Topological Galois spaces

by

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1. Introduction. In this paper we apply techniques like those of classical Galois theory to relate a topological space and its group of homeomorphisms. These techniques apply to a class of topological spaces which includes all manifolds (without boundary) and all 0-dimensional non-discrete homogeneous spaces.

If X is a topological space, we denote by $H(X)$ the group of all homeomorphisms from X onto itself. In [4], J. Whittaker has proved that if, X and Y are compact, locally Euclidean manifolds, with or without boundary, and $H(X)$ is isomorphic to $H(Y)$, then X is homeomorphic to Y. The techniques in [4] are complex and depend heavily upon the structure of compact locally Euclidean manifolds, but it is natural to hope that Whittaker's result could be extended to a larger class of topological spaces.

The purpose of this paper is to investigate a class of topological spaces called topological Galois spaces. Though there is only the scant evidence provided by Theorem 1 that Whittaker's result might hold in this class of topological spaces, topological Galois spaces are of interest in their own right since methods similar to those of classical Galois theory can be used in the study of these spaces. Indeed there is a simple proposition about topological Galois spaces which is analogous to the Fundamental Theorem of Galois Theory.

In section 2 we study the general topological properties of Galois spaces and in section 3 we study the relation between a Galois space and its group of homeomorphisms.

2. Galois spaces. We let $H(X)$ denote the group all homeomorphisms from a space X onto itself and let i denote the identity of $H(X)$. If $A \subset X$ then $A' = \{h\colon h \in H(X),\, h|A = i|A\}$; and if G is a subgroup of $H(X)$ then $G' = \{x\colon x \in X,\, g(x) = x$ for each $g \in G\}$. We will often write A'' for $(A')'$ and x' for $\{x\}'$.

DEFINITION. A space X is a *Galois space* if for each closed set C and each $p \in X - C$ there is $h \in C'$ such that $h(p) \neq p$.

PROPOSITION 1. *A space X is a Galois space if and only if for each closed set C, $C = C''$.*

With mild separation axioms we can prove stronger results.

PROPOSITION 2. *Let X be a Hausdorff Galois space and let $A \subset X$. Then A is closed if and only if $A = A''$.*

PROPOSITION 3. *Let X be a regular space. If X has a basis of Galois spaces then X is a Galois space.*

Proof. Let $\mathcal{B}$ be a basis of Galois subspaces of X. Let C be a closed subset of X and let $p \in X - C$. There is $B \in \mathcal{B}$ such that $p \in B \subset X - C$. Since X is regular there is an open set V such that $p \in V \subset \overline{V} \subset B$. Since B is a Galois space there is $h \in H(B)$ such that $h(p) \neq p$ and $h|(B-V) = i|(B-V)$. Then $h|\mathrm{Cl}(V) \cup i|(X-V)$ is the desired element of $H(X)$.

The converse of Proposition 3 is implied by

PROPOSITION 4. *Every non-empty open subset of a Galois space is a Galois space.*

Proof. Let $(X, \mathcal{T})$ be a Galois space and let $U \in \mathcal{T}$. Let V be an open subset of U. Then $V \in \mathcal{T}$. Let $p \in V$. There is $h \in (X-V)'$ such that $h(p) \neq p$. Clearly $h|V \in (U-V)'$.

DEFINITION. Let X be a space and G be a subgroup of $H(X)$. Let $p \in X$. Then $G(p) = \{g(p)|\ g \in G\}$.

PROPOSITION 5. *Let $(X, \mathcal{T})$ be a Galois space. Let $A \in \mathcal{T}$ and let $a \in A$. Then $(X-A)'(a)$ is a homogeneous non-degenerate Galois space.*

Proof. Clearly $a \in (X-A)'(a)$. If $(X-A)'(a) = \{a\}$, then $a \in (X-A)'' = X-A$. But $a \in A$. Hence $(X-A)'(a)$ is non-degenerate. Let V be an open subset of $(X-A)'(a)$. Then there is $U \in \mathcal{T}$ such that $V = U \cap \big((X-A)'(a)\big)$. Let $q \in V$. Then $q \in A$ and there is a function $h \in \big((X-A) \cup (X-U)\big)'$ such that $h(q) \neq q$. Since $h|(X-A)'(a) \in H\big((X-A)'(a)\big)$, by Proposition 1, $(X-A)'(a)$ is a Galois space. It is easy to verify that $(X-A)'(a)$ is a homogeneous space.

If X is a space with an isolated point, then X is not a Galois space. In particular any non-degenerate discrete space is a homogeneous space which is not a Galois space.

EXAMPLE 1. *A homogeneous space X which has no isolated points and which is not a Galois space.*

Let X be the plane. For each point $p = (a, b)$ let

$$D_{(p,\varepsilon)} = \{p\} \cup \{(x, y):\ y \neq b \text{ and } \sqrt{(x-a)^2+(y-b)^2} < \varepsilon\}.$$

Let $\mathcal{B} = \{D_{(p,\varepsilon)}:\ p \in X, \varepsilon > 0\}$. Then $\mathcal{B}$ is a base for a topology $\mathcal{T}$ and with this topology X is a homogeneous space. Suppose $(X, \mathcal{T})$ is a Galois space. Let $x = (0, 0)$. Then since $D_{(x,1)}$ is an open subset of $(X, \mathcal{T})$, by Proposition 4, $D_{(x,1)}$ is also a Galois space. For each real number α let L_α be the diameter of $D_{(x,1)}$ with slope α. Notice that for each α the relative topology of L_α with respect to $\mathcal{T}$ is the relative topology of this line segment in the usual topology. Hence $D_{(x,1)}$ is connected. Now $H(X)(x) = \{x\}$, since x is the only cut point of $D_{(x,1)}$. Thus $D_{(x,1)}$ is not a Galois space.

In spite of Example 1 there is an obvious connection between Galois spaces and strong local homogeneity.

DEFINITION. [1] A space X is *strongly locally homogeneous* if for every neighborhood of any point x, there exists a subneighborhood $U(x)$ such that for any $z \in U(x)$ there exists a homeomorphism g with $g(x) = z$ and with g equal to the identity on the complement of $U(x)$.

Every strongly locally homogeneous space without isolated points is a Galois space. Thus it follows from Theorems 4.2 and 4.3 of [1] that if X is a space without isolated points and X is either locally Euclidean and Hausdorff or homogeneous and 0-dimensional, then X is a Galois space.

3. Homeomorphism groups. In this section all spaces considered will be non-degenerate Hausdorff spaces.

DEFINITION. If X is a Galois space and G is a subgroup of $H(X)$, then G is *closed* provided $G = G''$.

Given a space X we let $\mathcal{C}(X)$ denote the collection of all closed sets of X and $\mathcal{C}\big(H(X)\big)$ denote the collection of all closed subgroups of $H(X)$. We now give a list of observations which will be of use subsequently.

PROPOSITION 6. *Let X be a Galois space.*

1) *If both $A, B \in \mathcal{C}(X)$ or both $A, B \in \mathcal{C}\big(H(X)\big)$ and $A \subset B$, then $B' \subset A'$.*

2) *If $A \subset X$ or A is a subgroup of $H(X)$, $A''' = A'$.*

3) *$H \in \mathcal{C}\big(H(X)\big)$ if and only if there is a set A such that $H = A'$.*

4) *$C \in \mathcal{C}(X)$ if and only if there is a subgroup H of $H(X)$ such that $C = H'$.*

5) *A is a dense subset of X if and only if $\{i\} = \bigcap \{a'|\ a \in A\}$.*

6) *If A, B are subsets of X, then $(A \cup B)' = A' \cap B'$.*

7) *If A, B are subgroups of $H(X)$, then $(A \vee B)' = A' \cap B'$.*

8) *If $A, B \in \mathcal{C}(X)$, then $A' \vee B' \subset (A \cap B)'$.*

9) *If $A, B \in \mathcal{C}\big(H(X)\big)$, then $A' \cup B' = (A \cap B)'$.*

10) *If $A, B \in \mathcal{C}\big(H(X)\big)$, then $A \cap B \in \mathcal{C}\big(H(X)\big)$.*

DEFINITION. Let A be a subset of a topological space X. Then A is *stable* provided, if $a \in A$ and $h \in H(X)$ then $h(a) \in A$.

LEMMA. *Let X be a space. If $f \in H(X)$ and H is a subgroup of $H(X)$, then $(fHf^{-1})' = f(H')$.*

Proof. Let $x \in (fHf^{-1})'$. Then, if $h \in H$, $fhf^{-1}(x) = x$ so that $h(f^{-1}(x)) = f^{-1}(x)$. Therefore $f^{-1}(x) \in H'$ and $x \in f(H')$. Thus $(fHf^{-1})' \subset f(H')$. Similarly $f(H') \subset (fHf^{-1})'$.

PROPOSITION 7. *Let X be a Galois space. The function $f\colon \mathcal{C}(X) \to \mathcal{C}(H(X))$ defined by $f(A) = A'$ for each $A \in \mathcal{C}(X)$ is a one-to-one function from $\mathcal{C}(X)$ onto $\mathcal{C}(H(X))$. If $C \in \mathcal{C}(X)$, $f(C)$ is a normal subgroup of $H(X)$ if and only if C is a stable subset of X.*

Proof. Let $A, B \in \mathcal{C}$. If $f(A) = f(B)$, then $A' = B'$ and $A = A'' = B'' = B$. Therefore f is a one-to-one function. Let $H \in \mathcal{C}(H(X))$. Then $H = H''$. By Proposition 6.4, $H' \in \mathcal{C}(X)$. Since $f(H') = H'' = H$, f maps $\mathcal{C}(X)$ onto $\mathcal{C}(H(X))$.

Let A be a stable subset of X and let $x \in A$. Let $h \in H(X)$ and let $g \in A'$. Then $h^{-1}(x) \in A$ so that $gh^{-1}(x) = h^{-1}(x)$ and $hgh^{-1}(x) = hh^{-1}(x) = x$. Therefore $hgh^{-1} \in A'$ and A' is a normal subgroup of $H(X)$.

Let N be a closed normal subgroup of $H(X)$ and let $h \in H(X)$. Since N is normal $(hNh^{-1})' = N'$. By the lemma, $(hNh^{-1})' = h(N')$. Therefore $N' = h(N')$ and N' is a stable set. Moreover $f(N') = N'' = N$.

As in classical Galois theory, we can use the one-to-one correspondence between closed sets and closed subgroups established in Proposition 7 to relate the topological structure of X to the group structure of $H(X)$. However in classical Galois theory the groups are finite and the correspondence between intermediate fields and subgroups is a bijection, while we will now prove that in our case the groups are infinite and the one-to-one correspondence is never a bijection.

First a Galois space must be infinite since a discrete space is never a Galois space. Let X be a Galois space. For each $x \in X$, $\{x\} = x''$ is not a stable set. Thus by Proposition 7, $H(X)$ has infinitely many non-normal subgroups and is infinite.

If every subgroup of $H(X)$ is closed, then the lattice of subgroups of $H(X)$ is distributive, which implies that $H(X)$ is abelian ([3], Theorem 4).

THEOREM 1. *Let X and Y be Galois spaces. Then X is homeomorphic to Y if and only if there is an isomorphism $\varphi\colon H(X) \to H(Y)$ such that $\mathcal{C}(H(Y)) = \{\varphi(C) \mid C \in \mathcal{C}(H(X))\}$.*

Proof. Suppose first that there is a homeomorphism $h\colon X \to Y$. Let $\varphi\colon H(X) \to H(Y)$ be defined by $\varphi(f) = hfh^{-1}$ for each $f \in H(X)$. Let $C \in \mathcal{C}(H(X))$. It follows immediately from the definitions that for any closed subset A of X, $hA'h^{-1} = h(A)'$. Thus if $C = C''$, then $\varphi(C) = h(C')'$ so that $\varphi(C)'' = \varphi(C)$. By symmetry if $\varphi(C)'' = \varphi(C)$, then $C = C''$.

Now suppose X and Y are Galois spaces and there is an isomorphism $\varphi\colon H(X) \to H(Y)$ such that $\mathcal{C}(H(Y)) = \{\varphi(C) \mid C \in \mathcal{C}(H(X))\}$. As partially ordered by inclusion the sets $\mathcal{C}(X)$ and $\mathcal{C}(H(X))$ are dually isomorphic. Since φ preserves inclusion, φ induces an isomorphism $\varphi^*\ \mathcal{C}(X) \to \mathcal{C}(Y)$ of partially ordered sets; and as any T_1 topological space S is determined by the lattice $\mathcal{C}(S)$, the isomorphism φ^* provides a homeomorphism $h\colon X \to Y$ given by the equation $h(x) = \varphi(x')'$.

In light of Theorem 1, the authors pose the following question: If X and Y are Galois spaces and $H(X)$ is isomorphic to $H(Y)$ is X homeomorphic to Y? If we modify the definition of closed subgroups by calling a group G closed provided there is a closed set A such that $G = A'$, then the proof of Theorem 1 extends to T_1 Galois spaces without change.

EXAMPLE 2. Let X be the real line, let $\mathcal{U}$ be the usual topology on X and let $\mathcal{V}$ be the subcollection of $\mathcal{U}$ containing only $\emptyset$, X, and sets whose complements are bounded. Then $\mathcal{V}$ is a topology on X such that $H(X, \mathcal{U}) = H(X, \mathcal{V})$ ([2], Theorem 3).

The above example shows that without the assumption that X and Y are Hausdorff the answer to our question is negative even if $H(X) = H(Y)$. This fact is surprising since in a Hausdorff Galois space the functions $\varphi\colon \mathcal{S}(X) \to \mathcal{S}(X)$ defined by $\varphi(A) = A''$ is the Kuratowski closure operation. Thus it is impossible for two Hausdorff Galois spaces with the same underlying set to have the same group of homeomorphisms.

Recall that a group G is complete provided G is centerless and every automorphism of G is inner. The following slight improvement on Theorem 1 might be useful in answering the question.

THEOREM 2. *Let X be a Galois space and suppose that every automorphism of $H(X)$ is inner. Then $H(X)$ is complete. Furthermore if Y is a Galois space and if φ is an isomorphism from $H(X)$ onto $H(Y)$ then X is homeomorphic to Y if and only if $\varphi\colon \mathcal{C}(H(X)) \to \mathcal{C}(H(Y))$ is a bijection.*

Proof. Let $f \in H(X)$. If $f \neq i$, there exists $x \in X$ such that $f(x) \neq x$. There is $g \in H(X)$ such that $g(x) = x$ and $g(f(x)) \neq f(x)$. Then $fg \neq gf$. Hence $H(X)$ is complete.

It remains to show that if X is homeomorphic to Y then $\varphi\colon \mathcal{C}(H(X)) \to \mathcal{C}(H(Y))$ is a bijection. By Theorem 1, since, X is homeomorphic to Y, there is a bijection $\psi\colon \mathcal{C}(H(X)) \to \mathcal{C}(H(Y))$. Since $\psi^{-1}\varphi$ is an automorphism of $H(X)$, there is $g \in H(X)$ such that $\psi^{-1}\varphi(f) = gfg^{-1}$ for each $f \in H(X)$. Now let $F \in \mathcal{C}(H(X))$. Then $\psi^{-1}\varphi(F) = gFg^{-1} \in \mathcal{C}(H(X))$. So $\psi(\psi^{-1}\varphi)(F) = \varphi(F) \in \mathcal{C}(H(Y))$. If $\varphi(F_1) = \varphi(F_2)$ certainly $F_1 = F_2$. Let $G \in \mathcal{C}(H(Y))$. Then $\varphi^{-1}(G) \in \mathcal{C}(H(X))$. Otherwise $(\psi^{-1}\varphi)\varphi(G) = \psi^{-1}(G) \notin \mathcal{C}(H(X))$.

Whittaker has shown that if X is a manifold then $H(X)$ is complete ([4], Theorem 4). Thus at least one large class of Galois spaces has complete groups.

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Bimeasurable maps*

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1. Introduction. All spaces considered in this paper are assumed to be metrizable and k will denote an infinite cardinal. We further assume the generalized continuum hypothesis.

A map f between two absolute Borel (metric) spaces is *bimeasurable* if both f and f^{-1} preserve absolute Borel sets. R. Purves [6] has shown the following:

THEOREM 1.1. *If f is a bimeasurable map between two separable absolute Borel metric spaces, then $f^{-1}(y)$ is countable except for at most countably many points in the range of f.*

The purpose of this paper is to obtain generalizations of this theorem for non-separable spaces. In place of countability we are led to considerations of the cardinality and σ-discreteness of the sets $f^{-1}(y)$. Summarizing Theorems 4.3, 4.4, 5.1, and 5.2, we obtain the following (definitions are given in Section 2):

THEOREM. *Let f be an a-bimeasurable map defined on an absolute Borel space X of weight k. Let*

$$B = \{y \in f(X)\colon f^{-1}(y) \text{ not } \sigma\text{-discrete}\}$$

and let

$$B^* = \{y \in f(X)\colon \operatorname{card} f^{-1}(y) > k\}.$$

Then

(i) $\operatorname{card} B \leqslant k$,

(ii) $\operatorname{card} B^* \leqslant k$,

(iii) *if B is absolutely $\aleph_0$-analytic, then B is σ-discrete,*

(iv) *if B^* is absolutely $\aleph_0$-analytic, then B^* is σ-discrete.*

Each of the four conclusions in this theorem reduces to the theorem of Purves if the spaces in question are separable, i.e. if $k = \aleph_0$.

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