Decompositions of a 3-cell

by

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The main result of this paper is a generalization of Theorem 3 of [2] and a result announced by R. H. Bing. It states that any 3-dimensional crumpled cube can be obtained from an upper semi-continuous decomposition of a 3-cell using arcs which are locally polyhedral except at one end-point, which is not on the boundary of the 3-cell, and the projection of the non-degenerate elements is a 0-dimensional set.

1. Terminology and notation. If A is a set in a topological space then $\mathrm{Cl}A$ will denote the closure of A. If S is a 2-sphere is R^3, then $\mathrm{Int}\,S$ and $\mathrm{Ext}\,S$ will denote the interior and exterior, respectively, of S. If A is an n-manifold-with-boundary then $\mathrm{Bd}A$ will denote the set of points in A which do not have neighborhoods homeomorphic to R^n and $\mathrm{Int}A$ will denote $A-\mathrm{Bd}A$. A set C is a crumpled cube if and only if C is homeomorphic to the union of a 2-sphere S and $\mathrm{Int}S$ in R^3.

If r is a positive number, $B(r)$ will denote the ball in R^3 with center at the origin and radius r. If S and T are 2-spheres in R^3 and $S\subset \mathrm{Int}\,T$, then $[S, T]$ will denote $\mathrm{Cl}\,(\mathrm{Int}T-\mathrm{Int}S)$. Suppose r and s are positive numbers and $r<s$. A cellular subdivision $\{E_1, \dots, E_q\}$ of $[\mathrm{Bd}B(r), \mathrm{Bd}B(s)]$ is a canonical subdivision of $[\mathrm{Bd}B(r), \mathrm{Bd}B(s)]$ if and only if there exist disks $D_1, \dots$ and D_q on $\mathrm{Bd}B(r)$ such that

(1) for each i, there exists a j such that if p is in E_i, then there exists a point t in D_j and a real number c such that $p=ct$,

(2) if $i\neq j$ and $D_i\cap D_j$ is non-empty, then $D_i\cap D_j\subset \mathrm{Bd}D_i\cap \mathrm{Bd}D_j$,

(3) for each i, $E_i\cap B(r)$ is non-empty,

(4) for each i and j, the diameter of D_i is at most twice the diameter of D_j.

Suppose S_1 and S_2 are tame 2-spheres in R^3 and S_2 is contained in the interior of S_1. A cellular subdivision $\{E_1, \dots, E_q\}$ of $[S_2, S_1]$ is an A-subdivision of $[S_2, S_1]$ if and only if there exists a homeomorphism h from $[S_2, S_1]$ onto $[\mathrm{Bd}B(1), \mathrm{Bd}B(2)]$ such that $\{h(E_1), \dots, h(E_q)\}$ is a canonical subdivision of $[\mathrm{Bd}B(1), \mathrm{Bd}B(2)]$. If X is a topological space and G is an upper semi-continuous decomposition of X, then X/G will denote the decomposition space associated with G.

THEOREM. *If C is a crumpled cube in R^3, $R^3 - \operatorname{Int} C$ is homeomorphic with the closure of the exterior of a tame sphere in R^3, and $\{a_i\}$ is a sequence of tame finite linear graphs such that for each i, $a_i \subset a_{i+1}$, $a_i \subset \operatorname{Bd} C$ and $\operatorname{Bd} C - (\bigcup_{i=1}^{\infty} a_i)$ is 0-dimensional, then there exists an upper semi-continuous decomposition G of $B(1)$ such that, each non-degenerate element of G is an arc which intersects $\operatorname{Bd} B(1)$ at one end-point and is locally polyhedral except possibly at the other end point, $B(1)/G$ is homeomorphic to C, and the projection from $B(1)$ to C takes no non-degenerate element of G to a point in $\bigcup_{i=1}^{\infty} a_i$.*

Proof. The proof is similar to the proof of Theorem 3 of [2]. Let S denote the boundary of C, and let e_1 denote $\min\{1, 1/10 \operatorname{diam} C\}$. There exists a polyhedral 2-sphere S_1 in $\operatorname{Ext} S$ such that S_1 is homeomorphically within e_1 of S. Now let e_{11} be $(1/2)d(C, S_1)$. By Theorem 2 of [1] (Theorem 2 of [1] holds if A is a tame finite linear graph), there exists a polyhedral 2-sphere S_{11} with the following properties:

(1) S_{11} is homeomorphically within e_{11} of S,

(2) there exist disjoint disks $D_{11}, \ldots$, and $D_{1n(1)}$ on S, each of diameter less than e_{11}, such that $\operatorname{Cl}(S - \bigcup_{i=1}^{n(1)} D_{1i})$ is contained in $\operatorname{Ext} S_{11}$,

(3) there exist disjoint disks $K_{11}, \ldots$, and $K_{1m(1)}$ on S_{11}, each of diameter less than e_{11}, such that $\operatorname{Cl}(S_{11} - \bigcup_{i=1}^{m(1)} K_{1i})$ is contained in $\operatorname{Int} S$,

(4) $(\bigcup_{i=1}^{n(1)} D_{1i}) \cup (\bigcup_{i=1}^{m(1)} K_{1i})$ is disjoint from a_1 and a_1 is contained in $\operatorname{Ext} S_{11}$.

Let e_{12} denote

$$(1/3)\min\Big\{\min\{d(D_{1i}, D_{1j})\colon i \neq j,\ 1 \leqslant i \leqslant n(1), \text{ and } 1 \leqslant j \leqslant n(1)\},\ \min\{d(K_{1i}, K_{1j})\colon i \neq j,\ 1 \leqslant i \leqslant m(1), \text{ and } 1 \leqslant j \leqslant m(1)\},\ \min\{d(D_{1i}, S_{11} - \bigcup_{j=1}^{m(1)} K_{1j})\colon 1 \leqslant i \leqslant n(1)\},\ d\big(a_1, (\bigcup_{i=1}^{n(1)} D_{1i}) \cup (\bigcup_{i=1}^{m(1)} K_{1i})\big)\Big\}.$$

Let T_1 be a triangulation of R^3 with mesh less than e_{12}, let N be the union of all 3-simplexes of T_1 which intersect $[S, S_1]$, and let M_{11} be the union of all 3-simplexes in the 2nd barycentric subdivision of T_1 which intersect N. Then M_{11} is a polyhedral 3-manifold-with-boundary and it can be assumed that $\operatorname{Bd} M_{11}$ and S_{11} are in relative general position.

Let M_1 denote the union of M_{11} and $[S_{11}, S_1]$. Now $M_1 - [S_{11}, S_1]$ has a finite number of components, and let $F_{11}, \ldots$, and $F_{1r(1)}$ be the components of $\operatorname{Cl}(M_1 - [S_{11}, S_1])$. It follows that for each i, F_{1i} has diameter less than $4e_1 + 2e_{12}$ which is less than $5e_1$.

There exists an A-subdivision $\{E_{11}, \ldots, E_{1q(1)}\}$ of $[S_{11}, S_1]$ such that

(1) for each i, $i \leqslant n(1)$, there exists a positive integer j such that $\operatorname{Bd} D_{1i} \subset \operatorname{Int} E_{1j}$,

(2) for each i, $i \leqslant q(1)$, the diameter of E_{1i} is less than $5e_1$,

(3) $\bigcup\{E_{1i}\colon E_{1i} \cap a_1 \neq \emptyset\}$ and $\bigcup\{E_{1i}\colon E_{1i} \cap F_{1j} \neq \emptyset$ for some $j\}$ are disjoint.

Let A_{1i}^* and A_{1i}, $i \leqslant q(1)$, be defined as follows: $A_{1i}^* = \bigcup\{X\colon X \in \{E_{11}, \ldots, E_{1q(1)}\}$ or $X \in \{F_{11}, \ldots, F_{1r(1)}\}$ and $d(X, E_{1i}) < e_1\}$ and A_{1i} is the component of A_{1i}^* that contains E_{1i}. Note that for each p in S, there exists an integer j such that p is in A_{1j}. For each p in $S - a_1$, denote $\bigcup\{E_{1i}\colon p \in E_{1i}\}$ by A_{p1}. Now for each i, the diameter of A_{p1} is at most $34e_1$.

Then there exists a homeomorphism h_1 from $S_1 \cup \operatorname{Int} S_1$ onto $B(1.1)$ with the following properties:

(1) $h_1([S_{11}, S_1]) = [\operatorname{Bd} B(.9), \operatorname{Bd} B(1.1)]$,

(2) $h_1(C)$ is contained in $\operatorname{Int} B(1.01)$,

(3) $\{h_1(E_{11}), \ldots, h_1(E_{1q(1)})\}$ is a canonical subdivision of $[\operatorname{Bd} B(.9), \operatorname{Bd} B(1.1)]$ into $q(1)$ cells,

(4) $h_1(a_1)$ is contained in $\operatorname{Bd} B(1)$,

(5) for each p, the image of the union of A_{p1} and a small product neighborhood of $A_{p1} \cap [S_{11}, S_1]$ under h_1 is the union of two sets, α_{p1} and β_{p1}, where $\operatorname{diam} \alpha_{p1} \leqslant 2 \operatorname{diam} A_{p1}$, β_{p1} is homeomorphic with the product of a subset of $\operatorname{Bd} \alpha_{p1}$ and $[0, 1]$, and $\beta_{p1} \cap \alpha_{p1}$ is homeomorphic with the cross-section of β_{p1}.

Now M_1 is a 3-manifold-with-boundary and S is a compact subset in $\operatorname{Int} M_1$. It follows that there exists a positive number e_{13} such that, if Q is a set with diameter less that $34e_{13}$ and the intersection of Q with S exists, then there exists a 3-cell R such that $Q \subset \operatorname{Int} R \subset R \subset \operatorname{Int} M_1$.

Let e_2 denote $\min\{d(S, \operatorname{Bd} M_1), e_{13}, 1/4\}$. There exists a polyhedral 2-sphere S_2 in $\operatorname{Ext} S$ such that S_2 is homeomorphically within e_2 of S. Let e_{21} be

$$1/3 \min\big\{d(S, S_2),\ \min\{d(\operatorname{Bd} D_{1i}, \operatorname{Bd} E_{1j})\colon \operatorname{Bd} D_{1i} \subset \operatorname{Int} E_{1j}\}\big\}.$$

By Theorem 2 of [1] there exists a polyhedral 2-sphere S_{22} such that

(1) S_{22} is homeomorphically within e_{21} of S,

(2) there exist disjoint disks $D_{21}, \ldots$, and $D_{2n(2)}$ on S, each of diameter less than e_{21}, such that $\operatorname{Cl}(S - \bigcup_{i=1}^{n(z)} D_{2i}) \subset \operatorname{Ext} S_{22}$,

(3) there exist disjoint disks $K_{21}, \ldots,$ and $K_{2m(2)}$ on S_{22}, each of diameter less than e_{21}, such that $\mathrm{Cl}(S_{22} - \bigcup_{i=1}^{m(2)} K_{2i}) \subset \mathrm{Int}\, S$,

(4) $(\bigcup_{i=1}^{n(2)} D_{2i}) \cup (\bigcup_{i=1}^{m(2)} K_{2i})$ is disjoint from a_2 and a_2 is contained in $\mathrm{Ext}\, S_{22}$.

Let e_{22} denote

$$1/3 \min\{\min\{d(D_{2i}, D_{2j}): i \neq j,\ 1 \leqslant i \leqslant n(2), \text{ and } 1 \leqslant j \leqslant n(2)\},$$
$$\min\{d(K_{2i}, K_{2j}): i \neq j,\ 1 \leqslant i \leqslant m(2), \text{ and } 1 \leqslant j \leqslant m(2)\},$$
$$\min\{d(D_{2i}, S_{22} - \bigcup_{j=1}^{m(2)} K_{2j}): 1 \leqslant i \leqslant n(2)\},\ d\big(a_2, (\bigcup_{i=1}^{n(2)} D_{2i}) \cup (\bigcup_{i=1}^{m(2)} K_{2i})\big)\}.$$

If

$$M'_{22} = \{z: z \in S_2 \cup \mathrm{Int}\, S_2 \text{ and } d(z, \mathrm{Ext}\, S) \leqslant e_{22}\}$$

then in the same way as M_{11} was obtained a polyhedral 3-manifold-with-boundary M_{22} exists such that $[S, S_2] \subset \mathrm{Int}\, M_{22}$ and $M_{22} \subset M'_{22}$. It can be assumed that $\mathrm{Bd}\, M_{22}$ and S_{22} are in relative general position. Let M_2 denote the union of M_{22} and $[S_{22}, S_2]$. Now $M_2 - [S_{22}, S_2]$ has a finite number of components, and let $F_{21}, \ldots,$ and $F_{2r(2)}$ be the components of $\mathrm{Cl}(M_2 - [S_{22}, S_2])$. It follows that for each i, the diameter of F_{2i} is at most $5e_2$. It follows from the definition of M_2 that $S \subset \mathrm{Int}\, M_2 \subset M_2 \subset \mathrm{Int}\, M_1$ and that $\{x: x \in C \text{ and } d(x, S) > e_{22}\}$ is disjoint from $\mathrm{Int}\, M_2$.

There exists an A-subdivision $\{E_{21}, \ldots, E_{2q(2)}\}$ of $[S_{22}, S_2]$ such that

(1) for each i, $1 \leqslant i \leqslant n(2)$, there exists a positive integer j such that $\mathrm{Bd}\, D_{2i} \subset \mathrm{Int}\, E_{2j}$,

(2) for each i, $1 \leqslant i \leqslant q(2)$, the diameter of E_{2i} is less than $5e_2$,

(3) $q(2) \geqslant 4q(1)$,

(4) $\bigcup\{E_{2i}: E_{2i} \cap a_2 \neq \emptyset\}$ and $\bigcup\{E_{2i}: E_{2i} \cap F_{2j} \neq \emptyset$ for some $j\}$ are disjoint.

Let A^*_{2i}, A_{2i}, and A_{p2} be defined in the same manner as A in the first construction. Then for each p in S, the diameter of A_{p2} is at most $34e_2$, and hence for each p in S, $A_{p2} \subset \mathrm{Int}\, A_{p1}$. Furthermore, if $p \in [S_{11}, S_1]$ and Z_{p1} is the component of $A_{p1} \cap [S_{11}, S_1]$ that contains p, then $[S_{22}, S_2] \cap A_{p2}$ is contained in Z_{p1}, and for each p in S, there exists a 3-cell C_{p2} such that $A_{p2} \subset \mathrm{Int}\, C_{p2} \subset C_{p2} \subset \mathrm{Int}\, A_{p1}$.

Now $S_2 \cap \{\bigcup_{i=1}^{q(1)} \mathrm{Bd}\, E_{1i}\}$ exists and has one and only one component with diameter greater than 1/2 the diameter of C. Denote this component by W_1. It follows that for each i, $1 \leqslant i \leqslant q(1)$, the intersection of W_1 and E_{1i} exists.

There exists a homeomorphism h_2 from $S_2 \cup \mathrm{Int}\, S_2$ onto $B(1.01)$ with the following properties:

(1) $h_2([S_{22}, S_2]) = [\mathrm{Bd}\, B(.99), \mathrm{Bd}\, B(1.01)]$,

(2) $h_2(C)$ is contained in $\mathrm{Int}\, B(1.001)$,

(3) if $x \in W_1$ then $h_2(x)$ is equal to the radial projection of $h_1(x)$ onto $\mathrm{Bd}\, B(1.01)$,

(4) if x is in $C - M_2$, then $h_2(x) = h_1(x)$,

(5) h_2 differs from h_1 only at points close to $[S_{22}, S_2]$,

(6) $\{h_2(E_{21}), \ldots, h_2(E_{2q(2)})\}$ is a canonical subdivision of $[\mathrm{Bd}\, B(.99), \mathrm{Bd}\, B(1.01)]$,

(7) $h_2(a_2)$ is contained in $\mathrm{Bd}\, B(1)$,

(8) if x is in a_1 then $h_2(x) = h_1(x)$,

(9) for each p, the image of the union of A_{p2} and a small product neighborhood of $A_{p2} \cap [S_{22}, S_2]$ under h_2 is the union of two sets, α_{p2} and β_{p2}, where $\mathrm{diam}\, \alpha_{p2} \leqslant 2\, \mathrm{diam}\, A_{p2}$, β_{p2} is homeomorphic with the product of a subset of $\mathrm{Bd}\, \alpha_{p2}$ and $[0, 1]$, $\beta_{p2} \cap \beta_{p1}$ is straight in β_{p1}, $\alpha_{p2} \subset \alpha_{p1}$, $\alpha_{p2} \cap \beta_{p2}$ is homeomorphic with the cross-section of β_{p2}, and the cross-sectional diameter of β_{p2} is at most the diameter of α_{p2}.

The process is continued by induction as in Theorem 3 of [2]. It follows that $B(1) = \bigcap_{i=1}^{\infty} h_i(S_i \cup \mathrm{Int}\, S_i)$. Let G be the following decomposition of $B(1)$: $G = \{g$: either (1) for some point p of $B(1)$ there exists a positive integer n such that $p \notin h_n(M_n)$ and $g = \{p\}$, or (2) there exists a q in S such that $g = \bigcap_{i=1}^{\infty} h_i(A_{qi})\}$. Then G is an upper semi-continuous decomposition of $B(1)$. It follows from the way the A_{pi}'s and the h_i's are defined that the intersection of each non-degenerate element of G with $\mathrm{Bd}\, B(1)$ is a one-point set.

If g is a non-degenerate element of G, then there exists a q in S such that $g = \bigcap_{i=i}^{\infty} h_i(A_{qi})$. But $\bigcap_{i=1}^{\infty} h_i(A_{qi})$ is equal to $\bigcap_{i=1}^{\infty} h_i(C_{qi})$ and hence is cellular. $\bigcap_{i=1}^{\infty} h_i(A_{qi})$ is also equal to $\bigcap_{i=1}^{\infty} (\alpha_{qi} \cup \beta_{qi})$ and hence is an arc which can be assumed to be locally polyhedral except at $\bigcap_{i=1}^{\infty} \alpha_{qi}$.

Define a function h from C into $B(1)/G$ as follows. If $x \in \mathrm{Int}\, C$, $h(x) = \{\lim h_i(x)\}$ and if $x \in S$, $h(x) = \bigcap_{i=1}^{\infty} h_i(A_{xi})$. Then h is one-to-one. If not there exist points p and q in C such that $p \neq q$ and $h(p) = h(q)$. Clearly, neither p nor q can be in $\mathrm{Int}\, C$. If p and q are in S then there exists an integer n such that A_{pn} and A_{qn} are disjoint, and hence $h_n(A_{pn})$

and $h_n(A_{qn})$ are disjoint. This is a contradiction. Clearly h is onto $B(1)/G$ and h is a homeomorphism on $\operatorname{Int} C$. To show h is a homeomorphism it suffices to show h is continuous at each point of S. Suppose p is in S, $\{b_i\}$ is a sequence of points in C, and $\{b_i\}$ converges to p. If U is an open set in $B(1)/G$ that contains $h(p)$, then there exists a positive integer k such that $h_k(A_{pk})$ is contained in U. Since $\{b_i\}$ converges to p, all but finitely many of the b_i's belong to A_{pi}. Therefore all but finitely many of the $h_i(b_i)$'s belong to $h_i(A_{pi})$. By the way the h_i's are constructed, if $j > i$ and $h_i(b_i) \in h_i(A_{pi})$, then $h_j(b_i) \in h_i(A_{pi})$, and hence all but finitely many of the $h(b_i)$'s belong to U. Therefore h is continuous at p and h is a homeomorphism.

If p is an element in $\bigcup_{i=1}^{\infty} a_i$ then $\bigcup_{i=1}^{\infty} h_i(A_{pi})$ is a one point set and hence the projection from $B(1)$ to C takes no non-degenerate element of G to a point in $\bigcup_{i=1}^{\infty} a_i$.

It follows that there exists a pseudo-isotopy H from R^3 onto R^3 such that

(1) $H\colon R^3 \times I \to R^3$,

(2) for each $t \in I$, $H(R^3 \times \{t\})$ is R^3,

(3) if $x \in R^3$ then $H(x, 0) = x$,

(4) if $0 \leqslant t \leqslant 1$ then $H|R^3 \times \{t\}$ is a homeomorphism,

(5) $H(x, 1) = H(y, 1)$ if and only if x and y are in the same element of G or $x = y$,

(6) $H(B(1), 1) = C$.

References

[1] D. Gillman, *Side approximation, missing an arc*, Amer. J. of Math. 85 (1963), pp. 459–476.

[2] L. Lininger, *Some results on crumpled cubes*, Trans. of the Amer. Math. Soc. 118 (1965), pp. 534–549.

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On subdirect embeddings in categories

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§ 1. In his paper [4] Suliński considers categories satisfying certain natural, although strong, additional conditions, and asks whether every object of such a category could be subdirectly embedded in a direct product of subdirectly irreducible objects. Such a theorem for universal algebras was proved by Birkhoff [1]. In the proof of this theorem it is implicitly assumed that the lattice of all congruence-relations of any universal algebra is a so-called algebraic lattice (1). However, the notion of congruence-relation cannot be formulated in a category-theoretical manner; it is possible to consider factor-objects instead of congruence-relations. Among the factor-objects one can define a partial ordering. Thus the condition that the congruence-relations form an algebraic lattice means that the factor-objects form a lattice and its dual lattice is algebraic.

After the preliminaries we consider a category satisfying weaker conditions than those of [4]. We assume, that every epimorphism is normal, but we do not suppose that every map has a kernel. (Related investigations are made in [5], where every map has a kernel, but an epimorphism need not be normal. There the possibility of dualization is also discussed.)

In § 3 we prove that an object of such a category can be subdirectly embedded in a direct product of subdirectly irreducible objects if the dual lattice of that of all factor-objects is algebraic. In § 4 we show that this condition is independent of all the conditions assumed by Suliński [4]; moreover, in the category $\mathcal{A}^*$, which is dual to the category of all abelian groups $\mathcal{A}$, there are objects which cannot be subdirectly embedded in a direct product of subdirectly irreducible objects.

§ 2. Let $\mathfrak{C}$ be a category whose objects and maps will be denoted by small Latin and Greek letters, respectively. By definition, the following axioms hold:

(1) Algebraic lattices are sometimes called compactly generated lattices.