,0)=x$;*

b. *If $H_t(x)=H(x,t)$ then for all $t<1$, H_t is a homeomorphism of E^{n+m} onto itself which is the identity outside a compact set;*

c. *H_1 maps E^{n+m} onto itself and maps each component of $A_\infty\times B_\infty$ onto a distinct point;*

d. *If $x\in E^{m+n}-(A_\infty\times B_\infty)$ then*
$$H_1^{-1}(H_1(x))=x\,.$$

Proof. Let $\varepsilon_0=\operatorname{diam}(A_1\times B_1)$ and $\varepsilon_i=(\frac12)^i$ for $i=1,2,\ldots$ A sequence of integers $1=N(1), N(2),\ldots$ and isotopies,
$$H^i\colon E^{n+m}\times\left[\frac{i-1}{i},\frac{i}{i+1}\right]\to E^{n+m}$$
for $i=1,2,\ldots$ which satisfy

1. $H^1(x,0)=x$,
2. $H^{i-1}\left(x,\dfrac{i-1}{i}\right)=H^i\left(x,\dfrac{i-1}{i}\right)$ for $i=2,3,\ldots$,
3. $\operatorname{diam} H^i\left(A'\times B',\dfrac{i}{i+1}\right)<\varepsilon_i$ for each component $A'\times B'\subset A_{N(i+1)}\times B_{N(i+1)}$,
4. $H^i(x,t)=H^{i-1}\left(x,\dfrac{i-1}{i}\right)$ for $x\in E^{n+m}-(A_{N(i)}\times B_{N(i)})$ and $i=2,3,\ldots$,
5. $\|H^i(x,t)-H^i(x,t')\|<\varepsilon_{i-1}$ for all $x\in E^{n+m}$ and $t,t'\in\left[\dfrac{i-1}{i},\dfrac{i}{i+1}\right]$

are defined inductively as follows. Let A_r and B_s of Lemma 4.1 be A_1 and B_1 respectively and let ε of Lemma 4.1 be ε_1. Then there exists a uniformly continuous isotopy
$$h_1\colon E^{n+m}\times I\to E^{n+m}$$
and an integer $N(2)$ such that
$$h_1(x,0)=x\,,$$
$$\operatorname{diam} h_1(A'\times B',1)<\varepsilon_1\quad\text{for each component}\,,$$
$$A'\times B'\subset A_{N(2)}\times B_{N(2)}\quad\text{and}\quad h_1(x,t)=x\ \text{on}\ E^{n+m}-(A_1\times B_1)\,.$$
Define $H'(x,t)=h_1(x,2t)$, $0\leqslant t\leqslant\frac12$.

Suppose H^k and N_{k+1} are defined. Since H^k_w is uniformly continuous for $w=\dfrac{k}{k+1}$ there exists a $\delta>0$ such that if the diameter of $V\subset E^{n+m}$ is less than δ then the diameter of $H^k_w(V)$ is less than ε_{k+1}. Lemma 4.1 implies the existence of an integer N_{k+2} and an isotopy such that
$$h_{k+1}(x,0)=x\quad\text{on}\quad E^{n+m},$$
$$h_{k+1}(x,t)=x\quad\text{on}\quad E^{n+m}-(A_{N(k+1)}\times B_{N(k+1)})\,,$$
$$\operatorname{diam}(A^*\times B^*,1)<\delta\quad\text{for each component}\,,$$
$$A^*\times B^*\subset A_{N(k+2)}\times B_{N(k+2)}\quad\text{and}\quad h_{k+1}\ \text{is uniformly continuous}\,.$$
Define
$$H^{k+1}(x,t)=H^k_w h_{k+1}\left(x,(k+1)(k+2)\left(t-\frac{k}{k+1}\right)\right)\quad\text{for}\quad\frac{k}{k+1}\leqslant t\leqslant\frac{k+1}{k+2}.$$
Clearly 1 and 2 are satisfied. Now
$$H^{k+1}\left(x,\frac{k+1}{k+2}\right)=H^k_w h_{k+1}(x,1)$$
thus by choice of δ condition 3 is satisfied. Further $h_{k+i}(x,t)=x$ for $x\in E^{n+m}-(A_{N(k+1)}\times B_{N(k+1)})$ hence condition 4 is satisfied. $h_{k+1}(A''\times B'',t)$ is contained in $A''\times B''$ for each component $A''\times B''\subset A_{N(k+1)}\times B_{N(k+1)}$. $\operatorname{Diam}\big(H^k_w(A''\times B'')\big)<\varepsilon_k$ be condition 3, thus condition 5 is satisfied.

Define
$$H(x,t)=H^i(x,t)\quad\text{on}\quad E^{n+m}\times\left[\frac{i+1}{i},\frac{i}{i+1}\right]\quad\text{for}\quad i=1,2,\ldots$$

and define $H(x,1) = H_1(x) = \lim_{t\to 1} H(x,t)$. $H_1(x)$ is a continuous map of E^{n+m} onto E^{n+m} by condition 5. Clearly 1 implies that a is satisfied by H. Condition 4 along with definition of H^1 implies b is satisfied by H. Suppose $\varepsilon > 0$ and $\alpha\times\beta$ is a component of $A_\infty \times B_\infty$ then there exists an integer p such that $(\frac{1}{2})^P = \varepsilon_p < \varepsilon$. For all $t > p/(p+1)$, $\operatorname{diam} H(A^*\times B^*, t) < \varepsilon_p$ where $A^*\times B^*$ is the component of $A_{N(p)}\times B_{N(p)}$ containing $\alpha\times\beta$. Thus $H(\alpha\times\beta, 1)$ is a point. Let $x\in E^{n+m} - A_\infty\times B_\infty$ then there exists an integer $N(q)$ such that $x \in E^{n+m} - (A_{N(q)}\times B_{N(q)})$. Thus 4 implies that $H(x,t) = H\left(x, \frac{q-1}{q}\right)$ for all $t > \frac{q-1}{q}$. But $H/E^{n+m}\times\left[0, \frac{q-1}{q}\right]$ is an isotopy thus $H_1^{-1}(H_1(x)) = x$ and d is satisfied by H. Let $\alpha_1\times\beta_1$ and $\alpha_2\times\beta_2$ be distinct components of $A_\infty\times B_\infty$ then there exists an integer $N(j)$ such that $\alpha_1\times\beta_1\subset A'\times B'$ and $\alpha_2\times\beta_2 \subset A''\times B''$, where $A'\times B'$ and $A''\times B''$ are distinct components of $A_{N(j)}\times B_{N(j)}$. Thus $H_1(\alpha_1\times\beta_1) \neq H_1(\alpha_2\times\beta_2)$ and c is satisfied. Therefore H is the desired pseudo-isotopy.

COROLLARY 4.2. *Suppose F is an upper semi-continuous decomposition of E^{n+m} consisting of the 2-cells $\alpha\times\beta$, where $\alpha\subset A_\infty$ and $\beta\subset B_\infty$ and the points of $E^{n+m} - (A_\infty\times B_\infty)$. If Z is the decomposition space associated with F then Z is topologically E^{n+m}. Moreover, there exists a uniformly continuous homeomorphism carrying Z onto E^{n+m}.*

THEOREM 4.3. *Suppose $X_n\in\Gamma_n$ and $X_m\in\Gamma_m$ then $X_n\times X_m$ is topologically E^{n+m}.*

Proof. By Corollary 4.2 there exists a pseudo-isotopy H of E^{n+m} onto itself which shrinks each of the 2-cells $\alpha\times\beta$ for $\alpha\subset A_\infty$ and $\beta\subset B_\infty$. Let $f = H_1$. The proof will be completed by constructing a pseudo-isotopy K of $f(E^{n+m})$ onto itself which shrinks each of the arcs $f(\alpha\times y)$, $f(z\times\beta)$ where α is an arc of A_∞, β is an arc of B_∞, $z\in E^n$ and $y\in E^m$.

Let

$$U_1 = \bigcup_i f\big(\operatorname{int} A_i\times(E^m - B_i)\big)$$

and

$$U_2 = \bigcup_i f\big((E^n - A_i)\times \operatorname{int} B_i\big)\,.$$

Note that each arc $f(\alpha\times y)\subset U_1$ and $f(z\times\beta)\subset U_2$. Also $U_1\cap U_2 = \emptyset$.

The pseudo-isotopy K can be constructed by amending the construction of the pseudo-isotopy in [7] as follows.

(1) Replace the compact neighborhoods T_i and T'_i with A_i and B_i respectively.

(2) In the proof of the Lemma replace Theorem 1 of [1] with Theorem 3.6 of this paper. And further replace the R'_i by R_i of Remark 3.4.

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