But if there were $r \in M-(L_x \cup \overline{H'_{v_0}} \cup L_y \cup \overline{H'_{w_0}})$, then we could apply Claim 2, Claim 3 in reference to H_r, and the fact that v_0 and $w_0 \notin H_r$ to arrive at a contradiction to M being C.C. This then completes the sufficiency part of this theorem.

To establish the necessity half of this theorem, we will assume that M is not C.C. but it does satisfy Condition 2 and deduce that it is of the form expressed in 2.

Lemma 6 shows that Condition 2 implies (2) of Theorem 1, and so assuming M is not C.C. means that there exists x and $y \in M$ such that $L_x \cap L_y = \emptyset$ and yet $\{x\} \cup \{y\}$ cut no two points of $M-(L_x \cup L_y)$.

Since $L_x \cap L_y = \emptyset$ we know that there exist two continua H and K which are irreducible between L_x and L_y and $M = L_x \cup H \cup L_y \cup K$. Letting $t \in \mathrm{Int}(H)$ and $z \in \mathrm{Int}(K)$ we see by the above comments that there exist a continuum C such that $t, z \in C$ and $x, y \notin C$. So by Lemma 8, $L_r = H$ for each $r \in \mathrm{Int}(H)$, therefore using the fact that $\overline{\mathrm{Int}(H)} = H$ (Result 3), we see that H is indecomposable. Analogous comments hold for K. Also we know from Lemma 8 that $H \cap K \neq \emptyset$.

Now $L_r = H$ for each $r \in \mathrm{Int}(H)$ implies by Lemma 5 that $x \in \mathrm{Int}(L_x)$ and $y \in \mathrm{Int}(L_y)$. It is not difficult to see that $\overline{\mathrm{Int}(L_x)} = L_x$ and $\overline{\mathrm{Int}(L_y)} = L_y$. So to finish the proof we only need show that for each $r \in \mathrm{Int}(L_x)$, $L_r = L_x$. Clearly $L_r \subseteq L_x$. Therefore, $L_r \cap L_y = \emptyset$. Let H' and K' be the continua associated with L_r and L_y mentioned in the statement of Condition 2. Clearly we can take $K \subseteq K'$ and $H \subseteq H'$. Letting $t \in \mathrm{Int}(K)$ and $v \in \mathrm{Int}(H)$ we know that $L_t \cap L_v \neq \emptyset$, by Lemma 5 r and $y \notin L_t \cap L_v$, and so by Lemma 8

$$H' = L_v = H \quad \text{and} \quad K' = L_t = K$$

But $M = H \cup L_r \cup K \cup L_z$ and so $\mathrm{Int}(L_x) \subseteq L_r$ which implies $L_x \subseteq L_r$. This then completes the proof of the theorem.

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Dilating mappings, implicit functions and fixed point theorems in finite-dimensional spaces

by

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It is the purpose of this paper to investigate some properties of non-linear mappings of a finite-dimensional Euclidean spaces into itself. The argument presented here consists in a combination of two facts: Borsuk's theorem on ε-mappings in the narrow sense and Banach's contraction principle. By means of this method several theorems concerning non-linear mappings of finite-dimensional Banach spaces into themselves are obtained. In particular, an implicit function theorem for dilating mappings, a generalization of the contraction principle and some results concerning the non-linear eigenvalue problem are included.

Let f be a continuous transformation of a finite-dimensional Euclidean space X into itself. The transformation f is called an *ε-mapping in the narrow sense* if it has the following property:

(B) there exist two positive numbers η and ε such that the condition

$$\|f(x')-f(x'')\| < \eta, \quad x', x'' \in X$$

implies

$$\|x'-x''\| < \varepsilon.$$

In paper [1] K. Borsuk proved the following

THEOREM. *If $f(x)$ has property* (B), *then f is a mapping onto, i.e.* $f(X) = X$.

Implicit functions. In order to make use of Borsuk's theorem let us observe that if the mapping f possesses the following property: there exists a positive number c such that

$$c\|x_1-x_2\| \leqslant \|f(x_1)-f(x_2)\|$$

for arbitrary x_1, x_2 of X, then f is an ε-mapping in the narrow sense and, consequently, $f(X) = X$. Moreover, f is a homeomorphism of X onto itself.

After this remark we shall prove the following implicit function theorem.

THEOREM 1. *Let f be a continuous mapping defined on the product space $X \times Y$ with values in X.*

Suppose that the following conditions are satisfied. There exists a positive number c such that

(1) $$c\|x_1-x_2\| \leqslant \|f(x_1, y)-f(x_2, y)\|$$

for arbitrary y of Y and x_1, x_2 of X.

There exists a positive number K such that

(2) $$\|f(x, y_1)-f(x, y_2)\| \leqslant K\|y_1-y_2\|$$

for arbitrary x of X and y_1, y_2 of an arbitrary metric space Y. Then there exists a unique continuous function $x = g(y)$ satisfying the equation

(3) $$f\big(g(y), y\big) = 0, \quad y \in Y.$$

Proof. It follows from (1) and from the above remark that for arbitrary fixed y of Y the mapping $f(x, y)$ is an ε-mapping in the narrow sense. Hence, there exists a unique element $x = g(y)$ satisfying relation (3). Conditions (1), (2) yield

$$c\|x_1-x_2\| \leqslant \|f(x_1, y_1)-f(x_2, y_1)\| = \|f(x_2, y_1)-f(x_2, y_2)\| \leqslant K\|y_1-y_2\|,$$

where $x_1 = g(y_1)$, $x_2 = g(y_2)$ and $f(x_1, y_1) = f(x_2, y_2) = 0$.

Thus, we have

$$\|g(y_1)-g(y_2)\| \leqslant Kc^{-1}\|y_1-y_2\|,$$

which completes the proof.

Let us remark that Y can be replaced by an arbitrary metric space and the variable y can be restricted to an arbitrary subset of Y.

Suppose that the mapping F of X into itself is strictly contractive, i.e. there exists a positive number $\alpha < 1$ such that

$$\|F(x_1)-F(x_2)\| \leqslant \alpha\|x_1-x_2\|$$

for arbitrary x_1, x_2 of X. It is easily seen that the mapping $f(x) = x-F(x)$ satisfies the relation

$$(1-\alpha)\|x_1-x_2\| \leqslant \|f(x_1)-f(x_2)\|$$

for arbitrary x_1, x_2 of X. Hence, it follows that $f(x)$ is an ε-mapping in the narrow sense and by Borsuk's theorem it is a mapping onto X.

Thus, the mapping f is a homeomorphism of X onto itself and, in particular, there exists a unique element x^* of X such that $f(x^*) = 0$, i.e. $x^* = F(x^*)$ and, consequently, x^* is the unique fixed point of F.

A mapping F of X into itself is said to be a *dilating mapping* if there exists a positive number $\alpha < 1$ such that

(4) $$\|x_1-x_2\| \leqslant \alpha\|F(x_1)-F(x_2)\|$$

for arbitrary x_1, x_2 of X. Consider the mapping $f(x) = x-F(x)$.

We have

(5) $$\|f(x_1)-f(x_2)\| \geqslant (\alpha^{-1}-1)\|x_1-x_2\|.$$

Thus, it follows that f is an ε-mapping in the narrow sense Theorem 1 implies the following

THEOREM 2. *Suppose that the mapping F with values in X is defined on the product space $X \times Y$. Let us assume that the following conditions are satisfied. There exist two positive numbers K and $\alpha < 1$ such that*

(6) $$\|x_1-x_2\| \leqslant \alpha\|F(x_1, y)-F(x_2, y)\|$$

for arbitrary x_1, x_2 of X and y of Y, and

(7) $$\|F(x, y_1)-F(x, y_2)\| \leqslant K\|y_1-y_2\|$$

for arbitrary x of X and y_1, y_2 of Y.

Then there exists a unique continuous function $x = g(y)$ satisfying the relation

(8) $$g(y) = F\big(g(y), y\big), \quad y \in Y.$$

Proof. The mapping $f(x, y) = x-F(x, y)$ satisfies relation (1), with $c = \alpha^{-1}-1$.

Further, we obtain by assumptions (6), (7)

$$\begin{aligned}\|x_1-x_2\| &\leqslant \alpha\|F(x_1, y_1)-F(x_2, y_1)\| \\ &\leqslant \alpha\big(\|F(x_1, y_1)-F(x_2, y_2)\|+\|F(x_2, y_2)-F(x_2, y_1)\|\big) \\ &\leqslant \alpha\|x_1-x_2\|+\alpha K\|y_1-y_2\|,\end{aligned}$$

where $x_1 = g(y_1)$ and $x_2 = g(y_2)$ satisfy relation (8). Hence, we obtain

(9) $$(1-\alpha)\|g(y_1)-g(y_2)\| \leqslant \alpha K\|y_1-y_2\|$$

The last inequality yields the continuity of $g(y)$.

The remark concerning the variable y in Theorem 1 is also valid in this case.

THEOREM 3. *Suppose that $Y = X$. If, in addition to the hypotheses of Theorem 2, the numbers α and K are subject to the restriction $K < \alpha-1$, then there exists a unique fixed point y^* such that*

$$g(y^*) = y^*, \quad \textit{i.e.} \quad y^* = F(y^*, y^*).$$

Proof. In virtue of (9) we have

$$\|g(y_1)-g(y_2)\| \leqslant \alpha K(1-\alpha)^{-1}\|y_1-y_2\|.$$

Hence, it follows that the mapping g is contractive and the assertion of the theorem results from the contraction principle.

A generalization of the contraction principle. Let $F(x)$ and $L(x)$ be two continuous mappings of X into itself and put $f(x) = x - F(x)$. The following theorem is a generalization of the well-known contraction principle.

THEOREM 4. *Suppose that $F(x)$ and $L(x)$ satisfy the following conditions: There exist two positive numbers c and K such that*

$$c\|x_1 - x_2\| \leqslant \|f(x_1) - f(x_2)\| \tag{10}$$

for arbitrary x_1, x_2 of X,

$$\|L(x_1) - L(x_2)\| \leqslant K\|x_1 - x_2\| \tag{11}$$

for arbitrary x_1, x_2 of X and

$$K < c\,. \tag{12}$$

Then

(a) *the mapping $F(x)+L(x)$ has a unique fixed point, i.e. there is a unique element x^* of X such that $x^* = F(x^*)+L(x^*)$,*

(b) *the mapping $y = f(x) - L(x) = x - F(x) - L(x)$ is a homeomorphism of X onto itself and*

(c) *the inverse mapping $x = x(y)$ is Lipschitz continuous with the constant $(c-K)^{-1}$, i.e. $\|x(y_1) - x(y_2)\| \leqslant (c-K)^{-1}\|y_1 - y_2\|$.*

Proof. Condition (10) implies that the mapping f is a homeomorphism of X onto itself. Let x be a fixed element of X. Then for $L(x)$ there exists a unique element Rx of X such that

$$f(Rx) = L(x)\,. \tag{13}$$

Consider now the mapping $x \to Rx$.

In virtue of (10), (13) and (11) we obtain

$$c\|Rx_1 - Rx_2\| \leqslant \|f(Rx_1) - f(Rx_2)\| = \|L(x_1) - L(x_2)\| \leqslant K\|x_1 - x_2\|\,.$$

Hence we have

$$\|Rx_1 - Rx_2\| \leqslant Kc^{-1}\|x_1 - x_2\| \tag{14}$$

for arbitrary $x_1, x_2 \in X$. Thus, we see that the mapping R is a contractive mapping, by (12). It follows that there exists a unique element x^* of X such that $Rx^* = x^*$. Hence, we obtain $f(x^*) = L(x^*)$, by (13), which completes the proof of assertion (a), since the uniqueness of x^* follows from relations (14), (12) by putting $Rx_1^* = x_1^*$, $Rx_2^* = x_2^*$. The proof of assertion (b) follows by means of the same argument by replacing $L(x)$ in (13) by $L(x)+y$ for fixed y of X. Thus we obtain

$$f(Rx) = L(x) + y \tag{15}$$

instead of (13). Relations (11), (14) remain unchanged while we replace the mapping $L(x)$ by the mapping $L(x)+y$ in assertion (a), provided that y is an arbitrary but fixed element of X. Consequently, we infer by means of the contraction principle for Rx that for arbitrary y of X there is a unique element x of X such that

$$x - F(x) - L(x) = y\,. \tag{16}$$

Hence, it follows from (10) that

$$c\|x_1 - x_2\| \leqslant \|f(x_1) - f(x_2)\| \leqslant \|L(x_1) - L(x_2)\| + \|y_1 - y_2\|\,.$$

Thus, we obtain

$$(c-K)\|x_1 - x_2\| \leqslant \|y_1 - y_2\|$$

in virtue of (11), where $x_1 = x(y_1)$, $x_2 = x(y_2)$ are solutions of equation (16) for y_1, y_2, respectively. Since condition (12) is satisfied by assumption, the last inequality proves assertion (c).

Remark. Putting $F(x) = 0$ in Theorem 4, we obtain $c = 1$ and $K < 1$, i.e. L is a contractive mapping. Thus, Theorem 4 generalizes the well-known contraction principle.

Let us observe that the proof of Theorem 4 can be reduced directly to the contraction principle by considering the contractive mapping $f^{-1}L(x)$, where f^{-1} denotes the inverse mapping. It follows from (10) that the inverse exists and is Lipschitz continuous with the constant c^{-1}.

Now suppose that $F(x)$ is a dilating mapping, i.e. that relation (4) is satisfied. Then Theorem 4 assumes the following formulation

THEOREM 5. *Let F be a dilating mapping and L a Lipschitz continuous mapping satisfying relation (11) with the Lipschitz constant K subject to the restriction $K < a^{-1}-1$. Then assertions (a), (b) and (c) of Theorem 4 hold, where $c = a^{-1}-1$.*

Proof. It follows from (5) that relation (10) is satisfied with $c = a^{-1}-1$. Since by assumption we have $K < a^{-1}-1 = c$, we conclude that all hypotheses of Theorem 4 are fulfilled.

On the basis of Theorem 4 we obtain the following implicit function theorem.

THEOREM 6. *Let F, L be two continuous mappings defined on the product space $X \times Y$ with values in X. Let us assume that $F(x, y)$, $L(x, y)$ satisfy the following conditions. There exist two positive numbers c and K such that*

$$c\|x_1 - x_2\| \leqslant \|f(x_1, y) - f(x_2, y)\| \tag{17}$$

for arbitrary x_1, x_2 of X and y of Y, where $f(x, y) = x - F(x, y)$.

$$\|L(x_1, y) - L(x_2, y)\| \leqslant K\|x_1 - x_2\| \tag{18}$$

for arbitrary x_1, x_2 *of* X *and* y *of* Y.

(19) $$K < c.$$

In addition, there exists a positive number a *such that*

(20) $$\|F(x, y_1) - F(x, y_2) + L(x, y_1) - L(x, y_2)\| \leqslant a\|y_1 - y_2\|$$

for arbitrary x *of* X *and* y_1, y_2 *of* Y.

Then there exists a unique continuous function $x = g(y)$ *satisfying the equation*

(21) $$g(y) = F(g(y), y) + L(g(y), y).$$

Proof. In virtue of Theorem 4, it follows from conditions (17)–(19) that for arbitrary fixed y of Y there exists a unique element $x = g(y)$ satisfying equation (21). We have, by (21),

$$g(y_1) - g(y_2) - F(g(y_1), y_1) + F(g(y_2), y_1)$$
$$= [F(g(y_2), y_1) - F(g(y_2), y_2) + L(g(y_2), y_1) - L(g(y_2), y_2)] +$$
$$+ [L(g(y_1), y_1) - L(g(y_2), y_1)].$$

Hence, it follows, by (17), (20) and (18),

$$c\|g(y_1) - g(y_2)\| \leqslant a\|y_1 - y_2\| + K\|g(y_1) - g(y_2)\|.$$

Thus, we obtain, by (19), the inequality

$$\|g(y_1) - g(y_2)\| \leqslant a(c-K)^{-1}\|y_1 - y_2\|,$$

which proves the Lipschitz continuity of $g(y)$.

Assumming that F is a dilating mapping with respect to x but uniformly in y, we obtain the following

THEOREM 7. *If in addition to relations* (18) *and* (20) *there exists a positive number* $\alpha < 1$ *such that*

(22) $$\|x_1 - x_2\| \geqslant \alpha\|F(x_1, y) - F(x_2, y)\|$$

for arbitrary x_1, x_2 *of* X *and* y *of* Y *and*

(23) $$K < \alpha^{-1} - 1,$$

then there exists a unique continuous function $g(y)$ *satisfying equation* (21).

Proof. It is easily seen that conditions (17) and (19) follow from conditions (22) and (23) with $c = \alpha^{-1} - 1$. Thus, all the hypotheses of Theorem 6 are fulfilled.

Let us remark that in Theorems 6, 7 Y can be an arbitrary metric space and the variable y can be restricted to an arbitrary subset of Y.

Resolvents. (A) On the basis of Theorem 4 it is possible to investigate some families of continuous mappings depending on a real parameter.

Put $y = T_\mu(x) = \mu x - F(x) - L(x)$, $x, y \in X$.

The real number μ is called a regular value if the mapping T_μ is a homeomorphism of X onto itself. The mapping R_μ is called the resolvent of T_μ if $y = T_\mu(R_\mu y)$ for arbitrary y of X. The real number μ is called an eigenvalue of the mapping $F(x) + L(x)$ if there exists a vector x of X such that

(24) $$\mu x = F(x) + L(x).$$

The vector x is called the eigenvector corresponding to the eigenvalue μ. If F and L are both linear mappings, the vector $x = 0$ is always an eigenvector. Thus, in the linear case the trivial zero eigenvector is excluded.

THEOREM 8. *Suppose that the mappings* F *and* L *satisfy conditions* (10)–(12).

Let us assume that μ *satisfies the condition*

(25) $$|1 - \mu| < c - K.$$

Then the resolvent R_μ *exists and satisfies the relations*

(26) $$\|R_\mu y_1 - R_\mu y_2\| \leqslant (c - K - |1-\mu|)^{-1}\|y_1 - y_2\|,$$

(27) $$\|R_\alpha y - R_\beta y\| \leqslant |\beta - \alpha|(c - K - |1-\alpha|)^{-1}\|R_\beta y\|,$$

where α *and* β *are subject to restriction* (25).

Moreover, for every μ *satisfying relation* (25) *there exists a unique eigenvector* x *corresponding to the eigenvalue* μ, *i.e.* μ *and* x *satisfy relation* (24).

Proof. Let us write

$$y = T_\mu(x) = \mu x - F(x) - L(x) = x - F(x) - [(1-\mu)x + L(x)].$$

Then, replacing in Theorem 4 $L(x)$ by $(1-\mu)x + L(x)$ and the Lipschitz constant K by $K + |1-\mu|$, we infer that T_μ is a homeomorphism of X onto itself. Thus, the resolvent R_μ exists and satisfies relation (26) in virtue of assertion (c) of Theorem 4. Hence, it results that $x = R_\mu y$ for $y = 0$ is the unique eigenvector corresponding to the eigenvalue μ for each μ satisfying inequality (25). It remains to prove relation (27). For α and β satisfying inequality (25) we have

$$y = T_\alpha(R_\alpha y) = R_\alpha y - F(R_\alpha y) - [(1-\alpha)R_\alpha y + L(R_\alpha y)],$$
$$y = T_\beta(R_\beta y) = R_\beta y - F(R_\beta y) - [(1-\beta)R_\beta y + L(R_\beta y)].$$

Hence, it follows that

$$R_\alpha y - F(R_\alpha y) - [R_\beta y - F(R_\beta y)]$$
$$= (1-\alpha)R_\alpha y - (1-\beta)R_\beta y - [L(R_\alpha y) - L(R_\beta y)],$$

and by (10) and (11) we obtain

$$(c-K)\|R_\alpha y - R_\beta y\| \leqslant \|(1-\alpha)R_\alpha y - (1-\beta)R_\beta y\|$$
$$= \|(1-\alpha)(R_\alpha y - R_\beta y) + (\beta-\alpha)R_\beta y\|.$$

Since α and β satisfy relation (25), we conclude from the last inequality that the relation

$$(c-K-|1-\alpha|)\|R_\alpha y - R_\beta y\| \leqslant |\beta-\alpha|\,\|R_\beta y\|$$

holds, which proves inequality (27).

Thus, we see that the resolvent R_μ is continuously dependent on the parameter μ in the sense that $R_\mu y$ converges toward $R_\beta y$ if $\alpha \to \beta$.

Remark. If F is a dilating mapping, i.e. relation (4) holds, then Theorem 8 is valid, where $c = \alpha^{-1}-1$ and K is subject to restriction (23).

(B) Another family of continuous mappings depending on a real parameter can be introduced as follows. Put

$$y = T_\lambda(x) = x - F(x) - \lambda L(x), \qquad x, y \in X.$$

The real number λ is called a *regular value* if the mapping T_λ is a homeomorphism of X onto itself. The mapping R'_λ is called the *resolvent* of T_λ if $y = T_\lambda(R'_\lambda y)$ for arbitrary y of X. The real number λ is called an *eigenvalue* if there exists a vector x of X such that

(28) $$x = F(x) + \lambda L(x).$$

The vector x is called the eigenvector corresponding to the eigenvalue λ.

THEOREM 9. *Suppose that the mappings F and L satisfy conditions* (10)–(11) *and let λ satisfy the condition*

(29) $$|\lambda| < cK^{-1}.$$

Then the resolvent R'_λ exists and satisfies the relations

(30) $$\|R'_\lambda y_1 - R'_\lambda y_2\| \leqslant (c-|\lambda|K)^{-1}\|y_1 - y_2\|,$$

(31) $$\|R'_\alpha y - R'_\beta y\| \leqslant |\alpha-\beta|(c-|\beta|K)^{-1}\|L(R'_\alpha y)\|,$$

where α and β are subject to restriction (29). *Moreover, for any λ satisfying relation* (29) *there exists a unique eigenvector x corresponding to the eigenvalue λ, i.e. λ and x satisfy relation* (28).

Proof. Replacing in Theorem 4 the mapping L by λL and condition (12) by (29), we infer that T_λ is a homeomorphism of X onto itself. Thus, the resolvent R'_λ exists and satisfies relation (30) in virtue of assertion (c) of Theorem 4. Hence, we have $x = R'_\lambda y$ if $y = 0$, is the unique eigenvector corresponding to the eigenvalue λ for each λ satisfying inequality (29). It remains to prove relation (31).

For α and β satisfying inequality (29) we have

$$y = T_\alpha(R'_\alpha y) = R'_\alpha y - F(R'_\alpha y) - \alpha L(R'_\alpha y),$$
$$y = T_\beta(R'_\beta y) = R'_\beta y - F(R'_\beta y) - \alpha L(R'_\beta y).$$

Hence, it follows that

$$R'_\alpha y - F(R'_\alpha y) - [R'_\beta y - F(R'_\beta y)] = (\alpha-\beta)L(R'_\alpha y) + \beta[L(R'_\alpha y) - L(R'_\beta y)]$$

and, by (10) and (11), we obtain the inequality

$$(c-|\beta|K)\|R'_\alpha y - R'_\beta y\| \leqslant |\alpha-\beta|\,\|L(R'_\alpha y)\|,$$

which proves relation (31).

Thus, we see that the resolvent R'_α is continuously dependent on the parameter α in the sense that $R'_\beta y$ converges toward $R'_\alpha y$ if $\beta \to \alpha$.

Remark. If F is a dilating mapping, i.e. relation (4) holds, then Theorem 9 is valid with $c = \alpha^{-1}-1$ if $K < \alpha^{-1}-1$.

(C) We shall now consider a family of continuous mappings depending on two real parameters μ and λ. Put

$$y = T_{\mu\lambda}(x) = \mu x - F(x) - \lambda L(x), \qquad x, y \in X.$$

The real numbers μ and λ form a regular value pair if the mapping $T_{\mu\lambda}$ is a homeomorphism of X onto itself. The mapping $R_{\mu\lambda}$ is called the *resolvent* of $T_{\mu\lambda}$ of $y = T_{\mu\lambda}(R_{\mu\lambda}y)$ for arbitrary y of X. The real numbers μ and λ form an eigenvalue pair if there exists a vector x of X such that

(32) $$\mu x = F(x) + \lambda L(x).$$

The vector x is then called the eigenvector corresponding to the eigenvalue pair (μ, λ)

THEOREM 10. *Suppose that the mappings F and L satisfy conditions* (10)–(12). *Let us assume that μ and λ satisfy the condition*

(33) $$c > |1-\mu| + |\lambda|K.$$

Then the resolvent $R_{\mu\lambda}$ exists and satisfies the relations

(34) $$\|R_{\mu\lambda}y_1 - R_{\mu\lambda}y_2\| \leqslant (c-|1-\mu|-|\lambda|K)^{-1}\|y_1 - y_2\|,$$

(35) $$\|R_{\mu\lambda}y - R_{\bar\mu\bar\lambda}y\| \leqslant (c-|1-\mu|-|\lambda|K)^{-1}[|\mu-\bar\mu|\,\|R_{\bar\mu\bar\lambda}y\| + |\lambda-\bar\lambda|\,\|L(R_{\bar\mu\bar\lambda}y)\|],$$

where (μ, λ) and $(\bar\mu, \bar\lambda)$ are subject to restriction (33). *Moreover, for every pair (μ, λ) satisfying relation* (33) *there exists a unique eigenvector x corresponding to the eigenvalue pair (μ, λ), i.e. relation* (32) *holds.*

Proof. Let us write

$$y = T_{\mu\lambda}(x) = \mu x - F(x) - \lambda L(x) = x - F(x) - [(1-\mu)x + \lambda L(x)].$$

Then replacing in Theorem 4 $L(x)$ by $(1-\mu)x+\lambda L(x)$ and the Lipschitz constant K by $|1-\mu|+|\lambda|K$, we infer from (33) that $T_{\mu\lambda}$ is a homeomorphism of X onto itself. Thus, the resolvent $R_{\mu\lambda}$ exists and satisfies relation (34) in virtue of assertion (c) of Theorem 4. Hence, it follows that $x = R_{\mu\lambda}y$ if $y = 0$, is the unique eigenvector corresponding to the eigenvalue pair (μ, λ) for each μ and λ satisfying inequality (33). It remains to prove relation (35). For the pairs (μ, λ) and $(\bar{\mu}, \bar{\lambda})$ satisfying relation (33) we have

$$y = T_{\mu\lambda}(R_{\mu\lambda}y) = R_{\mu\lambda}y - F(R_{\mu\lambda}y) - [(1-\mu)R_{\mu\lambda}y + \lambda L(R_{\mu\lambda}y)],$$

$$y = T_{\bar{\mu}\bar{\lambda}}(R_{\bar{\mu}\bar{\lambda}}y) = R_{\bar{\mu}\bar{\lambda}}y - F(R_{\bar{\mu}\bar{\lambda}}y) - [(1-\bar{\mu})R_{\bar{\mu}\bar{\lambda}}y + \bar{\lambda} L(R_{\bar{\mu}\bar{\lambda}}y)].$$

Hence, it follows that

$$\begin{aligned} R_{\mu\lambda}y - F(R_{\mu\lambda}y) - [R_{\bar{\mu}\bar{\lambda}}y - F(R_{\bar{\mu}\bar{\lambda}}y)] &= (1-\mu)(R_{\mu\lambda}y - R_{\bar{\mu}\bar{\lambda}}y) + (\bar{\mu}-\mu)R_{\bar{\mu}\bar{\lambda}}y + \\ &\quad + \lambda[L(R_{\mu\lambda}y) - L(R_{\bar{\mu}\bar{\lambda}}y)] + (\lambda-\bar{\lambda})L(R_{\bar{\mu}\bar{\lambda}}y). \end{aligned}$$

Hence, we obtain the following relation in virtue of (10), (11) and (33):

$$(c - |1-\mu| - |\lambda|K)\|R_{\mu\lambda}y - R_{\bar{\mu}\bar{\lambda}}y\| \leqslant |\bar{\mu}-\mu|\,\|R_{\bar{\mu}\bar{\lambda}}y\| + |\lambda-\bar{\lambda}|\,\|L(R_{\bar{\mu}\bar{\lambda}}y)\|,$$

which proves inequality (35).

Thus, we see that the resolvent $R_{\mu\lambda}$ is continuously dependent on the two parameters μ and λ in the sense that $R_{\mu\lambda}\,y$ converges toward $R_{\bar{\mu}\bar{\lambda}}y$ if $\mu \to \bar{\mu}$ and $\lambda \to \bar{\lambda}$.

Remark. If F is a dilating mapping, i.e. relation (4) holds, then Theorem 10 is valid, where $c = \alpha^{-1}-1$ and K is subject to restriction (23).

We shall now give two simple examples in order to illustrate the above theorems.

Let us consider the following system of non-linear scalar equations:

$$\text{(a)} \qquad f_i(x_i) - L_i(x_1, x_2, \ldots, x_n) = y_i, \qquad i = 1, \ldots, n,$$

where the real functions f_i $(i = 1, \ldots, n)$ of the real variables x_i have the same slope, i.e. there exists a positive number c such that

$$\text{(b)} \qquad c|x_i - \bar{x}_i| \leqslant |f_i(x_i) - f_i(\bar{x}_i)|$$

for arbitrary values x_i and $\bar{x}_i$. The function $f_i(x_i)$ is continuous for $i = 1, \ldots, n$.

The functions $L_i(x_1, \ldots, x_n)$ are Lipschitz continuous, i.e. there exists a positive constant K_0 such that

$$\text{(c)} \qquad |L_i(x_1, \ldots, x_n) - L_i(\bar{x}_1, \ldots, \bar{x}_n)| \leqslant K_0 \sum_{i=1}^{n} |x_i - \bar{x}_i|$$

for $i = 1, \ldots, n$ and arbitrary $x_i, \bar{x}_i$.

Put $L(x) = (L_1(x), \ldots, L_n(x))$, where $x = (x_1, \ldots, x_n)$ and $\|x\| = \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}$. Then we obtain from (c)

$$\text{(d)} \qquad \|L(x) - L(\bar{x})\| \leqslant K\|x - \bar{x}\|,$$

where $K = nK_0$.

For $x = (x_1, \ldots, x_n)$ put $f(x) = \big(f_1(x_1), \ldots, f_n(x_n)\big)$.

Then condition (b) yields

$$\text{(e)} \qquad c\|x - \bar{x}\| \leqslant \|f(x) - f(\bar{x})\|$$

for arbitrary $x = (x_1, \ldots, x_n)$ and $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n)$.

Let us suppose that $K < c$. Then the hypotheses of Theorem 4 are satisfied, and we can claim that the system (a) has a unique solution for arbitrary $y = (y_1, \ldots, y_n)$. If x and $\bar{x}$ are solutions of (a) corresponding to y and $\bar{y}$, respectively, then we have in virtue of assertion (c) of Theorem 4 the relation

$$\|x - \bar{x}\| \leqslant (c-K)^{-1}\|y - \bar{y}\|.$$

Another simple example is given by considering the system

$$(\mathrm{a}_1) \qquad x_i - F_i(x_i) - L_i(x_1, \ldots, x_n) = y_i, \qquad i = 1, \ldots, n,$$

where the real continuous functions $F_i(x_i)$ satisfy the condition

$$|x_i - \bar{x}_i| \leqslant \alpha|F(x_i) - F(\bar{x}_i)|$$

for some positive constant $\alpha < 1$ and arbitrary $x_i, \bar{x}_i$ $i = 1, \ldots, n$.

The assumptions concerning L_i are the same as in system (a).

Putting

$$f_i(x_i) = x_i - F_i(x_i)$$

one can reduce system (a_1) to system (a), where we shall have $c = \alpha^{-1}-1$.

One can also consider systems (a) and (a_1) introducing the parameter μ or λ or both of them.

The corresponding theorems for resolvents can also be formulated in this case.

The non-linear form for non-linear mappings as a generalization of the quadratic form for linear mappings. Let A be a non-linear continuous mapping of the Euclidean n-space X into itself. The expression

$$\frac{(u-v, Au-Av)}{\|u-v\|^2}, \qquad u \neq v,\ u, v \in X$$

will be called the non-linear form of A. Let us suppose that

$$a(A) = \sup_{u \neq v} \frac{(u-v, Au-Av)}{\|u-v\|^2}$$

is finite. We shall show that if $0 < a(A) < \frac{1}{2}$, then the mapping $x - Ax = y$ is a homeomorphism of X onto itself. Indeed, we have

$$-2a(A)\|u-v\|^2 \leqslant -2(u-v, Au-Av).$$

Putting $c = 1-2a(A)$, we obtain $c-1 = -2a(A)$ and

$$c\|u-v\|^2 \leqslant \|u-v\|^2 - 2(u-v, Au-Av)$$
$$\leqslant \|u-v\|^2 - 2(u-v, Au-Av) + \|Au-Av\|^2.$$

Hence follows the inequality

$$c\|u-v\|^2 \leqslant \|u-v-(Au-Av)\|^2.$$

The last inequality implies that the mapping $f(x) = x - Ax$ is an ε-mapping in the narrow sense. Thus, it follows that f is a homeomorphism of X onto itself.

It is obvious that if λ is a positive number, then $a(\lambda A) = \lambda a(A)$. Thus, we conclude that the mapping $x - \lambda Ax = y$ is a homeomorphism of X onto itself if $0 < \lambda < (2a(A))^{-1}$.

It follows that the non-linear form may be considered as a generalization of the quadratic form of a linear mapping.

Now let us consider the following case. Suppose that A satisfies the relation

$$(u-v, Au-Av) \leqslant \tfrac{1}{2}\|Au-Av\|^2$$

for arbitrary u, v of X. We shall show that $f(x) = x - Ax$ is a homeomorphism of X onto itself. Indeed, we have

$$0 \leqslant -2(u-v, Au-Av) + \|Au-Av\|^2.$$

Hence, we obtain

$$\|u-v\|^2 \leqslant \|u-v\|^2 - 2(u-v, Au-Av) + \|Au-Av\|^2$$

and, consequently, we have

$$\|u-v\|^2 \leqslant \|u-v-(Au-Av)\|^2.$$

This inequality shows that f is an ε-mapping in the narrow sense and, consequently, we obtain our assertion.

Suppose now that $b(A)$ is the smallest number a satisfying the relation

$$(u-v, Au-Av) \leqslant a\|Au-Av\|^2$$

for arbitrary u, v of X. It is clear that if $b(A)$ exists, then $b(\lambda A) = \frac{1}{\lambda} b(A)$ for positive λ. Thus, we conclude that $x - \lambda Ax = y$ is a homeomorphism of X onto itself if $\frac{b(A)}{\lambda} \leqslant \frac{1}{2}$, i.e. if $\lambda \geqslant 2b(A)$.

Extensions to Banach spaces of some of these results will be given elsewhere, including some additional results.

References

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