Inverse limit spaces defined by only finitely many distinct bonding maps

by

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1. Introduction. Prior to the advent of the paper by Anderson and Choquet [1], most topologists regarded inverse limit spaces as a sort of curiosity, void of real interest or applications, say, to continua problems. Even after its appearance, inverse limit spaces were largely ignored, since the descriptions of the bonding maps were regarded as being synonymous to the definitions of a chain construction. Moreover, these descriptions involved infinitely many distinct functions and consequently were at least as complicated to use as chains. This attitude was greatly shaken by the Henderson paper [4] in which he described the pseudo-arc as an inverse limit space on arcs where only one bonding map is used. In 1967, Mahavier [5] gave an example of a chainable continuum which is not homeomorphic to any inverse limit space on arcs with only one bonding map; in addition, by making use of a result by Schori [7], he showed that there is a universal chainable continuum, defined as an inverse limit space on arcs with only one bonding map, in which every chainable continuum can be embedded.

In view of these results, a natural question would be: Can chainable continua be subdivided into infinitely many distinct types according to the number of distinct bonding maps required for their description? Another question would be: If n is a positive integer, does there exist a chainable continuum which is not homeomorphic to any inverse limit space on arcs with only n distinct bonding maps? If the answer to this latter question were yes, then the property of being the inverse limit of n maps, but not of $n-1$ maps, would be a meaningfully descriptive property of chainable continua. However in this paper, we answer these question in the negative. In fact we show the following theorem:

There exist four maps such that every chainable continuum is homeomorphic to some inverse limit space on arcs which uses only these four maps.

By a *continuum* we mean a nondegenerate, compact, connected subset of a metric space. A *map* is a continuous single-valued function. It is well known that a *chainable continuum* may be regarded as an inverse limit space on arcs [3], [6]. In the case where $\{X_n\}$ is a sequence of topological spaces and $\{f_n\}$ is a sequence of functions such that for each positive integer $n, f_n\colon X_{n+1}\to X_n$, then the *inverse limit space* (which may be empty) defined by this system is the subspace X of the cartesian product $X_1\times X_2\times X_3\times\ldots$ (with the topology derived as the relative topology from the product topology) where the elements of this subspace are the sequences $x_1, x_2, x_3, \ldots$ such that for each positive integer n, $x_n\in X_n$ and $f_n(x_{n+1}) = x_n$. The space X_n is referred to as the nth *coordinate* space and the functions $f_1, f_2, f_3, \ldots$ are called the *bonding functions*. When f_n is continuous, it is said to be a *bonding map*. We will use the notation

$$X = \lim(X_1 \xleftarrow{f_1} X_2 \xleftarrow{f_2} X_3 \xleftarrow{f_3} \ldots)$$

to express the fact that X is the inverse limit space determined by this system of spaces and bonding functions. We will also denote the unit number interval $[0,1]$ by I. In view of an earlier remark, it is clear that all chainable continua may be regarded as inverse limit spaces on I.

2. Inverse limits using only four bonding maps. We now want to prove the theorem mentioned in the introduction, but first some definitions will be in order. Suppose that $0\leqslant x_1 < x_2\leqslant 1$, $0\leqslant y_1 < y_2\leqslant 1$ and g is a map from I into I. By a *copy* of g in $[x_1, x_2]\times[y_1, y_2]$, we mean the map h from $[x_1, x_2]$ into $[y_1, y_2]$ defined by

$$h(x) = y_1+(y_2-y_1)f[(x-x_1)/(x_2-x_1)].$$

Let $\{f_n\}$ denote a countable dense (in the topology of uniform convergence) subset of the collection of all maps of I into I. For each positive integer n, let $a_n = 2^{-2n}$, $b_n = 2^{1-2n}$ and h_n denote the copy of f_n in $[a_n, b_n]\times[0, a_n]$. Let j denote the map from I onto I defined by

(i) $j(0) = 0$ and $j(1) = 1$,

(ii) if $x\in[a_n, b_n]$, then $j(x) = h_n(x)$ and

(iii) j is linear on each interval $[b_n, a_{n-1}]$.

Further define the maps p, q and k of I into I by

$$p(x) = \text{minimum}\{4x, 1\},\qquad q(x) = x/4\qquad\text{and}\qquad k(x) = (x+1)/4.$$

THEOREM. *Every chainable continuum is homeomorphic to some inverse limit space on I which uses only the maps j, k, p and q.*

Proof. Suppose

$$X = \lim(I \xleftarrow{g_1} I \xleftarrow{g_2} I \xleftarrow{g_3} \ldots).$$

Now since the functions of $\{f_n\}$ are dense, it follows that for each positive integer m, g_m can be approximated arbitrarily closely by some f_n. According to Brown [2], there exists a sequence $n(1), n(2), n(3), \ldots$ of positive integers such that X is homeomorphic to the inverse limit space

$$Y = \lim(I \xleftarrow{f_{n(1)}} I \xleftarrow{f_{n(2)}} I \xleftarrow{f_{n(3)}} \ldots).$$

However a routine calculation reveals that $f_n = p^n j q^{n-1} k$. Since each f_n is a composition of the mappings j, k, p and q, X is homeomorphic to an inverse limit space

$$Y = \lim(I \xleftarrow{\mu_1} I \xleftarrow{\mu_2} I \xleftarrow{\mu_3} \ldots)$$

where each μ_n is one of the maps j, k, p and q. This completes the proof of the theorem.

COROLLARY 1. *Every chainable continuum is homeomorphic to some inverse limit space X defined by*

$$X = \lim(I \xleftarrow{\alpha_1 j\beta_1} I \xleftarrow{\alpha_2 j\beta_2} I \xleftarrow{\alpha_3 j\beta_3} \ldots)$$

where each α_n and β_n are monotone functions.

COROLLARY 2. *Every chainable continuum is homeomorphic to some inverse limit space X defined by*

$$X = \lim(I \xleftarrow{j\varphi_1} I \xleftarrow{j\varphi_2} I \xleftarrow{j\varphi_3} \ldots)$$

where each φ_n is a monotone function.

One of the questions which naturally arises is whether four is the minimal number of maps. We may modify the proof of the theorem to obtain a partial result in this direction.

COROLLARY 3. *Each chainable continuum may be regarded as an inverse limit on arcs using only two maps and a function with a single point of discontinuity.*

Proof. Modify the definition of j in the following way: let each h_n be a copy of f_n in $[a_n, b_n]\times[0,1]$. In this fashion, each chainable continuum may be obtained as an inverse limit using only the functions j, k and q.

References

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Reçu par la Rédaction le 27. 7. 1968

On continua which resemble simple closed curves

by

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1. Introduction. Consider the following well-known theorems concerning simple closed curves: (for the definition of simple closed curves and arcs, see Moore [3], pp. 44 and 33 respectively)

THEOREM 1. *M is a simple closed curve if and only if for $x, y \in M$, $M = H \cup K \cup \{x\} \cup \{y\}$ where H and K are irreducible continua between x and y and* $\mathrm{Int}(H) \cap \mathrm{Int}(K) = \emptyset$. (In fact, H and K are arcs.)

(It stands for "the interior of")

THEOREM 2. *M is a simple closed curve if and only if no single point of M cuts, but every collection of two points of M do cut.* (Bing [1], also refer to this reference for the definition of cut).

THEOREM 3. *Let M be a simple closed curve. If C is a subcontinuum of M and $\mathrm{Int}(C) \neq \emptyset$, then $\overline{M - C}$ is a continuum.* (For the purposes of this paper, we shall say that any continuum which satisfies the condition expressed in the conclusion of Theorem 3 is C.C.)

The first two of these theorems both express equivalent conditions for M to be a simple closed curve, whereas the third is a rather weak necessary condition for M to be a simple closed curve. The question that is poised in this paper is how "close" to a simple closed curve, i.e., how "ring-like", is a compact metric continuum M which is C.C. It will be shown that if one essentially replaces the point x by the set

$$L_x \doteq \{y \in M\colon \text{there is no continuum } C \subset M \text{ with } y \in \mathrm{Int}(C) \text{ and } x \notin C\}$$

in Theorem 2, then we have a characterization of C.C., and if we replace x by L_x in Theorem 1, then either M is C.C. or M belongs to a very special class of continua which "look" very "ringlike" indeed.

This idea of replacing a point x by L_x was used by Thomas [4], where he showed that if M is irreducible between two closed sets, and contains no indecomposable subcontinuum with interior, then a certain collection of these L_x's forms an upper semi continuous decomposition of M that is an arc.