Dilating mappings, implicit functions
and fixed point theorems in finite-dimensional spaces

by

M. Altman (Warszawa)

It is the purpose of this paper to investigate some properties of non-linear mappings of a finite-dimensional Euclidean spaces into itself. The argument presented here consists in a combination of two facts: Borsuk's theorem on \( \varepsilon \)-mappings in the narrow sense and Banach's contraction principle. By means of this method several theorems concerning non-linear mappings of finite-dimensional Banach spaces into themselves are obtained. In particular, an implicit function theorem for dilating mappings, a generalization of the contraction principle and some results concerning the non-linear eigenvalue problem are included.

Let \( f \) be a continuous transformation of a finite-dimensional Euclidean space \( X \) into itself. The transformation \( f \) is called an \( \varepsilon \)-mapping in the narrow sense if it has the following property:

\[
(B) \quad \text{there exist two positive numbers} \quad \eta, \varepsilon \quad \text{such that} \quad \|f(x') - f(x'')\| < \eta, \quad x', x'' \in X
\]

implies

\[
\|x' - x''\| < \varepsilon.
\]

In paper [1] K. Borsuk proved the following theorem. If \( f(\varepsilon) \) has property (B), then \( f \) is a mapping onto, i.e. \( f(X) = X \).

Implicit functions. In order to make use of Borsuk's theorem let us observe that if the mapping \( f \) possesses the following property: there exists a positive number \( c \) such that

\[
|f(x_1) - f(x_2)| \leq c|x_1 - x_2|
\]

for arbitrary \( x_1, x_2 \) of \( X \), then \( f \) is an \( \varepsilon \)-mapping in the narrow sense and, consequently, \( f(X) = X \). Moreover, \( f \) is a homeomorphism of \( X \) onto itself.

After this remark we shall prove the following implicit function theorem.
THEOREM 1. Let \( f \) be a continuous mapping defined on the product space \( X \times Y \) with values in \( X \).

Suppose that the following conditions are satisfied. There exists a positive number \( c \) such that
\[
   c|\|x_1 - x_2\| \leq \|f(x_1, y) - f(x_2, y)\|
\]
for arbitrary \( y \) of \( Y \) and \( x_1, x_2 \) of \( X \).

There exists a positive number \( K \) such that
\[
   \|f(x, y_1) - f(x, y_2)\| \leq K\|y_1 - y_2\|
\]
for arbitrary \( x \) of \( X \) and \( y_1, y_2 \) of an arbitrary metric space \( Y \). Then there exists a unique continuous function \( x = g(y) \) satisfying the equation
\[
   f(g(y), y) = 0, \quad y \in Y.
\]

Proof. It follows from (1) and from the above remark that for arbitrary fixed \( y \) of \( Y \) the mapping \( f(x, y) \) is an \( \varepsilon \)-mapping in the narrow sense. Hence, there exists a unique element \( x = g(y) \) satisfying relation (3). Conditions (1), (2) yield
\[
   c|\|x_1 - x_2\| \leq \|f(x_1, y) - f(x_2, y)\| = \|f(x_1, y) - f(x_1, y_1) - f(x_1, y_1) - f(x_2, y_2)\| \leq K\|y_1 - y_2\|,
\]
where \( x_1 = g(y_1), x_2 = g(y_2) \) and \( f(x_1, y_1) = f(x_2, y_1) = 0 \).

Thus, we have
\[
   \|f(x_1, y) - f(x_2, y)\| \leq Kc\|y_1 - y_2\|,
\]
which completes the proof.

Let us remark that \( Y \) can be replaced by an arbitrary metric space and the variable \( y \) can be restricted to an arbitrary subset of \( Y \).

Suppose that the mapping \( F \) of \( X \) into itself is strictly contractive, i.e., there exists a positive number \( a < 1 \) such that
\[
   \|F(x) - F(x_1)\| \leq a|x_1 - x_2|
\]
for arbitrary \( x_1, x_2 \) of \( X \). It is easily seen that the mapping \( f(x) = x - F(x) \) satisfies the relation
\[
   (1 - a|x_1 - x_2| \leq \|f(x_1) - f(x_2)\|)
\]
for arbitrary \( x_1, x_2 \) of \( X \). Hence, it follows that \( f(x) \) is an \( \varepsilon \)-mapping in the narrow sense and by Borsuk’s theorem it is a mapping onto \( X \).

Thus, the mapping \( f \) is a homeomorphism of \( X \) onto itself and, in particular, there exists a unique element \( x^* \) of \( X \) such that \( f(x^*) = 0 \), i.e., \( x^* = F(x^*) \) and, consequently, \( x^* \) is the unique fixed point of \( F \).

A mapping \( F \) of \( X \) into itself is said to be a dilating mapping if there exists a positive number \( a < 1 \) such that
\[
   |\|x_1 - x_2\| \leq a|\|F(x_1) - F(x_2)\|
\]
for arbitrary \( x_1, x_2 \) of \( X \). Consider the mapping \( f(x) = x - F(x) \).

We have
\[
   \|f(x_1) - f(x_2)\| \geq (a^{-1} - 1)|x_1 - x_2|.
\]

Thus, it follows that \( f \) is an \( \varepsilon \)-mapping in the narrow sense Theorem 1 implies the following

THEOREM 2. Suppose that the mapping \( F \) with values in \( X \) is defined on the product space \( X \times X \). Let us assume that the following conditions are satisfied. There exist two positive numbers \( K \) and \( a < 1 \) such that
\[
   |\|x_1 - x_2\| \leq a|\|F(x_1) - F(x_2)\|
\]
for arbitrary \( x_1, x_2 \) of \( X \) and \( y \) of \( Y \), and
\[
   \|F(x, y_1) - F(x, y_2)\| \leq K\|y_1 - y_2\|
\]
for arbitrary \( x \) of \( X \) and \( y_1, y_2 \) of \( Y \).

Then there exists a unique continuous function \( x = g(y) \) satisfying
\[
   f(g(y), y) = 0, \quad y \in Y.
\]

Proof. The mapping \( f(x, y) = x - F(x, y) \) satisfies relation (1) with \( c = a^{-1} - 1 \).

Further, we obtain by assumptions (6), (7)
\[
   |\|x_1 - x_2\| \leq a|\|F(x_1, y_1) - F(x_2, y_1)\|
\]
\[
   \leq a\left( |\|F(x_1, y_1) - F(x_2, y_1)\| + |\|F(x_1, y_2) - F(x_2, y_2)\| \right)
\]
\[
   \leq a|x_1 - x_2| + aK\|y_1 - y_2\|,
\]
where \( x_1 = g(y_1) \) and \( x_2 = g(y_2) \) satisfy relation (8). Hence, we obtain
\[
   (1 - a)|g(y_1) - g(y_2)| \leq aK\|y_1 - y_2\|
\]
The last inequality yields the continuity of \( g(y) \).

The remark concerning the variable \( y \) in Theorem 1 is also valid in this case.

THEOREM 3. Suppose that \( X = Y \). If, in addition to the hypotheses of Theorem 2, the numbers \( a \) and \( K \) are subject to the restriction \( K < a^{-1} \), then there exists a unique fixed point \( x^* \) such that
\[
   g(y^*) = y^*, \quad i.e., \quad y^* = F(y^*, y^*).
\]

Proof. In virtue of (9) we have
\[
   |\|g(y_1) - g(y_2)\| \leq aK(1 - a^{-1})\|y_1 - y_2\|.
\]

Hence, it follows that the mapping \( g \) is contractive and the assertion of the theorem results from the contraction principle.
A generalization of the contraction principle. Let \( F(x) \) and \( L(x) \) be two continuous mappings of \( X \) into itself and put \( f(x) = x - F(x) \). The following theorem is a generalization of the well-known contraction principle.

**Theorem 4.** Suppose that \( F(x) \) and \( L(x) \) satisfy the following conditions: There exist two positive numbers \( c \) and \( K \) such that

\[
0 < c < c < 1
\]

for arbitrary \( x_1, x_2 \) of \( X \),

\[
\|L(x_1) - L(x_2)\| \leq K\|x_1 - x_2\|
\]

for arbitrary \( x_1, x_2 \) of \( X \) and

\[
K < c.
\]

Then

(a) the mapping \( F(x) + L(x) \) has a unique fixed point, i.e., there is a unique element \( x^* \) of \( X \) such that \( x^* = F(x^*) + L(x^*) \),

(b) the mapping \( y = f(x) - L(x) = x - F(x) - L(x) \) is a homeomorphism of \( X \) onto itself and

(c) the inverse mapping \( x = y \) is Lipschitz continuous with the constant \( (c - K)^{-1} \), i.e., \( \|x - y\| \leq (c - K)^{-1}\|x_1 - x_2\| \).

**Proof.** Condition (10) implies that the mapping \( F(x) + L(x) \) is a homeomorphism of \( X \) onto itself. Let \( x \) be a fixed element of \( X \). Then for \( L(x) \) there exists a unique element \( L(x) = z \) of \( X \) such that

\[
f(R_x) = L(x).
\]

Consider now the mapping \( x \rightarrow R_x \).

In virtue of (10), (13) and (11) we obtain

\[
c\|R_{x_1} - R_{x_2}\| \geq \|f(R_{x_1}) - f(R_{x_2})\| = \|L(x_1) - L(x_2)\| \leq K\|x_1 - x_2\|.
\]

Hence we have

\[
\|R_{x_1} - R_{x_2}\| \leq Kc^{-1}\|x_1 - x_2\|
\]

for arbitrary \( x_1, x_2 \) of \( X \). Thus, we see that the mapping \( R \) is a contraction mapping, by (12), it follows that there exists a unique element \( x^* \) of \( X \) such that \( R(x^*) = x^* \). Hence, we obtain \( f(x^*) = L(x^*) \), by (13), which completes the proof of assertion (a), since the uniqueness of \( x^* \) follows from relations (14), (12) by putting \( R(x) = x_1, R(x) = x_2 \). The proof of assertion (b) follows by means of the same argument by replacing \( L(x) \) in (13) by \( L(x) + y \) for fixed \( y \) of \( X \). Thus we obtain

\[
f(R_x) = L(x) + y
\]

instead of (13). Relations (11), (14) remain unchanged while we replace the mapping \( L(x) \) by the mapping \( L(x) + y \) in assertion (a), provided that \( y \) is an arbitrary but fixed element of \( X \). Consequently, we infer by means of the contraction principle for \( R_x \) that for arbitrary \( y \) of \( X \) there is a unique element \( x \) of \( X \) such that

\[
0 < c < c < 1
\]

\[
\|x - f(x) - L(x) - y\| = 0
\]

Hence, it follows from (10) that

\[
\|x_1 - x_2\| \leq \|f(x_1) - f(x_2)\| \leq \|L(x_1) - L(x_2)\| \leq K\|x_1 - x_2\|.
\]

Then, we obtain

\[
(c - K)^{-1}\|x_1 - x_2\| \leq \|y_1 - y_2\|
\]

in virtue of (11), where \( x_1 = x(y_1), x_2 = x(y_2) \) are solutions of equation (16) for \( y_1, y_2 \), respectively. Since condition (12) is satisfied by assumption, the last inequality proves assertion (c).

**Remark.** Putting \( F(x) = 0 \) in Theorem 4, we obtain \( c = 1 \) and \( K < 1 \), i.e., \( L(x) \) is a contracting mapping. Thus, Theorem 4 generalizes the well-known contraction principle.

Let us observe that the proof of Theorem 4 can be reduced directly to the contraction principle by considering the contractive mapping \( f^{-1}L(x) \), where \( f^{-1} \) denotes the inverse mapping. It follows from (10) that the inverse exists and is Lipschitz continuous with the constant \( c^{-1} \).

Now suppose that \( F(x) \) is a dilating mapping, i.e., that relation (4) is satisfied. Then Theorem 4 assumes the following formulation;

**Theorem 5.** Let \( F \) be a dilating mapping and \( L \) a Lipschitz continuous mapping satisfying relation (11) with the Lipschitz constant \( K \) subject to the restriction \( K < a^{-1} - 1 \). Then assertions (a), (b) and (c) of Theorem 4 hold, where \( a = a^{-1} - 1 \).

**Proof.** It follows from (5) that relation (10) is satisfied with \( c = a^{-1} - 1 \). Since by assumption we have \( K < a^{-1} - 1 = c \), we conclude that all hypotheses of Theorem 4 are fulfilled.

On the basis of Theorem 4 we obtain the following implicit function theorem.

**Theorem 6.** Let \( F, L \) be two continuous mappings defined on the product space \( X \times X \) with values in \( X \). Let us assume that \( F(x, y), L(x, y) \) satisfy the following conditions. There exist two positive numbers \( c \) and \( K \) such that

\[
0 < c < c < 1
\]

\[
\|x - f(x, y) - L(x, y)\| = 0
\]

for arbitrary \( x, y \) of \( X \) and \( y \) of \( X \), where \( f(x, y) = x - F(x, y) \).

\[
\|L(x_1, y) - L(x_2, y)\| \leq K\|x_1 - x_2\|
\]

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for arbitrary \( x, y \) of \( X \) and \( \mu \) of \( Y \).

(19) \[ K < c. \]

In addition, there exists a positive number \( a \) such that

(20) \[ \|F(x, y) - F(x, y_0) + L(x, y_1) - L(x, y_2)\| \leq a\|y_1 - y_2\| \]

for arbitrary \( x \) of \( X \) and \( y_1, y_2 \) of \( Y \).

Then there exists a unique continuous function \( x = g(y) \) satisfying the equation

(21) \[ g(y) = F(g(y), y) + L(g(y), y). \]

Proof. In virtue of Theorem 4, it follows from conditions (17)-(19) that for arbitrary fixed \( y \) of \( Y \) there exists a unique element \( x = g(y) \) satisfying equation (21). We have, by (21),

\[
g(y_1) - g(y_2) - F(g(y_1), y_1) + F(g(y_2), y_2) = \bigg[ F(g(y_1), y_1) - F(g(y_2), y_2) + L(g(y_1), y_1) - L(g(y_2), y_2) \bigg] + \bigg[ L(g(y_1), y_1) - L(g(y_2), y_2) \bigg]
\]

Hence, it follows, by (17), (20) and (18),

\[ \|g(y_1) - g(y_2)\| \leq a\|y_1 - y_2\| + K\|g(y_1) - g(y_2)\|. \]

Thus, we obtain, by (19), the inequality

\[ \|g(y_1) - g(y_2)\| \leq a\|(0 - K)^{-1}\|y_1 - y_2\|, \]

which proves the Lipschitz continuity of \( g(y) \).

Assuming that \( F \) is a dilating mapping with respect to \( x \) but uniformly in \( y \), we obtain the following

**Theorem 7.** If in addition to relations (17) and (20) there exists a positive number \( a < 1 \) such that

(22) \[ \|x_1 - x_2\| \leq a\|F(x_1, y) - F(x_2, y)\| \]

for arbitrary \( x_1, x_2 \) of \( X \) and \( y \) of \( Y \) and

(23) \[ K < a^{-1} - 1, \]

then there exists a unique continuous function \( g(y) \) satisfying equation (21).

Proof. It is easily seen that conditions (17) and (19) follow from conditions (22) and (23) with \( c = a^{-1} - 1 \). Thus, all the hypotheses of Theorem 6 are fulfilled.

Let us remark that in Theorems 6, 7 \( Y \) can be an arbitrary metric space and the variable \( y \) can be restricted to an arbitrary subset of \( Y \).

**Resolvents.** (A) On the basis of Theorem 4 it is possible to investigate some families of continuous mappings depending on a real parameter.

Put \( y = T_\mu(x) = \mu x - F(x) - L(x), x \in X. \)

The real number \( \mu \) is called a regular value if the mapping \( T_\mu \) is a homeomorphism of \( X \) onto itself. The mapping \( R_\mu \) is called the resolvent of \( T_\mu \) if \( y = T_\mu(x, y) \) for arbitrary \( y \) of \( X \). The real number \( \mu \) is called an eigenvalue of the mapping \( F(x) + L(x) \) if there exists a vector \( x \) of \( X \) such that

(24) \[ \mu x = F(x) + L(x). \]

The vector \( x \) is called the eigenvector corresponding to the eigenvalue \( \mu \). If \( F \) and \( L \) are both linear mappings, the vector \( x = 0 \) is always an eigenvector. Thus, in the linear case the trivial zero eigenvector is excluded.

**Theorem 8.** Suppose that the mappings \( F \) and \( L \) satisfy conditions (10)-(12). Let us assume that \( \mu \) satisfies the condition

(25) \[ |1 - \mu| < c - K. \]

Then the resolvent \( R_\mu \) exists and satisfies the relations

(26) \[ \|R_\mu y - R_\gamma y\| \leq (c - K - |1 - \mu|)^{-1}\|y_1 - y_2\|, \]

(27) \[ \|R_\mu y - R_\gamma y\| \leq \beta |\mu - \gamma| (c - K - |1 - \mu|)^{-1}\|y_1 - y_2\|, \]

where \( \alpha, \beta \) are subject to restriction (22).

Moreover, for every \( \mu \) satisfying relation (25) there exists a unique eigenvector \( x \) corresponding to the eigenvalue \( \mu \), i.e., \( \mu x = \mu x + L(x) \).

**Proof.** Let us write

\[ y = T_\mu(x) = \mu x - F(x) - L(x) = x - F(x) - [(1 - \mu)x + L(x)]. \]

Then, replacing in Theorem 4 \( L(x) \) by \( (1 - \mu)x + L(x) \) and the Lipschitz constant \( K \) by \( K + |1 - \mu| \), we infer that \( T_\mu \) is a homeomorphism of \( X \) onto itself. Thus, the resolvent \( R_\mu \) exists and satisfies relation (26) in virtue of assertion (a) of Theorem 4. Hence, it results that \( x = R_\mu y \) for \( y \) is the unique eigenvector corresponding to the eigenvalue \( \mu \) for each \( \mu \) satisfying inequality (25). It remains to prove relation (27). For \( \alpha, \beta \) satisfying inequality (25) we have

\[
\begin{align*}
 y &= T_\mu(R_\mu y) = R_\mu y - F(R_\mu y) - [(1 - \alpha)R_\mu y + L(R_\mu y)], \\
 y &= T_\mu(R_\mu y) = R_\mu y - F(R_\mu y) - [(1 - \beta)R_\mu y + L(R_\mu y)].
\end{align*}
\]

Hence, it follows that

\[
\begin{align*}
 R_\mu y - F(R_\mu y) - [R_\mu y - F(R_\mu y)] &= (1 - \alpha)R_\mu y - (1 - \beta)R_\mu y - [L(R_\mu y) - L(R_\mu y)],
\end{align*}
\]
and by (10) and (11) we obtain
\[(c - K)\|R_y y - R_0 y\| \leq \|(1 - a) R_y y - (1 - \beta) R_0 y\| \leq \|(1 - a) R_y y - R_0 y\| + |\beta - a|\|R_0 y\|.
\]
Since $a$ and $\beta$ satisfy relation (25), we conclude from the last inequality that the relation
\[(c - K)(1 - a)\|R_y y - R_0 y\| \leq |\beta - a|\|R_0 y\|
\]
holds, which proves inequality (27).

Thus, we see that the resolvent $R_y$ is continuously dependent on the parameter $\mu$ in the sense that $R_0 y$ converges toward $R_y y$ if $\mu = \beta$.

**Remark.** If $F$ is a dilating mapping, i.e. relation (4) holds, then Theorem 8 is valid, where $c = a^{-1} - 1$ and $K$ is subject to restriction (23).

(B) Another family of continuous mappings depending on a real parameter can be introduced as follows. Put
\[y = T_1(x) = x - F(x) - \lambda L(x), \quad x, y \in X.
\]

The real number $\lambda$ is called a regular value if the mapping $T_1$ is a homeomorphism of $X$ onto itself. The mapping $R_1$ is called the resolvent of $T_1$ if $y = T_1(R_1(x))$ for arbitrary $y$ of $X$. The real number $\lambda$ is called an eigenvalue if there exists a vector $x$ of $X$ such that
\[x = F(x) + \lambda L(x).
\]

The vector $x$ is called the eigenvector corresponding to the eigenvalue $\lambda$.

**Theorem 9.** Suppose that the mappings $F$ and $L$ satisfy conditions (10)–(11) and let $\lambda$ satisfy the condition
\[|\lambda| < cK^{-1}.
\]

Then the resolvent $R_1$ exists and satisfies the relations
\[|R_1 y - R_0 y| \leq (c - |K|)^{-1}|y_1 - y_0|, \quad (30)
\]
\[|R_1 y - R_0 y| \leq |\beta - a| |R_0 y|, \quad (31)
\]
where $a$ and $\beta$ are subject to restriction (20). Moreover, for any $\lambda$ satisfying relation (29) there exists a unique eigenvector $x$ corresponding to the eigenvalue $\lambda$, i.e. $x$ and $\lambda$ satisfy relation (38).

**Proof.** Replacing in Theorem 4 the mapping $L$ by $\lambda L$ and condition (12) by (29), we infer that $T_1$ is a homeomorphism of $X$ onto itself. Thus, the resolvent $R_1$ exists and satisfies relation (30) in virtue of assertion (e) of Theorem 4. Hence, we have $x = R_1 y$ if $y = 0$, is the unique eigenvector corresponding to the eigenvalue $\lambda$ for each $\lambda$ satisfying inequality (29). It remains to prove relation (31).

For $\mu$ and $\beta$ satisfying inequality (29) we have
\[y = T_1(R_0 y) = R_0 y - F(R_0 y) - \lambda L(R_0 y), \quad (29)
\]
\[y = T_1(R_1 y) = R_1 y - F(R_1 y) - \lambda L(R_1 y).
\]

Hence, it follows that
\[R_1 y - F(R_1 y) - (R_0 y - F(R_0 y)) = (c - |K| |R_0 y| - |R_1 y|) = (c - |K| |R_0 y| - |R_1 y|)
\]
and, by (10) and (11), we obtain the inequality
\[(c - |K| |R_0 y| - |R_1 y|) \leq |\beta - a| \|L(R_0 y) - L(R_1 y)\|,
\]
which proves relation (31).

Thus, we see that the resolvent $R_1$ is continuously dependent on the parameter $\mu$ in the sense that $R_0 y$ converges toward $R_1 y$ if $\beta = a$.

**Remark.** If $F$ is a dilating mapping, i.e. relation (4) holds, then Theorem 9 is valid with $c = a^{-1} - 1$ if $K < a^{-1} - 1$.

(C) We shall now consider a family of continuous mappings depending on two real parameters $\mu$ and $\lambda$. Put
\[y = T_\mu(x) = \mu x - F(x) - \lambda L(x), \quad x, y \in X.
\]

The real numbers $\mu$ and $\lambda$ form a regular value pair if the mapping $T_\mu$ is a homeomorphism of $X$ onto itself. The mapping $R_\mu$ is called the resolvent of $T_\mu$, of $y = T_\mu(R_\mu y)$ for arbitrary $y$ of $X$. The real numbers $\mu$ and $\lambda$ form an eigenvalue pair if there exists a vector $x$ of $X$ such that
\[\mu x = F(x) + \lambda L(x).
\]

The vector $x$ is then called the eigenvector corresponding to the eigenvalue pair $(\mu, \lambda)$.

**Theorem 10.** Suppose that the mappings $F$ and $L$ satisfy conditions (10)–(12). Let us assume that $\mu$ and $\lambda$ satisfy the condition
\[c > |1 - \mu| + |\lambda K|.
\]

Then the resolvent $R_\mu$ exists and satisfies the relations
\[|R_\mu y - R_0 y| \leq (c - |1 - \mu| - |\lambda K|)^{-1}|y_1 - y_0|, \quad (34)
\]
\[|R_\mu y - R_0 y| \leq |\beta - a| |R_0 y| + |\lambda - \lambda_0| \|L(R_0 y)\|, \quad (35)
\]
where $(\mu, \lambda)$ and $(\mu_0, \lambda_0)$ are subject to restriction (33). Moreover, for every pair $(\mu, \lambda)$ satisfying relation (33) there exists a unique eigenvector $x$ corresponding to the eigenvalue pair $(\mu, \lambda)$, i.e. relation (32) holds.

**Proof.** Let us write
\[y = T_\mu(x) = \mu x - F(x) - \lambda L(x) = x - F(x) - (1 - \mu)x + (1 - \mu)\lambda L(x).
\]
Then replacing in Theorem 4 $L(x)$ by $(1-\mu)x+L(x)$ and the Lipschitz constant $K$ by $1-\mu+2\bar\lambda|K|$, we infer from (33) that $R_{\mu\lambda}$ is a holomorphic function of $X$ to itself. Thus, the resolvent $R_{\mu\lambda}$ exists and satisfies relation (34) in virtue of assertion (c) of Theorem 4. Hence, it follows that $x = R_{\mu\lambda}y$ if $y = 0$, is the unique eigenvector corresponding to the eigenvalue pair $(\mu, \lambda)$ for each $\mu$ and $\lambda$ satisfying inequality (33). It remains to prove relation (35). For the pairs $(\mu, \lambda)$ and $(\bar\mu, \bar\lambda)$ satisfying relation (33) we have
\[
y = T_{\mu\lambda}(R_{\mu\lambda}y) = R_{\mu\lambda}y - F(R_{\mu\lambda}y) - [(1-\mu)R_{\mu\lambda}y + \lambda L(R_{\mu\lambda}y)],
y = T_{\bar\mu\bar\lambda}(R_{\mu\lambda}y) = R_{\mu\lambda}y - F(R_{\mu\lambda}y) - [(1-\bar\mu)R_{\mu\lambda}y + \lambda L(R_{\mu\lambda}y)].\]

Hence, it follows that
\[
|E_{\mu\lambda}y - F(R_{\mu\lambda}y)| - |E_{\bar\mu\bar\lambda}y - F(R_{\bar\mu\bar\lambda}y)| = |E_{\mu\lambda}y - (\mu)E_{\mu\lambda}y + (\bar\mu)E_{\bar\mu\bar\lambda}y + \lambda L(R_{\mu\lambda}y) - L(R_{\bar\mu\bar\lambda}y)|.
\]

Hence, we obtain the following relation in virtue of (10), (11) and (33):
\[
(c-1)\mu - |\lambda|K||E_{\mu\lambda}y - E_{\bar\mu\bar\lambda}y|| \leq |\bar\lambda - \mu - |\lambda||E_{\mu\lambda}y - E_{\bar\mu\bar\lambda}y||L(R_{\mu\lambda}y)|,
\]

which proves inequality (35).

Thus, we see that the resolvent $R_{\mu\lambda}$ is continuous and depends on the two parameters $\mu$ and $\lambda$ in the sense that $R_{\mu\lambda}$ converges toward $E_{\mu\lambda}$ if $\mu \to \mu$ and $\lambda \to \lambda$.

Remark. If $F$ is a dilating mapping, i.e. relation (4) holds, then Theorem 10 is satisfied, where $c = \alpha^{-1} - 1$ and $K$ is subject to restriction (33).

We shall now give two simple examples in order to illustrate the above theorems.

Let us consider the following system of non-linear scalar equations:

(a) $f(x) = L(x)$, $x = x_1, \ldots, x_n$, where the real functions $f_i$ $(i = 1, \ldots, n)$ of the real variables $x$ have the same slope, i.e. there exists a positive number $c$ such that

(b) $|x - x_i| \leq |f(x) - f(x)|$

for arbitrary values $x$ and $x_i$. The function $f(x)$ is continuous for $i = 1, \ldots, n$.

The functions $L(x)$ are Lipschitz continuous, i.e. there exists a positive constant $K$ such that

(c) $|L(x) - L(y)| \leq K\sum_{i=1}^{n} |x_i - y_i|$

for $i = 1, \ldots, n$ and arbitrary $x, y$. Put $L(x) = (L_1(x), \ldots, L_n(x))$, where $x = (x_1, \ldots, x_n)$ and $\|x\| = \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}$.

Then we obtain from (c)

(d) $\|L(x) - L(y)\| \leq K\|x - y\|$.

where $K = \sum_{i=1}^{n} h_{i\lambda}$.

For $x = (x_1, \ldots, x_n)$ put $f(x) = (f_1(x_1), \ldots, f_n(x_n))$.

Then condition (b) yields

(e) $|x - x_i| \leq |f(x) - f(x_i)|$

for arbitrary $x = (x_1, \ldots, x_n)$ and $x_i = (x_i, \ldots, x_n)$.

Let us suppose that $K < c$. Then the hypotheses of Theorem 4 are satisfied, and we can claim that the system (a) has a unique solution for arbitrary $y = (y_1, \ldots, y_n)$. If $x = x_i$ are solutions of (a) corresponding to $y$ and $x_i$, respectively, then we have in virtue of assertion (c) of Theorem 4 that the relation

$\|x - x_i\| \leq (c - K)^{-1}\|y - x_i\|$.

Another simple example is given by considering the system

(a) $x_i = F(x_{i-1}, \ldots, x_n)$, $i = 1, \ldots, n$,

where the real continuous functions $F(x)$ satisfy the condition

$|x_i - x_i| \leq c|F(x)|$

for some positive constant $c < 1$ and arbitrary $x$, $x_i$, $i = 1, \ldots, n$.

The assumptions concerning $L_i$ are the same as in system (a).

Putting $f(x) = x_i - F(x)$

one can reduce system (a) to system (a), where we shall have $c = \alpha^{-1} - 1$.

One can also consider systems (a) and (a) introducing the parameter $\mu$ or $\lambda$ both of them.

The corresponding theorems for resolvents can also be formulated in this case.

The non-linear form for non-linear mappings as a generalization of the quadratic form for linear mappings. Let $A$ be a non-linear continuous mapping of the Euclidean $n$-space $X$ into itself. The expression

$\frac{(u - v, A(u - v))}{\|u - v\|^2}$

is a positive definite quadratic form.
will be called the non-linear form of \( A \). Let us suppose that

\[
a(A) = \sup_{u \neq 0} \frac{(u - v, Au - Av)}{\|u - v\|^2}
\]

is finite. We shall show that if \( 0 < a(A) < \frac{1}{2} \), then the mapping \( x - Ax = y \) is a homeomorphism of \( X \) onto itself. Indeed, we have

\[-2a(A)\|u - v\|^2 \leq -2(u - v, A u - A v).\]

Putting \( c = 1 - 2a(A) \), we obtain \( c = 1 - 2a(A) \) and

\[
ce\|u - v\|^2 \leq \|u - v\|^2 - 2(u - v, Au - Av)
\leq \|u - v\|^2 - 2(u - v, Au - Av) + \|Au - Av\|^2.
\]

Hence follows the inequality

\[
ce\|u - v\|^2 \leq \|u - v - (Au - Av)\|^2.
\]

The last inequality implies that the mapping \( f(x) = x - Ax \) is an \( \varepsilon \)-mapping in the narrow sense. Thus, it follows that \( f \) is a homeomorphism of \( X \) onto itself.

It is obvious that if \( \lambda \) is a positive number, then \( a(\lambda A) = \lambda a(A) \). Thus, we conclude that the mapping \( x - \lambda Ax = y \) is a homeomorphism of \( X \) onto itself if \( 0 < \lambda < \left[ 2a(A) \right]^{-1} \).

It follows that the non-linear form may be considered as a generalization of the quadratic form of a linear mapping.

Now let us consider the following case. Suppose that \( A \) satisfies the relation

\[
(u - v, Au - Av) \leq \|Au - Av\|^2
\]

for arbitrary \( u, v \) of \( X \). We shall show that \( f(x) = x - Ax \) is a homeomorphism of \( X \) onto itself. Indeed, we have

\[
0 \leq -2(u - v, Au - Av) + \|Au - Av\|^2.
\]

Hence, we obtain

\[
\|u - v\|^2 \leq \|u - v\|^2 - 2(u - v, Au - Av) + \|Au - Av\|^2
\]

and, consequently, we have

\[
\|u - v\|^2 \leq \|u - v - (Au - Av)\|^2.
\]

This inequality shows that \( f \) is an \( \varepsilon \)-mapping in the narrow sense and, consequently, we obtain our assertion.

Suppose now that \( b(A) \) is the smallest number \( \alpha \) satisfying the relation

\[
(u - v, Au - Av) \leq \alpha \|Au - Av\|^2
\]

for arbitrary \( u, v \) of \( X \). It is clear that if \( b(A) \) exists, then \( b(\lambda A) = \frac{1}{\lambda} b(A) \) for positive \( \lambda \). Thus, we conclude that \( x - \lambda Ax = y \) is a homeomorphism of \( X \) onto itself if \( \frac{1}{\lambda} \leq \frac{1}{2} \), i.e. if \( \lambda \geq 2b(A) \).

Extensions to Banach spaces of some of these results will be given elsewhere, including some additional results.

References


Institute of Mathematics Polish Academy of Sciences
(Institut Matematyczny PAN)

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