On continua which resemble simple closed curves

by

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1. Introduction. Consider the following well-known theorems concerning simple closed curves: (for the definition of simple closed curves and arcs, see Moore [3], pp. 44 and 33 respectively)

Theorem 1. $M$ is a simple closed curve if and only if for $x, y \in M$, $M = H \cup K \cup \{x\} \cup \{y\}$ where $H$ and $K$ are irreducible continua between $x$ and $y$ and $\text{Int}(H) \cup \text{Int}(K) = \emptyset$. (In fact, $H$ and $K$ are arcs.)

(It stands for "the interior of")

Theorem 2. $M$ is a simple closed curve if and only if no single point of $M$ cuts, but every collection of two points of $M$ do cut. (Bing [1], also refer to this reference for the definition of cut).

Theorem 3. Let $M$ be a simple closed curve. If $C$ is a subcontinuum of $M$ and $\text{Int}(C) \neq \emptyset$, then $M - C$ is a continuum. (For the purposes of this paper, we shall say that any continuum which satisfies the condition expressed in the conclusion of Theorem 3 is C.C.)

The first two of these theorems both express equivalent conditions for $M$ to be a simple closed curve, whereas the third is a rather weak necessary condition for $M$ to be a simple closed curve. The question that is posed in this paper is how "close" to a simple closed curve, i.e., how "ring-like", is a compact metric continuum $M$ which is C.C. It will be shown that if one essentially replaces the point $x$ by the set

$L_x = \{y \in M: \text{there is no continuum } C \subset M \text{ with } y \in \text{Int}(C) \text{ and } x \notin C\}$

in Theorem 2, then we have a characterization of C.C., and if we replace $x$ by $L_x$ in Theorem 1, then either $M$ is C.C. or $M$ belongs to a very special class of continua which "look" very "ringlike" indeed.

This idea of replacing a point $x$ by $L_x$ was used by Thomas [4], where he showed that if $M$ is irreducible between two closed sets, and contains no indecomposable subcontinuum with interior, then a certain collection of these $L_x$'s forms an upper semi continuous decomposition of $M$ that is an arc.
Discussions of the set $L_2$ may be found in Jones [2], but for our purposes, we shall only need the following fact given in this reference: if $M$ is a compact metric continuum, then $L_2$ is a continuum for all $x \in M$.

II. Definitions and necessary background. For the sake of completeness and convenience, let us define here all the notions we shall work with:

**Definition.** $M$ is said to satisfy **Condition 1** iff for $x \in M$, $x$ does not cut $M - L_2$ and yet if $x$ and $y \in M$ with $L_2 \cap L_2 = \emptyset$, then $(x) \cup (y)$ does cut $M - (L_2 \cup L_2)$.

**Definition.** $M$ is said to satisfy **Condition 2** iff for $x$ and $y \in M$ such that $L_2 \cap L_2 = \emptyset$, then $M = L_2 \cup H \cup K \cup L_2$ where $H$ and $K$ are irreducible subcontinua between $L_2$ and $L_2$ and Int$(H) \cap Int(K) = \emptyset$.

**Definition.** $M$ is said to be **Continuous Complemented**, i.e., C.C., iff for each subcontinuum $C$ with Int$(C) \neq \emptyset$, $M - C$ is a subcontinuum.

**Results** which we shall use extensively throughout this paper are the following: (In each result we take $M$ to be a compact metric space, and we take $H$ and $K$ to be two closed subsets of $M$.)

**Result 1.** If there does not exist a subcontinuum $C$ of $M$ such that $C \cap H \neq \emptyset$, and $C \cap K \neq \emptyset$, then $M = L \cup V$ where $L$ and $V$ are separated sets and $H \subseteq L$ and $K \subseteq V$.

**Result 2.** Let $C$ be a subcontinuum of $M$ such that $H \cap C \neq \emptyset$ and $K \cap C = \emptyset$. Then there exists an irreducible subcontinuum $C'$ of $C$ from $H$ to $K$. (For a definition of irreducible see Moore [3]).

**Result 3.** Let $C$ be an irreducible subcontinuum of $M$ from $H$ to $K$. Then $C \cap (H \cup K)$ is connected.

**Result 4.** Let $C$ be a subcontinuum of $M$, and let $U$ be an open set. Then every component of $U \cap C$ has a limit point in $\partial U$. (This means the boundary of $U$).

III. Equivalence of C.C. and Condition 1.

**Lemma 1.** Let $M$ be a compact metric continuum which is C.C. If $C_1$ and $C_2$ are two disjoint continua with nonvoid interiors, then there exist two disjoint continua, $H$ and $K$, which are irreducible between $C_1$ and $C_2$, and $M = C_1 \cup H \cup C_2 \cup K$.

**Proof.** Let $C_1$ and $C_2$ be two continua as in the hypothesis. It is elementary that since $M$ is a C.C. continuum, $M - C_1 \cup C_2$ is not a continuum and so let $M - C_1 \cup C_2 = H \cup K$ where $H$ and $K$ are two disjoint closed sets.

We will now show that $H$ is an irreducible subcontinuum from $C_1$ to $C_2$. The same arguments will apply to $K$ and then the lemma will be proved.

**Claim 1.** There does not exist a continuum $V \nsubseteq H$ such that $V \cap C_1 \neq \emptyset$ and $V \cap C_2 \neq \emptyset$.

Suppose the contrary holds. We note then that by the definition of $H$ and by the fact that $V \nsubseteq H$,

$$Int(H - V) \neq \emptyset.$$ 

From this we see that $M - C_1 \cup C_2 = K \cup (H - V)$ which, since $H$ and $K$ are disjoint, clearly contradicts $M$ being C.C.

**Claim 2.** $H$ is a continuum.

This claim is proved by considering the following two possible cases: $C_1 \cup K \cup C_2$ is a continuum and $C_1 \cup K \cup C_2$ is a continuum. In the last case, the claim immediately follows from $M$ being C.C. In the second case we assume the claim is false, then using standard arguments that include Result 1, we arrive at a contradiction.

(Note. It is not difficult to show the converse of the above lemma does not hold.)

**Lemma 2.** Let $M$ be a compact metric C.C. continuum. If $x$ and $y \in M$, and $L_2 \cap L_2 = \emptyset$, then there exists a continuum $C_x$ and $C_y$ such that $x \in Int(C_x)$, $y \in Int(C_y)$, and $C_x \cap C_y = \emptyset$.

**Proof.** If $x \in L_2$, then $x \notin L_2$ and so by the definition of $L_2$ we see that there exists a continuum $K_2$, such that $x \notin Int(K_2)$ and $y \notin K_2$.

Since $L_2$ is closed and therefore compact, we see that there exists a finite number of these $K_2$'s, $K_1, K_2, \ldots, K_n$ such that

$$L_2 \subseteq Int(K) \quad \text{and} \quad y \notin K,$$

where $K = \bigcup_{i=1}^{n} K_i$. Now $L_2$ is a continuum and so we see that $K$ is a continuum. So we let $C_x = M - K$ which is a continuum because $M$ is C.C. Also, by the definition of $K$ we see that $y \notin Int(C_x)$.

Now repeat the above argument where we replace $x$ by $y$ and $L_2$ by $C_y$, thereby getting a continuum $V$ such that $C_y \subseteq Int(V)$ and $x \notin V$.

Now let $C_x = M - V$, which we see is a continuum because $M$ is C.C. Clearly $C_x$ and $C_y$ satisfy the conclusion of the lemma.

**Theorem 1.** Let $M$ be a compact metric continuum. $M$ is C.C. if and only if $M$ satisfies Condition 1.

**Proof.** Only if. First we show that $x$ doesn't cut any two points of $M - L_2$ and therefore doesn't cut $M - L_2$.

Let $x$ and $y \in M - L_2$. By the definition of $L_2$ there exist continua $C_x$ and $C_y$ such that $x \notin Int(C_y)$, $y \notin Int(C_x)$, and $x \notin C_y \cup C_x$. If $C_x \cap C_y \neq \emptyset$,
then the continuum $C_a \cup C_b$ shows that $a$ doesn’t cut $v$ and $c$. If $C_a \cap C_b \neq \emptyset$, then the use of Lemma 1 guarantees that $a$ doesn’t cut $v$ and $g$.

Second, we show that if $L_a \cap L_b = \emptyset$ then $(x \vee y)$ cuts $M - (L_a \cup L_b)$.

Since $L_a \cap L_b = \emptyset$, we see by Lemma 2 that there exists a continuum $C_a$ and $C_b$, with $x \in \text{Int}(C_a)$ and $y \in \text{Int}(C_b)$ and $C_a \cap C_b = \emptyset$.

So by Lemma 1, there exist a continuum $H$ and $K$ irreducible between $C_a$ and $C_b$ and $M = C_a \cup H \cup C_b \cup K$. Let $g \in \text{Int}(H)$ and $v \in \text{Int}(K)$. If $(x \vee y)$ doesn’t cut $M - (L_a \cup L_b)$, then there exists a continuum $V$ such that $g$ and $v \in \text{Int}(V)$ and $x \neq y \in V$. Now $H \cup V \cup K$ is a continuum, but $R = M - H \cup V \cup K \subseteq C_a \cup C_b \cup K \cup C_a \cup R$ and $y \neq C_a \cup R$ which is a clear contradiction to $M$ being C.C.

If $a$ assume to the contrary, i.e., assume there exists a continuum $C$ with $\text{Int}(C) \neq \emptyset$ and $M - C = A \cup B$ where $A$ and $B$ are two closed disjoint sets. Let $x \in \text{Int}(A)$ and $y \in \text{Int}(B)$, then it is clear that $L_a \subseteq A$ and $L_b \subseteq B$, i.e., $L_a \cap L_b = \emptyset$. By hypothesis then there exists $r$ and $v \in M - (L_a \cup L_b)$ such that $(x \vee y)$ cuts $r$ and $v$.

Claim. If $r \in (C \cup L_a \cup L_b)$, then there exists a continuum $C_a$ such that $x \in C_a \cup C_b \subseteq A \neq \emptyset$ and $y \notin C_a$.

Without loss of generality, let $v \in \text{Int}(A)$. Let $g \in \text{Int}(C)$. Since $s$ and $g \notin L_a$ we see that the hypothesis implies the existence of a continuum $C'$ for which $s$ and $g$ are distinct and $s \notin C'$. Using the fact that $\text{Int}(A)$ is an open set, $\delta C \subseteq C'$, and results 4 we see that $C_a$ can be taken to be the closure of the component of $C' \cap \text{Int}(A)$ which contains $s$.

It is now a straightforward argument that shows the existence of a continuum $C''$ which contains $r$ and $v$ but not $x$ and $y$. This is a contradiction and completes the proof of the theorem.

IV. Relationship between C.C. and Condition 2.

Lemma 3. Let $M$ be a compact metric C.C. If $x \notin L_a \cup L_b$, then there exists a continuum $C_a$ such that $x \in \text{Int}(C_a)$ and $x \neq y \in C_a$.

Proof. Using the fact that $x \notin L_a \cup L_b$ and that $M$ is C.C., we have the existence of two continua $C_a$ and $C_b$ for which $x \in \text{Int}(C_a)$ and $y \notin C_a \cup C_b$.

If $C_a \cap C_b = \emptyset$, then $C_a$ is the conclusion to be $C_a \cup C_b$.

If $C_a \cap C_b = \emptyset$, then the choice of $C_a$ follows directly from Lemma 1.

Lemma 4. Let $M$ be a continuum, let $H$ and $K$ be two closed subsets of $M$, let $C_a$ and $C_b$ be two irreducible subcontinua between $H$ and $K$ such that

$$\text{Int}(C_a) \cap \text{Int}(C_b) = \emptyset$$

and $M = C_a \cup H \cup C_b \cup K$.

If $R$ is an irreducible continuum between $H$ and $K$, then $R = C_a$ or $R = C_b$.

Proof. Clear by Result 3.
Using now the hypothesis \( L \cap L_0 \neq \emptyset \), Result 4, and the irreducibility of \( H \), it becomes clear that \( x \notin H \).

**Lemma 8.** If there exists \( v \in \text{Int}(H) \), \( x \in \text{Int}(K) \), and a continuum \( C \) with \( \{x, y\} \subseteq C \) and \( \{x, y\} \subseteq M - C \); then

\[(1) \quad L_0 = H \text{ for all } r \in \text{Int}(H)\]

\[(2) \quad H \cap K = \emptyset.\]

**Proof.** As in the first note of Lemma 7, we see that \( L_0 \subseteq H \) for all \( r \in \text{Int}(H) \).

We first show \( L_0 = H \) before verifying that \( L_0 = H \) for all \( r \in \text{Int}(H) \).

**Claim 1.** \( L_0 \cap L \neq \emptyset. \)

Assume the claim false, and let \( H' \) and \( K' \) be the two continua associated with \( L_0 \) and \( L_0 \) as specified in Claim 2. By Lemma 4, we see that \( C \) contains either \( H' \) or \( K' \), and so let \( C \) contain \( H' \). By hypothesis \( \{x, y\} \subseteq M - C \) and Lemma 5 implies \( \{x, y\} \subseteq M - (L_0 \cap L) \) and so we see that \( L_0 \cap L \subseteq C \). Let \( K_0 \) be an irreducible subcontinuum of \( K' \) between \( L_0 \) and \( L_0 \). By Lemma 4, \( K_0 \cong K_0 \) and therefore \( v \in \text{Int}(K_0) \) which is a contradiction.

Since \( 2K \subseteq L_0 \cap L_0 \) we assume without loss of generality that \( L_0 \cap L_0 \subseteq L_0 \). Therefore by Lemma 7 we see that \( x \in H \cap K \).

**Claim 2.** \( L_0 \cap L_0 \neq \emptyset. \)

Again we assume the claim false and let \( H^* \) and \( K^* \) be the two continua associated with \( L_0 \) and \( L_0 \) as specified in Claim 2. From Claim 1, we see that \( L_0 \cap K = \emptyset \) and so let \( H_0 \cap K \) be an irreducible subcontinua of \( K, H \) between \( L_0 \) and \( L_0 \). By Lemma 4 we see that we can take \( K_0 = K^* \) and \( H_0 = H^* \). But \( M = L_0 \cap H_0 \cap K_0 \) and yet \( x \) is not an element of any of these sets — so the claim holds.

So \( L_0 \cap L_0 \neq \emptyset \), \( L_0 \cap L_0 \neq \emptyset, \), \( L_0 \) is a continuum and \( L_0 \subseteq H \). Therefore the irreducibility of \( H \) implies \( L_0 = H \).

Now let \( r \in \text{Int}(H) \subseteq L_0 \). By Lemma 5, \( v \in L_0 \). But then \( L_0 \cap L_0 \subseteq \emptyset \), \( L_0 \) is a continuum between \( r \) and \( z \) missing \( z \) and \( y \); so now we just let \( r \) take \( v \)'s place in the above arguments.

\[H \cap K = \emptyset \text{ follows from Claim 1.}\]

**Theorem 2.** Let \( M \) be a compact metric continuum. Then \( M \) satisfies Condition 2 iff

\[(1) \quad M \text{ is O.C.}\]

\[(2) \quad \text{There exists four indecomposable continua } (R_i)_4 = 1 \text{ such that } M = \bigcup_i R_i \text{ and } R_2 \text{ is irreducible between } R_0 \text{ and } R_1, \text{ and } R_1 \cap R_2 \cap R_3 \neq \emptyset.\]

**Proof.** Only if. Since (2) clearly implies that \( M \) satisfies Condition 2, we only need show that O.C. implies Condition 2. So we let \( x \) and \( y \in M \) be such that \( L_0 \cap L_0 = \emptyset \).

If \( \emptyset \), then define

\[H_0 = (m \in M: (x) \cup (y) \text{ does not cut } v \text{ and } m) .\]

It is clear that \( H_0 \) is connected and \( H_0 \cap H_0 \neq \emptyset \) implies \( H_0 = H \), for \( s, t \in m \).

This combined with Theorem 1 implies that if \( v \in M - (L_0 \cup L_0) \) then

\[(*) \quad H_0 \cap M = (L_0 \cup L_0) .\]

**Claim 1.** \( v \in M = (L_0 \cup L_0) \), then \( \emptyset \subseteq L_0 \subseteq L_0 \).

Suppose there exists \( \emptyset \subseteq L_0 \subseteq L_0 \). Then by Lemma 3, there exists a continuum \( C_0 \) with \( v \in \text{Int}(C_0) \) and \( x \) and \( y \in C_0 \). Clearly then by the definition of \( H_0 \), \( \text{Int}(C_0) \subseteq H_0 \) which is impossible since \( v \in H_0 \).

**Claim 2.** \( v \in (L_0 \cup L_0) \) are such that \( H_0 = H_0 \) then: there exist continua \( C_0, C_0, K, H \) such that \( v \in \text{Int}(C_0), \emptyset \subseteq \text{Int}(C_0), H \) and \( K \) are irreducible subcontinua between \( C_0 \) and \( C_0 \), \( H \cap K = \emptyset, L_0 \subseteq H, L_0 \subseteq K \)

\[M = H \cup C_0 \cup K \cup C_0.\]

By Lemma 3, there are continua \( C_0 \) and \( C_0 \), with \( v \in \text{Int}(C_0), v \in C_0 \) and \( x \) and \( y \in C_0 \). Since \( H_0 \neq H_0 \), we know that \( H_0 \cap H_0 = \emptyset \) which in turn implies \( C_0 \cap C_0 = \emptyset \). Using Lemma 1, the fact \( v \in H_0 \), and the assumption of \( M \) being O.C., we see that the claim is valid.

At this point it is convenient to introduce the following notation:

\[H_0 \supseteq H_0 - (L_0 \cup L_0) \]

**Claim 3.** \( v \in M - (L_0 \cup L_0) \), then \( H_0 \) is an irreducible continuum between \( L_0 \) and \( L_0 \).

From Claim 1 we know that \( \emptyset \subseteq L_0 \subseteq L_0 \). Assume for the moment that \( \emptyset \subseteq L_0 \subseteq L_0 \). Then by (e) we know there exists \( v \in M - (L_0 \cup L_0) \). Now an application of Claim 2 and a recollection of the definition of \( H_0 \), we find a contradiction. Therefore, \( H_0 \) is a continuum between \( L_0 \) and \( L_0 \). By Result 2 we know there exists a subcontinuum \( H_0 \) of \( H_0 \) which is irreducible between \( L_0 \) and \( L_0 \). Now if we assume \( H_0 \neq H_0 \) and combine this with Claim 2 and the definition of O.C. we lead to a contradiction. We thus see that the claim holds.

Theorem 1 says that there exist two points \( v_0 \) and \( v_0 \in M - (L_0 \cup L_0) \) which are cut by \( x \) and \( y \). Therefore by Claim 3, \( H_0 \supseteq H_0 \) and \( H_0 \supseteq H_0 \) will serve as the \( H_0 \) and \( K \) in the definition of Condition 2 if we can show that

\[M = L_0 \cup \text{Int}(L_0) \cup H_0 \cup H_0.\]
But if there were \( r \in M - (L_0 \cup H_0 \cup L_0' \cup H_0') \), then we could apply Claim 2, Claim 3 in reference to \( H_0 \), and the fact that \( v_0 \) and \( v_0' \in H_0 \) to arrive at a contradiction to \( M \) being C.C. This then completes the sufficiency part of this theorem.

To establish the necessity half of this theorem, we will assume that \( M \) is not C.C. but it does satisfy Condition 2 and deduce that it is of the form expressed in 2.

Lemma 6 shows that Condition 2 implies (2) of Theorem 1, and so assuming \( M \) is not C.C. means that there exists \( x \) and \( y \in M \) such that \( L_0 \cap L_0' = \emptyset \) and yet \( x \circ (y) \) cut no two points of \( M - (L_0 \cup L_0') \).

Since \( L_0 \cap L_0' = \emptyset \) we know that there exist two continua \( R \) and \( K \) which are irreducible between \( L_0 \) and \( L_0' \) and \( M = L_0 \cup H \cup L_0' \cup K \).

Letting \( t \in \text{Int}(H) \) and \( x \in \text{Int}(K) \) we see by the above comments that there exist a continuum \( C \) such that \( t, x \in C \) and \( x, y \notin C \). So by Lemma 8, \( L_0 = H \) for each \( r \in \text{Int}(H) \), therefore using the fact that \( \text{Int}(H) = H \) (Result 3), we see that \( H \) is indecomposable. Analogous comments hold for \( K \). Also we know from Lemma 8 that \( H \cap K \neq \emptyset \).

Now \( L_0 = H \) for each \( r \in \text{Int}(H) \) implies by Lemma 5 that \( x \in \text{Int}(L_0) \) and \( y \in \text{Int}(L_0) \). It is not difficult to see that \( \text{Int}(L_0) = L_0 \) and \( \text{Int}(L_0) = L_0' \).

Let \( H' \) and \( K' \) be the continua associated with \( L_0 \) and \( L_0' \) mentioned in the statement of Condition 2.

Clearly we can take \( K \subseteq K' \) and \( H \subseteq H' \). Letting \( t \in \text{Int}(K) \) and \( x \in \text{Int}(H) \) we know that \( L_0 \cap L_0' \neq \emptyset \), by Lemma 5 \( r \in \text{Int}(L_0) \), and so by Lemma 8

\[
H' = L_0 = H \quad \text{and} \quad K' = L_0' = K
\]

But \( M = H \cup L_0 \cup K \cup L_0' \) so \( \text{Int}(L) \subseteq L_0 \) which implies \( L_0 \subseteq L_0' \).

This then completes the proof of the theorem.

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References: