Inverse limit spaces defined by only finitely many distinct bonding maps

by

R. F. JOLLY and J. T. ROGERS, Jr. (Riverside, Cal.)

1. Introduction. Prior to the advent of the paper by Anderson and Choquet [1], most topologists regarded inverse limit spaces as a sort of curiosity, void of real interest or applications, say, to continua problems. Even after its appearance, inverse limit spaces were largely ignored, since the descriptions of the bonding maps were regarded as being synonymous to the definitions of a chain construction. Moreover, these descriptions involved infinitely many distinct functions and consequently were at least as complicated to use as chains. This attitude was greatly shaken by the Henderson paper [4] in which he described the pseudo-arc as an inverse limit space on arcs where only one bonding map is used. In 1967, Mahavier [5] gave an example of a chainable continuum which is not homeomorphic to any inverse limit space on arcs with only one bonding map; in addition, by making use of a result by Schori [7], he showed that there is a universal chainable continuum, defined as an inverse limit space on arcs with only one bonding map, in which every chainable continuum can be embedded.

In view of these results, a natural question would be: Can chainable continua be subdivided into infinitely many distinct types according to the number of distinct bonding maps required for their description? Another question would be: If n is a positive integer, does there exist a chainable continuum which is not homeomorphic to any inverse limit space on arcs with only n distinct bonding maps? If the answer to the latter question were yes, then the property of being the inverse limit of n maps, but not of n−1 maps, would be a meaningfully descriptive property of chainable continua. However in this paper, we answer these question in the negative. In fact we show the following theorem:

There exist four maps such that every chainable continuum is homeomorphic to some inverse limit space on arcs which uses only these four maps.

By a continuum we mean a nondegenerate, compact, connected subset of a metric space. A map is a continuous single-valued function. It is well known that a chainable continuum may be regarded as an inverse
limit space on arcs \([3], [6]\). In the case where \((X_n)\) is a sequence of topological spaces and \((f_n)\) is a sequence of functions such that for each positive integer \(n, f_n : X_{n+1} \to X_n\), then the inverse limit space (which may be empty) defined by this system is the subspace \(X\) of the cartesian product \(X_1 \times X_2 \times X_3 \times \ldots\) (with the topology derived as the relative topology from the product topology) where the elements of this subspace are the sequences \(x_1, x_2, x_3, \ldots\) such that for each positive integer \(n, x_n \in X_n\) and \(f_n(x_{n+1}) = x_n\). The space \(X\) is referred to as the \(n\)th coordinate space and the functions \(f_1, f_2, f_3, \ldots\) are called the bonding functions. When \(f_n\) is continuous, it is said to be a bonding map. We will use the notation

\[X = \lim(I \leftarrow I \leftarrow I \leftarrow I \leftarrow I \leftarrow \ldots)\]

to express the fact that \(X\) is the inverse limit space determined by this system of spaces and bonding functions. We will also denote the unit number interval \([0, 1]\) by \(I\). In view of an earlier remark, it is clear that all chainable continua may be regarded as inverse limit spaces on \(I\).

2. Inverse limits using only bonding maps. We now want to prove the theorem mentioned in the introduction, but first some definitions will be in order. Suppose that \(0 \leq x_1 < x_2 \leq 1\) and \(g\) is a map from \(I\) into \(I\). By a copy of \(g\) in \([x_1, a_1]\times[y_1, y_2]\), we mean the map \(h\) from \([x_1, a_1]\) into \([y_1, y_2]\) defined by

\[h(x) = y_1 + (y_2 - y_1)f([x-x_1]/(a_0-a_1)].\]

Let \((f_n)\) denote a countable dense (in the topology of uniform convergence) subset of the collection of all maps of \(I\) into \(I\). For each positive integer \(n\), let \(a_n = 2^{-2n}\), \(b_n = 2^{-2n} + 1\) and \(h_n\) denote the copy of \(f_n\) in \([a_n, b_n]\times[0, a_n]\). Let \(j\) denote the map from \(I\) onto \(I\) defined by

(i) \(j(0) = 0\) and \(j(1) = 1,\)
(ii) \(j(x) = b_n(x)\) and \(h_n\) is linear on each interval \([a_n, a_{n+1}].\)

Further define the maps \(p, q, h, p, q, k, h, p, q\) of \(I\) into \(I\) by

\[p(x) = \text{minimum}(4x, 1), \quad q(x) = x/4, \quad k(x) = (x+1)/4.\]

Theorem. Every chainable continuum is homeomorphic to some inverse limit space on \(I\) which uses only the maps \(j, h, p, q\).

Proof. Suppose

\[X = \lim(I \leftarrow I \leftarrow I \leftarrow I \leftarrow \ldots)\].

Now since the functions of \((f_n)\) are dense, it follows that for each positive integer \(m, g_m\) can be approximated arbitrarily closely by some \(f_n\). Acc-
On continua which resemble simple closed curves

by

H. H. Stratton (Albany, N. Y.)

1. Introduction. Consider the following well-known theorems concerning simple closed curves: (for the definition of simple closed curves and arcs, see Moore [3], pp. 44 and 33 respectively)

Theorem 1. \( M \) is a simple closed curve if and only if for \( x, y \in M \), 
\[ M = H \cup K \cup \{x\} \cup \{y\} \] 
where \( H \) and \( K \) are irreducible continua between \( x \) and \( y \) and \( \text{Int}(H) \cap \text{Int}(K) = \emptyset \). (In fact, \( H \) and \( K \) are arcs.)

(\( \text{It stands for "the interior of\( \)} \)

Theorem 2. \( M \) is a simple closed curve if and only if no single point of \( M \) cuts, but every collection of two points of \( M \) do cut. (Bing [1], also refer to this reference for the definition of cut).

Theorem 3. Let \( M \) be a simple closed curve. If \( C \) is a subcontinuum of \( M \) and \( \text{Int}(C) \neq \emptyset \), then \( M - C \) is a continuum. (For the purposes of this paper, we shall say that any continuum which satisfies the condition expressed in the conclusion of Theorem 3 is C.C.)

The first two of these theorems both express equivalent conditions for \( M \) to be a simple closed curve, whereas the third is a rather weak necessary condition for \( M \) to be a simple closed curve. The question that is posed in this paper is how "close" to a simple closed curve, i.e., how "ring-like", is a compact metric continuum \( M \) which is C.C. It will be shown that if one essentially replaces the point \( x \) by the set

\[ L_x = \{y \in M : \text{there is no continuum } C \subset M \text{ with } y \in \text{Int}(C) \text{ and } x \notin C\} \]

in Theorem 2, then we have a characterization of C.C., and if we replace \( x \) by \( L_x \) in Theorem 1, then either \( M \) is C.C. or \( M \) belongs to a very special class of continua which "look" very "ringlike" indeed.

This idea of replacing a point \( x \) by \( L_x \) was used by Thomas [4], where he showed that if \( M \) is irreducible between two closed sets, and contains no indecomposable subcontinuum with interior, then a certain collection of these \( L_x \)'s forms an upper semi continuous decomposition of \( M \) that is an arc.