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Concerning the ordering of shapes of compacta

by

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The purpose of this note is to give answers to one question concerning the existence of maximal shapes (majorants) and to another question, concerning the existence of continuously ordered families of shapes.

§ 1. Basic definitions. Let X, Y be two compacta lying in the Hilbert cube Q. A sequence of maps $f_k\colon Q \to Q$ is said to be a *fundamental sequence from X to Y* (notation: $\{f_k, X, Y\}$, or $\underline{f}\colon X \to Y$. Compare [2], p. 225) if for every neighborhood V of Y there is a neighborhood U of X such that

$$f_k/U \simeq f_{k+1}/U \quad \text{in } V \text{ for almost all } k\,.$$

In particular, if f_k is the identity map of Q onto itself for every $k = 1, 2, \ldots$, then $\{f_k, X, X\}$ is said to be the *fundamental identity sequence* $\underline{i}_X$. Two fundamental sequences $\underline{f} = \{f_k, X, Y\}$ and $\underline{g} = \{g_k, X, Y\}$ are said to be *homotopic* (notation: $\underline{f} \simeq \underline{g}$) if for every neighborhood V of Y there is a neighborhood U of X such that

$$f_k/U \simeq g_k/U \quad \text{in } V \text{ for almost all } k\,.$$

The family of all fundamental sequences homotopic to a given fundamental sequence $\underline{f}\colon X \to Y$ is said to be the *fundamental class* $[\underline{f}]$ *from X to Y*.

If X, Y, Z are compacta lying in Q and if $\underline{f} = \{f_k, X, Y\}$, $\underline{g} = \{g_k, Y, Z\}$ are fundamental sequences, then $\{g_k f_k, X, Z\}$ is a fundamental sequence, called the *composition* of $\underline{f}$ and $\underline{g}$; it is denoted by $\underline{g}\underline{f}$.

Two compacta X, Y (not necessarily lying in Q) are said to be of the same *shape* (notation: $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$ (see [4]) if there are two compacta $X', Y' \subset Q$, homeomorphic to X and Y respectively, and two fundamental sequences $\underline{f}\colon X' \to Y'$, $\underline{g}\colon Y' \to X'$ such that $\underline{g}\underline{f} \simeq \underline{i}_{X'}$ and $\underline{f}\underline{g} \simeq \underline{i}_{Y'}$. If we assume only that $\underline{f}$ and $\underline{g}$ satisfy the first of those homotopies, then $\mathrm{Sh}(X)$ is said to be *not greater* than $\mathrm{Sh}(Y)$ (notation: $\mathrm{Sh}(X) \leqslant \mathrm{Sh}(Y)$). If $\mathrm{Sh}(X) \leqslant \mathrm{Sh}(Y)$, but the relation $\mathrm{Sh}(Y) \leqslant \mathrm{Sh}(X)$ does not hold, then we say that $\mathrm{Sh}(X)$ is *less* than $\mathrm{Sh}(Y)$ and we write $\mathrm{Sh}(X) < \mathrm{Sh}(Y)$.

It is known that the relation "$<$" is a partial order in the collection of all shapes of compacta. We do not claim that $\mathrm{Sh}(X) \leqslant \mathrm{Sh}(Y)$ and $\mathrm{Sh}(Y) \leqslant \mathrm{Sh}(X)$ imply $\mathrm{Sh}(X) = \mathrm{Sh}(Y)$.

§ 2. Formulation of the results. If Γ is a family of shapes (of non-empty compacta), then $\mathrm{Sh}(Y)$ is said to be a *majorant* for Γ if $\mathrm{Sh}(X) \leqslant \mathrm{Sh}(Y)$ for every $\mathrm{Sh}(X) \in \Gamma$. Let us observe that

(2.1) *For every at most countable family Γ of shapes there exists a majorant.*

In order to show it, let us order the shapes belonging to Γ into a sequence $\mathrm{Sh}(X_1), \mathrm{Sh}(X_2), \ldots$ It is clear that there is in Q a sequence of disjoint compacta $X_1', X_2', \ldots$ such that the diameter of X_n' is less than $1/n$ and that X_n' is homeomorphic to X_n for every $n = 1, 2, \ldots$, and that the sets X_n' converge to a point $a \in X_1'$. Evidently

$$X = \bigcup_{n=1}^{\infty} X_n'$$

is a compactum for which every set X_n' is a retract. Since it is known (compare [4]) that

(2.2) *If Y_0 is a retract of Y, then* $\mathrm{Sh}(Y_0) \leqslant \mathrm{Sh}(Y)$,

we infer that $\mathrm{Sh}(X_n) = \mathrm{Sh}(X_n') \leqslant \mathrm{Sh}(X)$ for every $n = 1, 2, \ldots$ Hence $\mathrm{Sh}(X)$ is a majorant for the family Γ.

The question arises whether (2.1) holds also without the hypothesis that the family Γ is countable. The negative answer to this question is given by the following

(2.3) THEOREM. *For the family of all (generalized) solenoids a majorant does not exist.*

Let us observe that (2.3) implies the following

(2.4) COROLLARY. *For every compactum $X \neq \emptyset$ there is a compactum Y such that* $\mathrm{Sh}(X) < \mathrm{Sh}(Y)$.

In fact, by (2.3) there exists a solenoid S such that the relation $\mathrm{Sh}(S) \leqslant \mathrm{Sh}(X)$ does not hold. We may assume that X and S are subsets of Q and that $X \cap S$ consists of only one point. Setting $Y = X \cup S$, let us observe that X and S are retracts of Y, whence $\mathrm{Sh}(X) \leqslant \mathrm{Sh}(Y)$ and $\mathrm{Sh}(S) \leqslant \mathrm{Sh}(Y)$. Hence the relation $\mathrm{Sh}(Y) \leqslant \mathrm{Sh}(X)$ does not hold and consequently $\mathrm{Sh}(X) < \mathrm{Sh}(Y)$. Let us add that in the case where X is a continuum our construction gives a continuum Y with $\mathrm{Sh}(X) < \mathrm{Sh}(Y)$.

The proof of Theorem (2.3), being the first result of this note, is given in § 5.

In order to formulate the second result, let us mention that in the shape-theory an important role is played by the notion of movability.

A compactum X is said to be *movable* if there exists a compactum $X' \subset Q$ homeomorphic to X with the property that for every neighborhood U of X' (in Q) there is a neighborhood U_0 of X' such that U_0 can be homotopically deformed in U onto a subset of every neighborhood of X. It is known ([3], pp. 137, 138 and 145) that all compact ANR-sets and also all plane compacta are movable, but every solenoid is non-movable.

The movability of a compactum is a property depending only on the shape of this compactum. Thus we can speak about *movable and non-movable shapes*. Moreover, it is known ([3], p. 140) that if $\mathrm{Sh}(X) \leqslant \mathrm{Sh}(Y)$ and if $\mathrm{Sh}(Y)$ is movable, then $\mathrm{Sh}(X)$ is also movable.

The second result of this note, which is proved in § 6, is the following theorem, which casts a light on the richness of the class of movable shapes:

(2.5) THEOREM. *There exists a function assigning to every real number t a locally connected, movable continuum C_t, so that $t < t'$ implies* $\mathrm{Sh}(C_t) < \mathrm{Sh}(C_{t'})$.

§ 3. A lemma concerning homology groups. Let X be a compactum and $\mathfrak{A}$ an abelian group. By $H_n(X, \mathfrak{A})$ we denote the nth homology group of X (in the sense of Vietoris) over the group of coefficients $\mathfrak{A}$. It is known ([2], p. 240) that every fundamental sequence $\underline{f} = \{f_k, X, Y\}$ induces a homomorphism

$$\underline{f}_*\colon H_n(X, \mathfrak{A}) \to H_n(Y, \mathfrak{A}),$$

defined as follows: If $\underline{\gamma} = \{\gamma_k\}$ is a true cycle in X which is a representative of an element $(\underline{\gamma})$ of $H_n(X, \mathfrak{A})$, then there exists an increasing sequence $\{i_k\}$ of indices such that for every sequence of indices $\{j_k\}$ satisfying the inequality $j_k \geqslant i_k$ for $k = 1, 2, \ldots$, the sequence $\{f_k(\gamma_{j_k})\}$ is a true cycle $\underline{\gamma}'$ in Y over $\mathfrak{A}$.

The homomorphism $\underline{f}_*$ assigns $(\underline{\gamma}')$ to $(\underline{\gamma})$.

Now let us assume that $X \subset Q$ and that $\mathfrak{A}$ is a compact topological group. In this case, the group $H_n(X, \mathfrak{A})$ may be considered as a topological (compact) group ([8], p. 000) if we define the base of neighborhoods about the zero point 0 of $H_n(X, \mathfrak{A})$ as the family of all sets

$$G_X(U, W) \subset H_n(X, \mathfrak{A}),$$

where U is a neighborhood of X (in Q) and W is a neighborhood (in $\mathfrak{A}$) of the element $0 \in \mathfrak{A}$ and $G_X(U, W)$ consists of all elements $(\underline{\gamma})$ of the group $H_n(X, \mathfrak{A})$ for which there exists a representative $\underline{\gamma} = \{\gamma_k\}$ lying in U (i.e. such that there is a sequence $\{\varkappa_k\}$ of chains satisfying the condition $\partial\varkappa_k = \gamma_k - \gamma_{k+1}$ for $k = 1, 2, \ldots$ and with geometric realizations of all simplexes of $\varkappa_k$ and of γ_k contained in U for every $k = 1, 2, \ldots$) and such that

$$\gamma_1 = a_1\sigma_1 + a_2\sigma_2 + \ldots + a_m\sigma_m ,$$

where every linear combination $n_1a_1 + n_2a_2 + \ldots + n_ma_m$ with coefficients n_i equal to 0 or to 1 or to -1 (for $i = 1, 2, \ldots, m$) belongs to W.

It is known [(5)] that

(3.1) *If $\mathfrak{A}$ is a compact, topological, abelian group, then for every fundamental sequence $\underline{f}$: $X \to Y$ the induced homomorphisms*

$$\underline{f}_*: H_n(X, \mathfrak{A}) \to H_n(Y, \mathfrak{A})$$

are continuous.

§ 4. Some properties of solenoids. It is well known (see [7], p. 261) that the Vietoris homology theory with coefficients belonging to a compact abelian group is continuous, that is, if a compactum X is the inverse limit of an inverse sequence

$$X_1 \underset{a_1}{\leftarrow} X_2 \underset{a_2}{\leftarrow} X_3 \underset{a_3}{\leftarrow} \ldots$$

of compacta X_i, then $H_n(X, \mathfrak{A})$ is the inverse limit of the inverse sequence of groups

$$H_n(X_1, \mathfrak{A}) \underset{a_{1*}}{\leftarrow} H_n(X_2, \mathfrak{A}) \underset{a_{2*}}{\leftarrow} \ldots ,$$

where a_{k*} denotes the homomorphism of $H_n(X_{k+1}, \mathfrak{A})$ into $H_n(X_k, \mathfrak{A})$ induced by the map a_k.

Since every compactum is the inverse limit of a sequence of polyhedra and since the inverse limit of a sequence of compact metrizable spaces is a compact metrizable space, we infer that

(4.1) *If X is a compactum and $\mathfrak{A}$ is a compact metrisable abelian group, then $H_n(X, \mathfrak{A})$ is a compact metrisable group.*

By a (generalized) *solenoid* we understand every space $S = S(m_1, m_2, \ldots)$ which is the inverse limit of the inverse sequence

$$K \underset{\chi_{m_1}}{\longleftarrow} K \underset{\chi_{m_2}}{\longleftarrow} K \underset{\chi_{m_3}}{\longleftarrow} \ldots ,$$

where K denotes the circle defined as the multiplicative group of complex numbers z with $|z| = 1$, the indices $m_1, m_2, \ldots$ are natural numbers $\geqslant 2$ and χ_{m_i}: $K \to K$ is given by the formula

$$\chi_{m_i}(z) = z^{m_i} \quad \text{for every } z \in K .$$

It follows that K is a compact topological group. Moreover, it is clear that K is isomorphic with the group $H_1(K, K)$. We can identify K with $H_1(K, K)$ and thus we may consider the homomorphism

$$\chi_{m_i*}: H_1(K, K) \to H_1(K, K)$$

as identic with the homomorphism χ_{m_i}. It follows, by the continuity of the homology theory, that

(4.2) $$H_1(S(m_1, m_2, \ldots), K) = S(m_1, m_2, \ldots) .$$

Let $(k_1, k_2, \ldots)$ and $(l_1, l_2, \ldots)$ be two sequences of natural numbers $\geqslant 2$. If there exists an index i_0 such that for every $i \geqslant i_0$ there exists a natural number m such that $k_{i_0} \cdot k_{i_0+1} \cdot \ldots \cdot k_i$ is a factor of $l_1 \cdot l_2 \cdot \ldots \cdot l_m$, then the sequence $(k_1, k_2, \ldots)$ is said to be a *factorant* of the sequence $(l_1, l_2, \ldots)$. By a theorem of H. Cook ([6], p. 236 and p. 238),

(4.3) *$S(k_1, k_2, \ldots)$ is a continuous image of $S(l_1, l_2, \ldots)$ if and only if $(k_1, k_2, \ldots)$ is a factorant of $(l_1, l_2, \ldots)$.*

Since every proper subcontinuum of a solenoid is an arc or a point, we infer by (4.3) that

(4.4) *If $(k_1, k_2, \ldots)$ is not a factorant of $(l_1, l_2, \ldots)$, then every homomorphism of $S(l_1, l_2, \ldots)$ into $S(k_1, k_2, \ldots)$ is trivial.*

Now let us order all prime numbers into an increasing sequence $p_1, p_2, \ldots$ and all rational numbers into a sequence $w_1, w_2, \ldots$ with $w_i \neq w_j$ for $i \neq j$. For every positive number t, let us denote by M_t the set of all numbers p_n with indices n satisfying the condition $-1/t < w_n < t$. It is clear that $0 < t < t'$ implies that the sets $M_t - M_{t'}$ and $M_{t'} - M_t$ are both countable.

Let $(p_{t,1}, p_{t,2}, \ldots)$ denote the increasing sequence consisting of all numbers belonging to M_t. Then the inequality $t < t'$ implies that each of the sequences $(p_{t,1}, p_{t,2}, \ldots)$ and $(p_{t',1}, p_{t',2}, \ldots)$ contains an infinite number of elements which do not belong to the other. Setting

$$S_t = S(p_{t,1}, p_{t,2}, \ldots) \quad \text{for every } t > 0 ,$$

we get a family consisting of $2^{\aleph_0}$ solenoids with the property that all homomorphisms of one of them into another are trivial.

§ 5. Proof of Theorem (2.3). Keeping the notation of § 4, let us suppose that there exists a compactum X such that

$$\mathrm{Sh}(S_t) \leqslant \mathrm{Sh}(X) \quad \text{for every positive } t .$$

This means that for every positive t there exist two fundamental sequences

$$\underline{f_t}: X \to S_t \quad \text{and } \underline{g_t}: S_t \to X$$

such that the induced homomorphisms

$$f_{t*}\colon H_1(X, K)\to H_1(S_t, K)\,, \qquad g_{t*}\colon H_1(S_t, K)\to H_1(X, K)$$

satisfy the condition

$$f_{t*}g_{t*} = 1\,.$$

Then the group $\mathfrak{A}_t = g_{t*}[H_1(S_t, K)]$ is a subgroup of the group $H_1(X, K)$ isomorphic to $H_1(S_t, K)$, whence non-trivial.

Let us select in each of the groups $\mathfrak{A}_t$ an element $a_t \neq 0$. Since the group $H_1(X, K)$ is (by (4.1) compact and metrizable, we infer that there exists a sequence $t_0, t_1, t_2, \ldots$ of positive numbers such that $t_n \neq t_0$ for every $n = 1, 2, \ldots$ and that $\lim_{n=\infty} a_{t_n} = a_{t_0}$. Let $a_t = g_{t*}(b_t)$, where $b_t \in H_1(S_t, K)$.

Then $b_t \neq 0$ and $f_{t_0*}(a_{t_n}) = f_{t_0*}g_{t_n*}(b_{t_n}) = 0$, because the homomorphism

$$f_{t_0*}g_{t_n*}\colon H_1(S_{t_n}, K)\to H_1(S_{t_0}, K)$$

is trivial, as follows by (4.4). Since, by (3.1) f_{t_0*} is continuous, we infer that

$$f_{t_0*}(a_{t_0}) = \lim_{n=\infty} f_{t_0*}g_{t_n*}(a_{t_n}) = 0\,,$$

which contradicts the relation

$$f_{t_0*}(a_{t_0}) = f_{t_0*}g_{t_0*}(b_{t_0}) = b_{t_0} \neq 0\,.$$

Thus the supposition that the family of all solenoids S_t has a majorant is contradictory, whence the proof of Theorem (2.3) is finished.

§ 6. Proof of Theorem (2.5). Consider in the Euclidean 5-space E^5 a sequence of 5-dimensional simplices $\Delta_1, \Delta_2, \ldots$ having one common vertex a and, apart from it, disjoint. Let us assume that $\lim_{n=\infty} \delta(\Delta_n) = 0$. Consider a disk D in the plane of complex numbers z given by the inequality $|z| \leqslant 1$. Given a natural number n, let us identify all points of the form $e^{2\pi i(\vartheta + k/n)}$ for $k = 0, 1, \ldots, (n-1)$. By this identification the disk D passes onto a 2-dimensional polyhedron A_n (called a *pseudo-projective plan* [1] p. 266) with the first Betti group $H_1(A_n)$ isomorphic to the additive group $\mathfrak{N}_n$ of integers modulo n.

It is clear that for every $n = 1, 2, \ldots$ there is a set $B_n \subset \Delta_n$ homeomorphic with A_n and such that $a \in B_n$.

As in § 5, let us consider an increasing sequence $p_1, p_2, p_3, \ldots$ of prime numbers and a sequence $w_1, w_2, \ldots$ consisting of all rational numbers such that $w_i \neq w_j$ for $i \neq j$. Let us assign to every real number t the set N_t consisting of all prime numbers p_n with indices n satisfying the condition $w_n < t$. It is clear that $t < t'$ implies that N_t is a proper subset of $N_{t'}$.

Setting

$$C_t = \bigcup_{p \in N_t} B_p \quad \text{for every real number } t\,,$$

$$C_\infty = \bigcup_{n=1}^{\infty} B_n\,,$$

let us observe that C_t and C_∞ are locally connected continua.

Now let us assign to every real number t the function $r_t\colon C_\infty \to C_t$ given by the formulas:

$$r_t(x) = \begin{cases} x & \text{if} \quad x \in C_t\,, \\ a & \text{if} \quad x \in C_\infty - C_t\,. \end{cases}$$

It is clear that r_t is a retraction, and if $t < t'$, then the restriction $r_t^{t'} = r_t/C_{t'}$ is a retraction of $C_{t'}$ to C_t. We infer by (2.2) that

(6.1) *If* $t \leqslant t'$ *then* $\mathrm{Sh}(C_t) \leqslant \mathrm{Sh}(C_{t'})$.

Moreover, if $p \in N_t$, then setting

$$s_p(x) = \begin{cases} x & \text{for every point } x \in B_p\,, \\ a & \text{for every point } x \in C_t - B_p\,, \end{cases}$$

we get a retraction $s_p\colon C_t \to B_p$. But B_p contains a 1-dimensional true cycle γ with integers as coefficients, such that $\gamma \not\sim 0$ in B_p and $p\cdot\gamma \sim 0$ in B_p. Since B_p is a retract of C_t, we infer that

(6.2) $\gamma \not\sim 0$ in C_t and $p\cdot\gamma \sim 0$ in C_t.

If, however, p' is a prime number which does not belong to N_t and γ' is a 1-dimensional true cycle in C_t with integers as coefficients, such that $p'\cdot\gamma' \sim 0$ in C_t, then for every $p \in N_t$ the true cycle $s_p(\gamma')$ (that is, the part of γ' lying in B_p) is a true cycle in B_p satisfying the condition $p'\cdot s_p(\gamma') \sim 0$. Since $H_1(B_p)$ is isomorphic with $\mathfrak{N}_p$, we infer that $s_p(\gamma') \sim 0$. It follows that, without changing the homology class of γ', we can cancel in it the part lying in B_p for every $p \in N_t$. Since the diameters of the sets B_n converge to 0, we infer that the true cycle γ' is homologous in C_t to a true cycle lying in an arbitrarily small neighborhood of the point a, whence $\gamma' \sim 0$ in C_t.

Thus we have shown that if p' is a prime number which does not belong to N_t, then $H_1(C_t)$ does not contain any element of order p'. If we recall that for $t < t'$ the set $N_{t'} - N_t$ is not empty, we infer by (6.2) that the group $H_1(C_t)$ does not contain any subgroup isomorphic to $H_1(C_{t'})$ and consequently $H_1(C_{t'})$ is not an r-image of the group $H_1(C_t)$. Hence it is not true that $\mathrm{Sh}(C_{t'}) \leqslant \mathrm{Sh}(C_t)$ and we get by (6.1)

(6.3) *If* $t < t'$ *then* $\mathrm{Sh}(C_t) < \mathrm{Sh}(C_{t'})$.

In order to finish the proof of Theorem (2.5), it remains to show that all sets C_t are movable. Since C_t is a retract of C_∞, it suffices to prove that C_∞ is movable.

First let us prove the following proposition:

(6.4) *If $X \subset Q$ is a compactum such that for every $\varepsilon > 0$ there is a movable compactum $X_\varepsilon \subset X$ and a map $f_\varepsilon\colon X \to X_\varepsilon$ satisfying the inequality $\varrho(x, f_\varepsilon(x)) < \varepsilon$ for every $x \in X$, then X is movable.*

In order to prove it, consider a neighborhood U of X. Then there is a positive ε such that

(6.5) $$\text{If } \varrho(x, X) < 2\varepsilon \text{ then } x \in U.$$

Since X_ε is movable, there is a neighborhood V_ε of X_ε which can be homotopically deformed in U to a subset of every neighborhood of X_ε. Now, let us observe that for the map $f_\varepsilon\colon X \to X_\varepsilon$ there exists a map $\bar{f}_\varepsilon\colon Q \to Q$ such that $\bar{f}_\varepsilon(x) = f_\varepsilon(x)$ for every point $x \in X$. It is clear that there is a neighborhood U_0 of X such that

$$\varrho(x, X) < \varepsilon \quad \text{and} \quad \varrho(x, \bar{f}_\varepsilon(x)) < \varepsilon \quad \text{for every point } x \in U_0$$

and

$$\bar{f}_\varepsilon(U_0) \subset V_\varepsilon .$$

It follows that $\varrho(\bar{f}_\varepsilon(x), X) < 2\varepsilon$ for every point $x \in U_0$. We infer by (6.5) that setting

$$\varphi(x, u) = u \cdot \bar{f}_\varepsilon(x) + (1-u)\cdot x \quad \text{for every } (x, u) \in U_0 \times \langle 0, 1\rangle ,$$

we get a homotopy which deforms in U the set U_0 to the set $\bar{f}_\varepsilon(U_0) \subset V_\varepsilon$. By the definition of V_ε, we can deform the set $\bar{f}_\varepsilon(U_0)$ by a homotopy in U onto a subset of each neighborhood of the set X_ε, whence also onto a subset of each neighborhood of X. Thus X is movable and the proof of (6.4) is finished.

Now it suffices to observe that for every positive ε there is an index n_0 such that the diameters of all sets B_n with $n > n_0$ are less than ε. Setting

$$f_\varepsilon(x) = x \quad \text{for every } x \in \bigcup_{n=1}^{n_0} B_n ,$$

$$f_\varepsilon(x) = a \quad \text{for every } x \in \bigcup_{n>n_0} B_n ,$$

we get a map (a retraction) $f_\varepsilon\colon C_\infty \to \bigcup_{n=1}^{n_0} B_n$ satisfying the condition $\varrho(x, f_\varepsilon(x)) < \varepsilon$ for every point $x \in C_\infty$. But the set $\bigcup_{n=1}^{n_0} B_n$ is a polyhedron, whence a movable compactum. Thus we infer by (6.4) that C_∞ is movable and the proof of Theorem (2.5) is finished.

(6.6) Problem. *Does exist a majorant for the family of all movable shapes?*

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