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On continua which resemble simple closed curves

by

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1. Introduction. Consider the following well-known theorems concerning simple closed curves: (for the definition of simple closed curves and arcs, see Moore [3], pp. 44 and 33 respectively)

THEOREM 1. *M is a simple closed curve if and only if for $x, y \in M$, $M = H \cup K \cup \{x\} \cup \{y\}$ where H and K are irreducible continua between x and y and* $\mathrm{Int}(H) \cap \mathrm{Int}(K) = \emptyset$. (In fact, H and K are arcs.)

(It stands for "the interior of")

THEOREM 2. *M is a simple closed curve if and only if no single point of M cuts, but every collection of two points of M do cut.* (Bing [1], also refer to this reference for the definition of cut).

THEOREM 3. *Let M be a simple closed curve. If C is a subcontinuum of M and $\mathrm{Int}(C) \neq \emptyset$, then $\overline{M - C}$ is a continuum.* (For the purposes of this paper, we shall say that any continuum which satisfies the condition expressed in the conclusion of Theorem 3 is C.C.)

The first two of these theorems both express equivalent conditions for M to be a simple closed curve, whereas the third is a rather weak necessary condition for M to be a simple closed curve. The question that is poised in this paper is how "close" to a simple closed curve, i.e., how "ring-like", is a compact metric continuum M which is C.C. It will be shown that if one essentially replaces the point x by the set

$$L_x \doteq \{y \in M: \text{there is no continuum } C \subset M \text{ with } y \in \mathrm{Int}(C) \text{ and } x \notin C\}$$

in Theorem 2, then we have a characterization of C.C., and if we replace x by L_x in Theorem 1, then either M is C.C. or M belongs to a very special class of continua which "look" very "ringlike" indeed.

This idea of replacing a point x by L_x was used by Thomas [4], where he showed that if M is irreducible between two closed sets, and contains no indecomposable subcontinuum with interior, then a certain collection of these L_x's forms an upper semi continuous decomposition of M that is an arc.

Discussions of the set L_x may be found in Jones [2], but for our purposes, we shall only need the following fact given in this reference: if M is a compact metric continuum, then L_x is a continuum for all $x \in M$.

II. Definitions and necessary background. For the sake of completeness and convenience, let us define here all the notions we shall work with:

DEFINITION. M is said to satisfy *Condition* 1 iff for $x \in M$, x does not cut $M-L_x$ and yet if x and $y \in M$ with $L_x \cap L_y = \emptyset$, then $\{x\} \cup \{y\}$ does cut $M-(L_x \cup L_y)$.

DEFINITION. M is said to satisfy *Condition* 2 iff for x and $y \in M$ such that $L_x \cap L_y = \emptyset$, then $M = L_x \cup H \cup K \cup L_y$ where H and K are irreducible subcontinua between L_x and L_y and $\operatorname{Int}(H) \cap \operatorname{Int}(K) = \emptyset$.

DEFINITION. M is said to be *Continuum Complemented*, i.e., C.C., iff for each subcontinuum C with $\operatorname{Int}(C) \neq \emptyset$, $\overline{M-C}$ is a subcontinuum.

Results which we shall use extensively throughout this paper are the following: (In each result we take M to be a compact metric space, and we take H and K to be two closed subsets of M.)

RESULT 1. If there does not exist a subcontinuum C of M such that $C \cap H \neq \emptyset$, and $C \cap K \neq \emptyset$, then $M = L \cup V$ where L and V are separated sets and $H \subseteq L$ and $K \subseteq V$.

RESULT 2. Let C be a subcontinuum of M such that $H \cap C \neq \emptyset$ and $K \cap C \neq \emptyset$. Then there exists an irreducible subcontinuum C' of C from H to K. (For a definition of irreducible see Moore [3].)

RESULT 3. Let C be an irreducible subcontinuum of M from H to K. Then $C-(H \cup K)$ is connected.

RESULT 4. Let C be a subcontinuum of M, and let U be an open set. Then every component of $U \cap C$ has a limit point in ∂U. (∂U means the boundary of U).

III. Equivalence of C. C. and Condition 1.

LEMMA 1. *Let M be a compact metric continuum which is* C.C. *If C_1 and C_2 are two disjoint continua with nonvoid interiors, then there exist two disjoint continua, H and K, which are irreducible between C_1 and C_2, and*

$$M = C_1 \cup H \cup C_2 \cup K$$

Proof. Let C_1 and C_2 be two continua as in the hypothesis. It is elementary that since M is a C.C. continuum, $\overline{M-C_1 \cup C_2}$ is not a continuum and so let

$$\overline{M-C_1 \cup C_2} = H \cup K$$

where H and K are two disjoint closed sets.

We will now show that H is an irreducible subcontinuum from C_1 to C_2. The same arguments will apply to K and then the lemma will be proved.

Claim 1. There does not exist a continuum $V \not\subseteq H$ such that $V \cap C_1 \neq \emptyset$ and $V \cap C_2 \neq \emptyset$.

Suppose the contrary holds. We note then that by the definition of H and by the fact that $V \not\subseteq H$

$$\operatorname{Int}(H-V) \neq \emptyset .$$

From this we see that $\overline{M-C_1 \cup C_2} = K \cup \overline{(H-V)}$ which, since H and K are disjoint, clearly contradicts M being C.C.

Claim 2. H is a continuum.

This claim is proved by considering the following two possible cases: $C_1 \cup K \cup C_2$ is a continuum and $C_1 \cup K \cup C_2$ is not a continuum. In the first case the claim immediately follows from M being C.C. In the second case we assume the claim is false, then using standard arguments that include Result 1, we arrive at a contradiction.

(Note. It is not difficult to show the converse of the above lemma does not hold.)

LEMMA 2. *Let M be a compact metric* C.C. *continuum. If x and $y \in M$, and $L_x \cap L_y = \emptyset$, then there exists continua C_x and C_y such that $x \in \operatorname{Int}(C_x)$, $y \in \operatorname{Int}(C_y)$ and $C_x \cap C_y = \emptyset$.*

Proof. If $z \in L_x$, then $z \notin L_y$ and so by the definition of L_y we see that there exists a continuum K_z such that $z \in \operatorname{Int}(K_z)$ and $y \in K_z$. Since L_x is closed and therefore compact, we see that there exists a finite number of these K_z's, $K_1, K_2, \ldots, K_n$ such that

$$L_x \subseteq \operatorname{Int}(K) \quad \text{and} \quad y \notin K .$$

where $K = \bigcup_{i=1}^{n} K_i$. Now L_x is a continuum and so we see that K is a continuum. So we let $C_y = \overline{M-K}$ which is a continuum because M is C.C. Also, by the definition of K we see that $y \in \operatorname{Int}(C_y)$.

Now repeat the above argument where we replace x by y and L_x by C_y, thereby getting a continuum V such that $C_y \subseteq \operatorname{Int}(V)$ and $x \notin V$. Now let $C_x = \overline{M-V}$ which we see is a continuum because M is C.C. Clearly C_x and C_y satisfy the conclusion of the lemma.

THEOREM 1. *Let M be a compact metric continuum. M is* C.C. *if and only if M satisfies Condition* 1.

Proof. Only if. First we show that x doesn't cut any two points of $M-L_x$ and therefore doesn't cut $M-L_x$.

Let v and $g \in M-L_x$. By the definition of L_x, there exist continua C_v and C_g such that $v \in \operatorname{Int}(C_v)$, $g \in \operatorname{Int}(C_g)$ and $x \notin C_v \cup C_g$. If $C_g \cap C_v \neq \emptyset$, then the continuum $C_g \cup C_v$ shows that x doesn't cut g and v. If $C_x \cap C_v \neq \emptyset$, then the use of Lemma 1 guarantees that x doesn't cut v and g.

Second, we show that if $L_x \cap L_y = \emptyset$, then $\{x\} \cup \{y\}$ cuts $M-(L_x \cup L_y)$.

Since $L_x \cap L_y = \emptyset$, we see by Lemma 2 that there exists two continua C_x, and C_y, with $x \in \mathrm{Int}(C_x)$, $y \in \mathrm{Int}(C_y)$ and $C_x \cap C_y = \emptyset$.

So by Lemma 1, there exist two continua H and K irreducible between C_x and C_y and $M = C_x \cup H \cup C_y \cup K$. Let $g \in \mathrm{Int}(H)$ and $v \in \mathrm{Int}(K)$. If $\{x\} \cup \{y\}$ doesn't cut $M-(L_x \cup L_y)$, then there exists a continuum V such that g and $v \in V$, and x and $y \notin V$. Now $H \cup V \cup K$ s a continuum, but $R = \overline{M-H \cup V \cup K} \subseteq C_x \cap C_y$ with $x \in C_x \cup R$ and $y \in C_y \cup R$ which is a clear contradiction to M being C.C.

If. Assume to the contrary, i.e., assume there exists a continuum C with $\mathrm{Int}(C) \neq \emptyset$ and $\overline{M-C} = A+B$ where A and B are two closed disjoint sets. Let $x \in \mathrm{Int}(A)$ and $y \in \mathrm{Int}(B)$, then it is clear that $L_x \subseteq A$ and $L_y \subseteq B$, i.e., $L_x \cap L_y = \emptyset$. By hypothesis then there exists r and $v \in M-(L_x \cup L_y)$ such that $\{x\} \cup \{y\}$ cuts r and v.

Claim. If $z \in M-(C \cup L_x \cup L_y)$, then there exists a continuum C_z such that $z \in C_z$, $C_z \cap C \neq \emptyset$ and x and $y \notin C_z$.

Without loss of generality, let $z \in \mathrm{Int}(A)$. Let $q \in \mathrm{Int}(C)$. Since z and $q \notin L_x$ we see that the hypothesis implies the existence of a continuum C' for which z and $q \in C'$ and $x \notin C'$. Now using the fact that $\mathrm{Int}(A)$ is an open set, $\partial C' \subseteq C$, and result 4 we see that C_z can be taken to be the closure of the component of $C' \cap \mathrm{Int}(A)$ which contains z.

It is now a straightforward argument that shows the existence of a continuum C'' which contains r and v but not x and y. This is a contradiction and completes the proof of the theorem.

IV. Relationship between C.C. and Condition 2.

LEMMA 3. *Let M be a compact metric* C.C. *If $z \notin L_x \cup L_y$, then there exists a continuum C_z such that $z \in \mathrm{Int}(C_z)$ and x and $y \notin C_z$.*

Proof. Using the fact that $z \notin L_x \cup L_y$ and that M is C.C., we have the existence of two continua C_x and C_y for which $x \in \mathrm{Int}(C_x)$, $y \in \mathrm{Int}(C_y)$ and $z \notin C_x \cup C_y$.

If $C_x \cap C_y \neq \emptyset$, then take C_z of the conclusion to be $C_x \cup C_y$.

If $C_x \cap C_y = \emptyset$, then the choice of C_z follows directly from Lemma 1.

LEMMA 4. *Let M be a continuum, let H and G be two closed subsets of M, let C_1 and C_2 be two irreducible subcontinua between H and G such that*

$$\mathrm{Int}(C_1) \cap \mathrm{Int}(C_2) = \emptyset \quad \text{and} \quad M = C_1 \cup H \cup C_2 \cup G.$$

If R is an irreducible continuum between H and G, then $R = C_1$ or $R = C_2$.

Proof. Clear by Result 3.

LEMMA 5. *Let M be a continuum satisfying Condition 2. Then $x \in L_v$ implies $v \in L_x$.*

Proof. Suppose $v \notin L_x$. Then there exists a continuum C_v such that $v \in \mathrm{Int}(C_v)$ and $x \notin C_v$. Since $x \in L_v$ we have

$$\overline{M-C_v} = R \cup S$$

where $R \cap S = \emptyset$. Without loss of generality let $x \in R$. Choose $y \in \mathrm{Int}(S)$. $C_v \cup R$ and $C_v \cup S$ are continua and so $L_x \subseteq R$ and $L_y \subseteq S$, i.e., $L_x \cap L_y = \emptyset$. Let H and K be the two continua, mentioned in Condition 2, which are irreducible between L_x and L_y. Without loss of generality let $v \in \mathrm{Int}(H)$. Because $x \in L_v$ and $K \cup L_x \cup L_y$ is a continuum we have that $x \in H$.

Case 1. $C_v \cap (L_x \cup L_y) \neq \emptyset$.

$\partial H \subseteq L_y \cup L_y$ and so by Result 4 each of the components of $C_v \cap \mathrm{Int}(H)$ have a limit point in either L_x or L_y. Define then

$$H_x \equiv L_x \cup \{\text{all components of } C_v \cap \mathrm{Int}(H) \text{ with limit points in } L_x\},$$

$$H_y \equiv L_y \cup \{\text{all components of } C_v \cap \mathrm{Int}(H) \text{ with limit points in } L_y\}.$$

Since $x \in H$, $x \notin C_v$ and H is irreducible we see that H_x and H_y are disjoint continua. (Without loss of generality, let $v \in H_x$.) This combined with Result 1 shows that $\overline{H-C_v}$ is a continuum and therefore so is $\overline{H-C_v} \cup L_x \cup K \cup L_y \cup H_y$ which violates $x \in L_v$.

Case 2. $C_v \cap (L_x \cup L_y) = \emptyset$.

H being irreducible quickly implies that $\overline{H-C_v} = R \cup S$ where R and S are disjoint continua with $R \cap L_x \neq \emptyset$ and $S \cap L_v \neq \emptyset$. But then $R \cup L_x \cup K \cup L_y \cup S$ is a continuum that violates $x \in L_v$.

LEMMA 6. *Let M be a compact metric continuum which satisfies Condition 2. Then x cuts $\{v\} \cup \{w\}$ implies v or $w \in L_x$.*

Proof. Assume to the contrary. Then there exists v and $w \in M-L_x$ for which x cuts $\{v\} \cup \{w\}$. By Lemma 5, $x \notin L_v \cup L_w$. But L_v and L_w are continua, and so $L_v \cap L_w = \emptyset$. Now a straightforward application of Condition 2 gives a contradiction.

In the next two lemmas, let M be a compact metric continuum satisfying Condition 2. Also, let x and $y \in M$ with $L_x \cap L_y = \emptyset$. Finally, let H and K be the two irreducible continua between L_x and L_y as guaranteed to us by Condition 2.

LEMMA 7. *If $v \in \mathrm{Int}(H)$ and $L_v \cap L_x = \emptyset$, then $x \notin H$.*

Proof. First we note that $\mathrm{Int}(H) \cap (L_x \cup K \cup L_y) = \emptyset$ and so we see that $L_v \subseteq H$.

Second we note by Lemma 5 that since $v \notin L_x$, then $x \notin L_v$.

Third we note that since $v \notin L_x$ there exists a continuum C such that $v \in \mathrm{Int}(C)$ and $x \notin C$.

Using now the hypothesis $L_x \cap L_y \neq \emptyset$, Result 4, and the irreducibility of H, it becomes clear that $x \notin H$.

Lemma 8. *If there exists $v \in \operatorname{Int}(H)$, $z \in \operatorname{Int}(K)$, and a continuum C with $\{v, z\} \subseteq C$ and $\{x, y\} \subseteq M - C$, then*

(1) *$L_r = H$ for all $r \in \operatorname{Int}(H)$*

(2) $H \cap K \neq \emptyset$.

Proof. As in the first note of Lemma 7, we see that $L_r \subseteq H$ for all $r \in \operatorname{Int}(H)$.

We first show $L_v = H$ before verifying that $L_r = H$ for all $r \in \operatorname{Int}(H)$.

Claim 1. $L_v \cap L_z \neq \emptyset$.

Assume the claim false, and let H' and K' be the two continua associated with L_v and L_z as specified in Condition 2. By Lemma 4, we see that C contains either H' or K' and so let it contain H'. By hypothesis $\{x, y\} \subseteq M - C$, and Lemma 5 implies $\{x, y\} \subseteq M - (L_v \cap L_z)$ and so we see that $L_x \cup L_y \subseteq K'$. Let K_0 be an irreducible subcontinuum of K' between L_x and L_y. By Lemma 4, $K = K_0$ and therefore $v \in \operatorname{Int}(K)$ which is a contradiction.

Since $\partial K \subseteq L_x \cup L_y$ we assume without loss of generality that $L_v \cap L_z \subseteq L_x$. Therefore by Lemma 7 we see that $x \notin H \cup K$.

Claim 2. $L_v \cap L_y \neq \emptyset$.

Again we assume the claim is false and let H^* and K^* be the two continua associated with L_v and L_y as specified in Condition 2. From Claim 1, we see that $L_v \cap K \neq \emptyset$ and so let H_0, K_0 be irreducible subcontinua of K, H between L_v and L_y. By Lemma 4 we see that we can take $K_0 = K^*$ and $H_0 = H^*$. But $M = L_y \cup H^* \cup L_v \cup K^*$ and yet x is not an element of any of these sets—so the claim holds.

So $L_v \cap L_y \neq \emptyset$, $L_v \cap L_x \neq \emptyset$, L_v is a continuum and $L_v \subseteq H$. Therefore the irreducibility of H implies $L_v = H$.

Now let $r \in \operatorname{Int}(H) \subseteq L_v$. By Lemma 5, $v \in L_r$. But then $C \cup L_r$ is a continuum between r and z missing x and y—so now we just let r take v's place in the above arguments.

$H \cap K \neq \emptyset$ follows from Claim 1.

Theorem 2. *Let M be a compact metric continuum. Then M satisfies Condition 2 iff*

(1) *M is* C.C.

or

(2) *There exists four indecomposable continua $\{R_i\}_{i=1}^{4}$ such that $M = \bigcup_{i=1}^{4} R_i$, R_1 and R_2 are irreducible between R_3 and R_4, and $R_1 \cap R_2 \cap R_3 \neq \emptyset$.*

Proof. Only if. Since (2) clearly implies that M satisfies Condition 2, we only need show that C.C. implies Condition 2. So we let x and $y \in M$ be such that $L_x \cap L_y = \emptyset$.

If $v \in M$, then define

$$H_v = \{m \in M\colon \{x\} \cup \{y\} \text{ does not cut } v \text{ and } m\}\,.$$

It is clear that H_v is connected and $H_t \cap H_s \neq \emptyset$ implies $H_s = H_t$ for $s, t \in m$.

This combined with Theorem 1 implies that if $v \in M - (L_x \cup L_y)$ then

$$(*) \qquad H_v \not\supseteq M - (L_x \cup L_y)\,.$$

Claim 1. If $v \in M = (L_x \cup L_y)$, then $\partial H_v \subseteq L_x \cup L_y$.

Suppose there exists $w \in \partial H_v - L_x \cup L_y$. Then by Lemma 3, there exists a continuum C_w with $w \in \operatorname{Int}(C_w)$ and x and $y \notin C_w$. Clearly then by the definition of H_v, $\operatorname{Int}(C_w) \subseteq H_v$ which is impossible since $w \in \partial H_v$.

Claim 2. If v and $w \in M - (L_x \cup L_y)$ are such that $H_v \neq H_w$ then: there exist continua C_v, C_w, K and H such that $v \in \operatorname{Int}(C_v)$, $w \in \operatorname{Int}(C_w)$, H and K are irreducible subcontinua between C_v and C_w, $H \cap K = \emptyset$, $C_v \cap C_w = \emptyset$, $L_x \subseteq H$, $L_y \subseteq K$ and

$$M = H \cup C_v \cup K \cup C_w\,.$$

By Lemma 3, there are continua C_v and C_w, with $v \in \operatorname{Int}(C_v)$, $w \in C_w$ and x and $y \notin C_v \cup C_w$. Since $H_v \neq H_w$ we know that $H_v \cap H_w = \emptyset$ which in turn implies $C_v \cap C_w = \emptyset$. Using Lemma 1, the fact $v \notin H_w$, and the assumption of M being C.C., we see that the claim is valid.

At this point it is convenient to introduce the following notation:

$$H'_v \doteq H_v - (L_x \cup L_y)\,.$$

Claim 3. If $v \in M - (L_x \cup L_y)$, then $\overline{H'_v}$ is an irreducible continuum between L_x and L_y.

From Claim 1 we know that $\partial H_v \subseteq L_x \cup L_y$. Assume for the moment that $\partial H_v \subseteq L_x$. Then by $(*)$ we know there exists $w \in M - (L_x \cup L_y \cup H_v)$. Now an application of Claim 2 and a recollection of the definition of H_v, we find a contradiction. Therefore, $\overline{H_v}$ is a continuum between L_x and L_y. By Result 2 we know there exists a subcontinuum H_0 of $\overline{H_v}$ which is irreducible between L_x and L_y. Now if we assume $H_0 \neq \overline{H'_v}$ and combine this with Claim 2 and the definition of C.C. we are lead to a contradiction. We thus see that the claim holds.

Theorem 1 says that there exist two points v_0 and $w_0 \in M - (L_x \cap L_y)$ which are cut by x and y. Therefore by Claim 3, $\overline{H'_v}$ and $\overline{H'_w}$ will serve as the H and K in the definition of Condition 2 if we can show that

$$M = L_x \cup \overline{H'_{v_0}} \cup L_y \cup \overline{H'_{w_0}}\,.$$

But if there were $r \in M-(L_x \cup \overline{H'_{v_0}} \cup L_y \cup \overline{H'_{w_0}})$, then we could apply Claim 2, Claim 3 in reference to H_r, and the fact that v_0 and $w_0 \notin H_r$ to arrive at a contradiction to M being C.C. This then completes the sufficiency part of this theorem.

To establish the necessity half of this theorem, we will assume that M is not C.C. but it does satisfy Condition 2 and deduce that it is of the form expressed in 2.

Lemma 6 shows that Condition 2 implies (2) of Theorem 1, and so assuming M is not C.C. means that there exists x and $y \in M$ such that $L_x \cap L_y = \emptyset$ and yet $\{x\} \cup \{y\}$ cut no two points of $M-(L_x \cup L_y)$.

Since $L_x \cap L_y = \emptyset$ we know that there exist two continua H and K which are irreducible between L_x and L_y and $M = L_x \cup H \cup L_y \cup K$. Letting $t \in \mathrm{Int}(H)$ and $z \in \mathrm{Int}(K)$ we see by the above comments that there exist a continuum C such that $t, z \in C$ and $x, y \notin C$. So by Lemma 8, $L_r = H$ for each $r \in \mathrm{Int}(H)$, therefore using the fact that $\overline{\mathrm{Int}(H)} = H$ (Result 3), we see that H is indecomposable. Analogous comments hold for K. Also we know from Lemma 8 that $H \cap K \neq \emptyset$.

Now $L_r = H$ for each $r \in \mathrm{Int}(H)$ implies by Lemma 5 that $x \in \mathrm{Int}(L_x)$ and $y \in \mathrm{Int}(L_y)$. It is not difficult to see that $\overline{\mathrm{Int}(L_x)} = L_x$ and $\overline{\mathrm{Int}(L_y)} = L_y$. So to finish the proof we only need show that for each $r \in \mathrm{Int}(L_x)$, $L_r = L_x$. Clearly $L_r \subseteq L_x$. Therefore, $L_r \cap L_y = \emptyset$. Let H' and K' be the continua associated with L_r and L_y mentioned in the statement of Condition 2. Clearly we can take $K \subseteq K'$ and $H \subseteq H'$. Letting $t \in \mathrm{Int}(K)$ and $v \in \mathrm{Int}(H)$ we know that $L_t \cap L_v \neq \emptyset$, by Lemma 5 r and $y \notin L_t \cap L_v$, and so by Lemma 8

$$H' = L_v = H \quad \text{and} \quad K' = L_t = K$$

But $M = H \cup L_r \cup K \cup L_z$ and so $\mathrm{Int}(L_x) \subseteq L_r$ which implies $L_x \subseteq L_r$. This then completes the proof of the theorem.

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Dilating mappings, implicit functions and fixed point theorems in finite-dimensional spaces

by

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It is the purpose of this paper to investigate some properties of non-linear mappings of a finite-dimensional Euclidean spaces into itself. The argument presented here consists in a combination of two facts: Borsuk's theorem on ε-mappings in the narrow sense and Banach's contraction principle. By means of this method several theorems concerning non-linear mappings of finite-dimensional Banach spaces into themselves are obtained. In particular, an implicit function theorem for dilating mappings, a generalization of the contraction principle and some results concerning the non-linear eigenvalue problem are included.

Let f be a continuous transformation of a finite-dimensional Euclidean space X into itself. The transformation f is called an *ε-mapping in the narrow sense* if it has the following property:

(B) there exist two positive numbers η and ε such that the condition

$$\|f(x')-f(x'')\| < \eta\,, \qquad x', x'' \in X$$

implies

$$\|x'-x''\| < \varepsilon\,.$$

In paper [1] K. Borsuk proved the following

THEOREM. *If $f(x)$ has property* (B), *then f is a mapping onto, i.e.* $f(X) = X$.

Implicit functions. In order to make use of Borsuk's theorem let us observe that if the mapping f possesses the following property: there exists a positive number c such that

$$c\|x_1-x_2\| \leqslant \|f(x_1)-f(x_2)\|$$

for arbitrary x_1, x_2 of X, then f is an ε-mapping in the narrow sense and, consequently, $f(X) = X$. Moreover, f is a homeomorphism of X onto itself.

After this remark we shall prove the following implicit function theorem.