Orbits of denumerable models of complete theories

by

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Let $S, T, \ldots$ be first order theories with infinite models. We denote by $M(S)$ the set of all models of S whose universes are the set of natural numbers. We can introduce a topology on $M(S)$ which, roughly speaking, expresses elementary types of elements of models [6]. Consider, for example, the theory S_ω of arithmetic which is closed under ω-rule. The theory S_ω is necessarily complete. The set M_ω of ω-standard models of S_ω forms a co-meager G_δ-set [6]. By orbits we mean quotient classes of models in $M(S)$ with respect to isomorphisms. We say that a model $\mathfrak{A}$ generates an orbit, if $\mathfrak{A}$ is in that orbit. All ω-standard models of S_ω are isomorphic and, therefore, they form an orbit which is a co-meager G_δ-set in the space $M(S_\omega)$. A model $\mathfrak{A}$ of a complete theory T is called prime, if it can be elementarily embedded into arbitrary models of T. The prime models are denumerable and mutually isomorphic [14]. Therefore, they form an orbit, which we shall call prime, in the space $M(T)$. In the example above, the orbit M_ω is really prime. In many examples of complete theories with prime models, the proof which we gave above in the case of theory S_ω, does not seem to work in order to decide whether the prime orbits are co-meager. Therefore, the following two questions naturally arise for complete theories: (1) Can a non-prime orbit be co-meager? (2) Does the prime orbit always form a co-meager set? The main purpose of this paper is to answer these questions. We shall prove, first, that each non-prime orbit forms a meager set and, second, that the prime orbit forms a co-meager G_δ-set. The idea of proof is to combine the following two facts: (1) We can suitably generalize the notions of ω-closedness of theories and ω-standardness of models. (2) A model is prime if and only if it is denumerable and atomic [14].

We should like to remark that the presentation of this paper is reversed. This work was started from a problem of Mostowski, i.e. "What

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topological properties does the space of denumerable β-models have?" The result of the final section was first obtained.

1. The space of models. We shall recall some notions and results from [6]. Let S be a theory with an infinite model. We denote by $M(S)$ the set of models $\mathfrak{A}$ of S whose universes $|\mathfrak{A}|$ coincide with the set of natural numbers. Let Δ be the mapping of the variables into natural numbers in such a way that $\Delta(v_i) = i$. We denote by $[\varphi]$ the set of models $\mathfrak{A}$ in $M(S)$ such that $\models_{\mathfrak{A}}\varphi[\Delta]$. The family $[\varphi]$ generates a topology on $M(S)$. When we speak about a topology on $M(S)$, we mean always this topology introduced by formulas. The set $M(S)$ is a 0-dimentional, separable Hausdorff space which carries a complete metric.

Let us denote by $F_m(S)$ the set of all formulas whose free variables are among $v_0, \ldots v_{m-1}$. We can introduce a Boolean structure, i.e. Lindenbaum's algebra, on $F_m(S)$. Speaking strictly, this Boolean structure is not defined on the set $F_m(S)$ itself, but on the quotient classes of $F_m(S)$ with respect to the theory S, however we speak as if it were introduced on $F_m(S)$. Let $\Pi = \{\pi_n\}_{n\in\omega}$ be a sub-set of $F_{m+1}(S)$. A model $\mathfrak{A}$ of S is called Π-standard, if for any sequence $a_0, \ldots, a_m$ of elements of $|\mathfrak{A}|$ there exists an n such that $\models_{\mathfrak{A}}\pi_n[a_0, \ldots, a_m]$. Let e be a sequence of natural numbers of length k and ψ be a formula in $F(S)$. We denote by $\psi(e)$ the formula $\psi(v_{e(0)}, \ldots, v_{e(k-1)})$. The theory S is called Π-closed, if the following holds: For any sequence e of natural numbers of length $m+1$ and for any formula φ of S, the condition that $\vdash_S \pi_n(e)\to\varphi$ for $n = 0, 1 \ldots$ implies that $\vdash_S\varphi$. If a sequence $\Pi^0, \Pi^1, \ldots$ is given, we can define similarly $(\Pi^l)_{l\in\omega}$-standardness of a model and $(\Pi^l)_{l\in\omega}$-closedness of a theory. A model $\mathfrak{A}$ is $(\Pi^l)_{l\in\omega}$-standard, if it is Π^l-standard for each l and a theory S is called $(\Pi^l)_{l\in\omega}$-closed if it is Π^l-closed for each l. As in [6], we can prove the following

THEOREM 1. *If S is Π-closed, then the set $M_\Pi(S)$ of its Π-standard models in $M(S)$ is a co-meager G_δ-set. Similarily, if S is $(\Pi^l)_{l\in\omega}$-closed, the set of $(\Pi^l)_{l\in\omega}$-standard models is a co-meager G_δ-set in $M(S)$* [1].

Proof. The proof of the Theorem is verbally the same with that in [6], but we shall repeat it here for the convenience of readers. Let us assume that Π is a sub-set of $F_{m+1}(S)$.

$$M_\Pi = \bigcap_{e\in\omega^{m+1}} \bigcup_n [\pi_n(e)] .$$

It is sufficient to show that each of the open sets $\bigcup_n[\pi_n(e)]$ is dense in $M(S)$, i.e., that for no φ this set is disjoint from $[\varphi]$ unless $[\varphi] = 0$.

[1] A similar version of the ω-completeness theorem for the case when $\pi^l \subseteq F_1(S)$ appeared in [7].

Otherwise we would have $0 = [\varphi] \cap [\pi_n(e)] = [\varphi \,\&\, \pi_n(e)]$ for $n = 0, 1, \ldots$ whence we would obtain for every $\mathfrak{A}$ in $M(S)$

$$\models_{\mathfrak{A}} (v_{e(0)}) \ldots (v_{e(m)})\big(\pi_n(e) \to \sim\varphi\big) .$$

By the completeness theorem and the assumption that S is Π-closed, we would obtain $\vdash_S \sim\varphi$ and $[\varphi]$ would be void. The set of the $(\Pi^l)_{l\in\omega}$-standard models is identical to the set $\bigcap_l M_{\Pi^l}$. A countable intersection of co-meager G_δ-sets is evidently a co-meager G_δ-set. The theorem is therefore proved.

The following Lemma is sometimes useful.

LEMMA 1. *Let T be a complete theory with a Π-standard model. Then T is Π-closed.*

Proof. Let $\mathfrak{A}$ be a Π-standard model. Let us assume that $\vdash_T \pi_n(e) \to \varphi$ for $n = 0, 1, \ldots$ Since $\mathfrak{A}$ is a model of T, $\models_{\mathfrak{A}}(v_{e(0)}) \ldots (v_{e(m)})\big(\pi_n(e) \to \varphi\big)$ for $n = 0, 1, \ldots$ Since $\mathfrak{A}$ is Π-standard and T is complete, we have $\vdash_T \varphi$.

2. The orbits of denumerable models. Henceforth T denotes a complete theory with an infinite model. A model $\mathfrak{A}$ of T is called atomic, if each finite sequence of elements of $|\mathfrak{A}|$ of any length $m+1$ satisfies in $\mathfrak{A}$ an atom of the Lindenbaum algebra $F_{m+1}(T)$. A model $\mathfrak{A}$ is prime if and only if it is denumerable and atomic [14].

THEOREM 2. *Each non-prime orbit forms a meager set in $M(T)$.*

Proof. Let us consider an orbit which is generated by a non-prime model $\mathfrak{A}$ in $M(T)$. Since $\mathfrak{A}$ is not prime, it is not atomic. There exists, by definition, a finite sequence $a_0, \ldots, a_m$ of elements of $|\mathfrak{A}|$ which satisfies in $\mathfrak{A}$ no atom of $F_{m+1}(T)$. Consider the prime filter $P = \{\varphi;\ \models_{\mathfrak{A}}\varphi[a_0, \ldots a_m]\ \&\ \varphi \in F_{m+1}(T)\}$. Since P does not contain atoms of $F_{m+1}(T)$, P is a non-principal filter. By the Theorem of Ehrenfeucht [11], there is a denumerable model $\mathfrak{B}$ of T which omits the filter P, i.e. for any sequence $b_0, \ldots b_m$ of elements of $|\mathfrak{B}|$ there is a φ in P which is not satisfied by the sequence $b_0, \ldots b_m$ in $\mathfrak{B}$. Let Π be the set of the formulas $\sim\varphi$ for $\varphi \in P$. By our choice of $\mathfrak{B}$, $\mathfrak{B}$ is a Π-standard model. T is complete and hence T is Π-closed by the Lemma 1. The set $M_\Pi(T)$ of Π-standard models forms a co-meager set by the Theorem 1. Since the orbit of $\mathfrak{A}$ is disjoint from $M_\Pi(T)$, it is meager in $M(T)$.

We should like to mention the fact that the above proof is a restatement in another terms of the proof of a Theorem in [14] which shows that prime models are atomic. We have the following

COROLLARY. *A theory which has no prime model has uncountably many non-isomorphic denumerable models* [2].

[2] This corollary is not the best possible. Cf. [14], Theorem 5.1, for sharper results originating from Mostowski's talks in Paris.

Proof. If the theory T has no prime model, then each orbit is meager. Since the space $M(T)$ carries a complete metric, by Baire's category argument, it cannot be meager on itself. Therefore there exist uncountably many orbits, i.e. uncountably many non-isomorphic denumerable models.

We shall prove a theorem which is inverse to the Theorem 2.

THEOREM 3. *The prime orbit forms a co-meager G_δ-set on $M(T)$.*

Proof. If $\mathfrak{A}$ is prime, it is denumerable and atomic [14]. Let $\Pi^m = \{\pi_n^m\}_{n\in\omega}$ be an enumeration of all the atoms of $F_{m+1}(T)$. Since $\mathfrak{A}$ is atomic, $\mathfrak{A}$ is $(\Pi^m)_{m\in\omega}$-standard. By Lemma 1, the theory T is $(\Pi^m)_{m\in\omega}$-closed and, by Theorem 1, the set of the $(\Pi^m)_{m\in\omega}$-standard models forms a co-meager G_δ-set. Let $\mathfrak{B}$ be a $(\Pi^m)_{m\in\omega}$-standard model. By the definition of $(\Pi^m)_{m\in\omega}$-standardness, any sequence $b_0, \ldots b_m$ of elements of $|\mathfrak{B}|$ of any finite length $m+1$ satisfies π_n^m in $\mathfrak{B}$ for some n. Since Π_n^m was an atom of $F_{m+1}(T)$, $\mathfrak{B}$ is atomic. Therefore the set of the $(\Pi^m)_{m\in\omega}$-standard models is nothing but the orbit of the prime model $\mathfrak{A}$.

Complete theories without prime models are known [5]. Mostowski conjectured that the theory T described below has no prime models. We shall establish his conjecture: By the height of a complete model $\mathfrak{N}$ of ZF, we mean the set of the ordinal numbers in $\mathfrak{N}$. Let $\mathfrak{M}$ be a denumerable complete model for $\mathrm{ZF}+\mathrm{V}=\mathrm{L}$ of the minimal height. Indeed $\mathfrak{M}$ is the minimal model for ZF [2,10]. Let us consider the generic extension $\mathfrak{M}(a)$ where a is a generic set of natural numbers over $\mathfrak{M}$ [1]. $\mathfrak{M}(a)$ determines a completion T of ZF. We must notice that the complete theory T is determined independently of the choice of generic sets [4]. T is an example of a complete theory without prime models. We can show that a stronger assertion holds for T, i.e., T has no model which can be isomorphically embedded into any model of T. Let $\mathfrak{A}$ be such a model of T. Certainly $\mathfrak{A}$ is well-founded and therefore we can assume that $\mathfrak{A}$ is a complete model. By our choice of $\mathfrak{M}$, the height of $\mathfrak{A}$ is the same as that of $\mathfrak{M}$. Let us recall the notion of generic sets (over $\mathfrak{M}$). A set D of sets of conditions is said dense, if for any set p of conditions, there is an extension p of p which is in D. A set a of natural numbers is generic, if for any dense set D constructible in $\mathfrak{M}$, there is a set p of conditions which is compatible with a [12].

For a complete model $\mathfrak{N}$ of the same height as that of $\mathfrak{M}$, "D is a dense set in $\mathfrak{M}$" is equivalent to "D is a dense constructible set". Therefore, the notion of generic sets is absolute for such a model $\mathfrak{N}$. Consider the formal statement $(\mathrm{E}x)\psi(x)$ where $\psi(x)$ is intended to mean "x is a generic set of natural numbers and every set is constructible from x". The statement $(\mathrm{E}x)\psi(x)$ holds in $\mathfrak{M}(a)$ and also in $\mathfrak{A}$. Consider an element b in $\mathfrak{A}$ which satisfies $\psi(x)$ in $\mathfrak{A}$. By the absoluteness of notions mentioned

above, b is really a generic set of natural numbers over $\mathfrak{M}$ and, therefore, $\mathfrak{A}$ is a generic extension $\mathfrak{M}(b)$. Let b_0 be the intersection of b with the set of even numbers. By the transform Lemma [4], the set b is not constructible from b_0. The set b_0 is generic and hence $\mathfrak{M}(b_0)$ is a model of T. $\mathfrak{M}(b)$ can not be embedded into $\mathfrak{M}(b_0)$. This result implies a known fact that there is no definable well-ordering of reals in $\mathfrak{M}(a)$, since the existence of such a definable well-ordering implies the existence of prime models for the theory T (see, p. 6).

3. β-models. The Theorems 2 and 3 were concerned with complete theories with infinite models. Nevertheless we can apply it to some non-complete theories.

Let A be the theory of analysis as formulated in [8]. We denote by A_ω the set of all statements valid in all ω-models of A. Similarily, the set A_β is the set of all statements valid in the β-models of A. For any consistent extension S of A, we denote by $M^\omega(S)$ and $M^\beta(S)$ the sets of the ω-models and of the β-models of S whose universes are the set of integers. By calculating hyper-arithmetical degrees of theories, we can prove that $M^\beta(A)$ is nowhere dense in $M(A)$ and, similarily, that $M^\beta(A_\omega)$ is nowhere dense in $M(A_\omega)$ [13]. This shows simply some defect of the theories A and A_ω in the consideration of β-models and this proof does not go through for the theory A_β. In fact, we can show that $M^\beta(A_\beta)$ is co-meager on some open set of $M(A_\beta)$ and therefore, that $M^\beta(A_\beta)$ is not meager on $M(A_\beta)$. In order to prove this, we need a lemma.

Let $\mathfrak{A}$ be a model of T. We shall say a formula δ in $F_1(T)$ defines an element a in $\mathfrak{A}$, if $\vdash_T(\mathrm{E}!v_0)\delta$ and δ is satisfied by a in $\mathfrak{A}$. We shall say an element a of $|\mathfrak{A}|$ is definable in $\mathfrak{A}$, if it is definable by some formula in $\mathfrak{A}$. We can prove the following:

LEMMA 2. *If all elements of $\mathfrak{A}$ are definable in $\mathfrak{A}$, then $\mathfrak{A}$ is a prime model.*

Proof. It is clear that $\mathfrak{A}$ is an atomic model and, therefore, it is a prime model [14].

Remark. We can prove directly that the models in which all elements are definable are isomorphic and that they coincide with the $\{\delta_i\}_{i\in\omega}$-standard models for some sequence $\{\delta_i\}_{i\in\omega}$ of formulas from $F_1(T)$. Therefore, we can prove that the orbit of such a model forms a co-meager G_δ-set, by applying the Theorem of [6] to the sequence $\{\delta_i\}_{i\in\omega}$, without relying on the result of Section 2.

We can prove the following

THEOREM 4. *The set $M^\beta(A_\beta)$ is co-meager on some non-empty open set of $M(A_\beta)$.*

Proof. The existence of a statement ψ which satisfies the following two conditions is known [8] (3): (1) $A_\beta+\{\psi\}$ has β-models which are unique up to isomorphisms. (2) ψ implies a version of the axiom of constructibility to hold. The first property implies clearly that the theory $T = A_\beta+\{\psi\}$ is consistent and complete. Let $\mathfrak{A}$ be a β-model of T. The second property implies the existence of a definable well-ordering of the universe $|\mathfrak{A}|$. If we take Skolem-hull $\mathfrak{C}$ of $\mathfrak{A}$ with respect to this definable well-ordering in $\mathfrak{A}$, every element c of $\mathfrak{C}$ is definable in $\mathfrak{A}$. Since $\mathfrak{C}$ is an elementary sub-structure of $\mathfrak{A}$, every element c of $\mathfrak{C}$ is definable even in $\mathfrak{C}$ itself. Since $\mathfrak{C}$ is an elementary sub-structure of a β-model $\mathfrak{A}$, it is a β-model of T, too [8]. By the first property, $\mathfrak{C}$ is isomorphic to $\mathfrak{A}$. Since every element was definable in $\mathfrak{C}$, every element of $\mathfrak{A}$ which is an isomorphic image of $\mathfrak{C}$, is definable in $\mathfrak{A}$. By Lemma 2 and Theorem 3, the β-models form a co-meager set on the non-empty open set $[\psi]$.

Theorem 4 was also proved by Mostowski. Let us make some remarks on it. By the completness theorem, it is clear that there exist models A_β which are not ω-models, i.e. $M^\omega(A_\beta) \subset M(A_\beta)$. However, we can not exclude ω-models which are not β-models even for the theory $A_\beta+\{\psi\}$ [7, 9].

Finally, we remark that all the considerations in this section can be parallelled for models of Zermelo—Fraenkel set theory [Cf. 7, 9]. In order to prove Theorem 4 for set theory, we take as ψ the statement which asserts the minimality of the universe [2, 10].

Note added on February 5, 1969. Professor R. L. Vaught kindly called my attention to the fact that a similar work on prime models like his [14] was also done independently by Professor L. Svenonius in Teoria 25 (1959), pp. 82—84.

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(3) The proof of this fact in [8] contains an error (cf. [3]), but it is possible to correct it.

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