If furthermore any of these conditions holds, the mapping φ *of* (iii) *and* (iv) *is unique.*

Proof. Obviously, (ii) implies (i). By Lemma 11 and Definition 14, (i) implies (iii). By Theorem 8, (iii) is equivalent to (iv). It remains to show that (iii) implies (ii). By Lemma 11, $F \subseteq E$ implies that H_f is reduced to f for any $f \in F$. Since $(R, \cdot)$ is a union of groups, we have

$$R = \bigcup_{e \in E} H_e = \bigcup_{e \in E-F} H_e \cup \bigcup_{f \in F} H_f = \big(\bigcup_{e \in E-F} H_e\big) \cup F = \bigcup_{e \in E-F} \big(H_e \cup \varphi(e)\big)$$

and this is obviously an F-disjoint union of subdivision rings of R, which completes the proof.

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An example of a monostratiform λ-dendroid

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A metric compact continuum is said to be a *dendroid* if it is hereditarily unicoherent and arcwise connected. It follows that it is hereditarily decomposable (see [2], (47), p. 239). A hereditarily unicoherent and hereditarily decomposable continuum is called a λ-*dendroid*. Note that every subcontinuum of a λ-dendroid is also a λ-dendroid.

It is proved in [3]. Corollary 2, p. 29, that for every λ-dendroid X there exists a unique decomposition $\mathfrak{D}$ of X (called the *canonical decomposition*):

$$X = \bigcup \{S_d \colon d \in \Delta(X)\}$$

such that

(i) $\mathfrak{D}$ is upper semicontinuous,

(ii) the elements S_d of $\mathfrak{D}$ are continua,

(iii) the hyperspace $\Delta(X)$ of $\mathfrak{D}$ is a dendroid,

(iv) $\mathfrak{D}$ is the finest possible decomposition among all decompositions satisfying (i), (ii) and (iii).

The elements S_d of $\mathfrak{D}$ are called *strata* of X. The question arises whether there exists a λ-dendroid X with trivial canonical decomposition, i.e. such that X has only one stratum.

The purpose of this paper is to give the affirmative answer to the above question.

Call a λ-dendroid to be *monostratiform* if it consists of only one stratum. Thus the hyperspace of the canonical decomposition of a monostratiform λ-dendroid is a point. It follows from [3], Theorem 7, p. 29 that:

(1) A λ-dendroid X is monostratiform if and only if every monotone mapping onto a dendroid is trivial, i.e. the whole X goes onto a point.

(See also [4], Corollaries 1 and 2, p. 933).

Construction. The decription of the example is based upon the description of Lelek's example of a dendroid with 1-dimenional set of end points (see [9], § 9, p. 314).

To describe the monostratiform λ-dendroid X which we are going to construct, the following geometrical procedure is needed:

By an *oriented triangle* T we mean a triangle (i.e. a 2-cell) in which an ordering $\prec$ of vertices is distinguished. If a, b, c are vertices of T and this ordering is just $a \prec b \prec c$, then we write $T = T(abc)$.

Let $T(abc)$ be a fixed oriented triangle lying in an Euclidean plane with the ordinary metric ϱ, and let a_i, where $i = 1, 2, \ldots$, be points such that for all $k = 1, 2, \ldots$

$$(2) \qquad a_{2k-1} \in ac \quad \text{and} \quad a_{2k} \in bc$$

$$(3) \qquad \varrho(a, a_{2k-1}) = \varrho(a, c)/2k \quad \text{and} \quad \varrho(b, a_{2k}) = \varrho(b, c)/2k\,.$$

Denote the centre of the straight segment $a_i a_{i+1}$ by b_i, where $i = 1, 2, \ldots$ and let for every $k = 1, 2, \ldots$ points $d_1^k, d_2^k, d_3^k, d_4^k$ be centres of straight segments $a_{2k-1}b_{2k-1}$, $b_{2k-1}a_{2k}$, $a_{2k}b_{2k}$ and $b_{2k}a_{2k+1}$ correspondingly. Thus, for every natural k, points d_1^k and d_2^k lie in the side $a_{2k-1}a_{2k}$ as well as points d_3^k and d_4^k lie in the side $a_{2k}a_{2k+1}$ of the triangle with vertices a_{2k-1}, a_{2k} and a_{2k+1}. Divide each of two straight segments $d_1^k d_4^k$ and $d_2^k d_3^k$ into three equal parts and define points $c_{4k-3}, c_{4k-2}, c_{4k-1}, c_{4k}$ of this division as follows:

$$\begin{array}{llcl}
c_{4k-3} \in d_1^k d_4^k & \text{and} \quad \varrho(c_{4k-3}, d_1^k) & = & \varrho(d_1^k, d_4^k)/3\,, \\
c_{4k-2} \in d_2^k d_3^k & \text{and} \quad \varrho(c_{4k-2}, d_2^k) & = & \varrho(d_2^k, d_3^k)/3\,, \\
c_{4k-1} \in d_2^k d_3^k & \text{and} \quad \varrho(c_{4k-1}, d_3^k) & = & \varrho(d_2^k, d_3^k)/3\,, \\
c_{4k} \in d_1^k d_4^k & \text{and} \quad \varrho(c_{4k}, d_4^k) & = & \varrho(d_1^k, d_4^k)/3\,.
\end{array}$$

Now, for every $k = 1, 2, \ldots$, take four oriented triangles

$$(4) \qquad \begin{cases} T_{4k-3} = T(a_{2k-1}b_{2k-1}c_{4k-3})\,, \\ T_{4k-2} = T(a_{2k}b_{2k-1}c_{4k-2})\,, \\ T_{4k-1} = T(a_{2k}b_{2k}c_{4k-1})\,, \\ T_{4k} \;\;\; = T(a_{2k+1}b_{2k}c_{4k}) \end{cases}$$

lying inside the triangle with vertices a_{2k-1}, a_{2k} and a_{2k+1}.

We denote by $\mathcal{T}(abc)$ the sequence $\{T_i\}_{i=1,2,\ldots}$ of oriented triangles defined above (see Fig. 1). Therefore for any two triangles $T_i, T_j \in \mathcal{T}(abc)$ we have

$$(5) \qquad T_{2i-1} \cap T_{2i} = b_i \quad \text{and} \quad T_{2i} \cap T_{2i+1} = a_{i+1} \quad \text{for } i = 1, 2, \ldots$$

by (4), and

$$(6) \qquad T_i \cap T_j = \emptyset \quad \text{for } |i-j| > 1\,.$$

Let $\delta(S)$ denote the diameter of a set S. Now we shall prove that

$$(7) \qquad \delta(T_i) \leqslant \tfrac{3}{4}\delta\big(T(abc)\big) \quad \text{for } i = 1, 2, \ldots$$

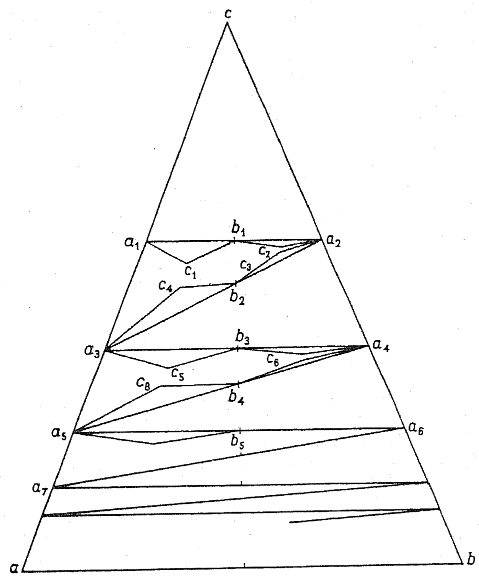

Fig. 1

With this in view let us denote by q the centre of the segment ab and consider two trapezia: A with vertices a, a_1, b_1, q and B with vertices b, a_2, b_1, q. The diameter of A is the maximum of the six numbers: four of them are lengths of the sides of A, the other two are lengths of its diagonals. Put

$$(8) \qquad \delta\big(T(abc)\big) = \delta_0\,.$$

To estimate $\delta(A)$ let us observe that we have following inequalities for the sides of A.

$$\begin{aligned}
\varrho(a, a_1) &= \tfrac{1}{2}\varrho(a, c) \leqslant \tfrac{1}{2}\delta_0\,, \\
\varrho(a_1, b_1) &= \tfrac{1}{2}\varrho(a_1, a_2) = \tfrac{1}{4}\varrho(a, b) \leqslant \tfrac{1}{4}\delta_0\,, \\
\varrho(b_1, q) &= \varrho(b_1, c) = \tfrac{1}{2}\varrho(c, q) \leqslant \tfrac{1}{2}\delta_0\,, \\
\varrho(q, a) &= \tfrac{1}{2}\varrho(a, b) \leqslant \tfrac{1}{2}\delta_0\,.
\end{aligned}$$

For the diagonals of A we obtain

$$\begin{aligned}
\varrho(a_1, q) &= \tfrac{1}{2}\varrho(b, c) \leqslant \tfrac{1}{2}\delta_0\,, \\
\varrho(a, b_1) &\leqslant \varrho(a, a_1) + \varrho(a_1, b_1) \leqslant \tfrac{1}{2}\delta_0 + \tfrac{1}{4}\delta_0 = \tfrac{3}{4}\delta_0\,.
\end{aligned}$$

Therefore $\delta(A) \leqslant \frac{3}{4}\delta_0$. In the same manner we can prove that $\delta(B) \leqslant \frac{3}{4}\delta_0$. Further, note by construction that for $i = 1, 2, \ldots$ we have either $T_i \subset A$ or $T_i \subset B$, which leads to $\delta(T_i) \leqslant \frac{3}{4}\delta_0$, and (7) is proved by (8).

If $\mathcal{A}$ is a family of sets, let $\mathcal{A}^*$ denote the union of all members of $\mathcal{A}$. So we see that

$$\mathcal{T}^*(abc) = \bigcup_{i=1}^{\infty} T_i ,$$

which is a connected set by (5). By construction

(9) $$\operatorname*{Ls}_{i\to\infty} T_i = ab = \overline{\mathcal{T}^*(abc)} \setminus \mathcal{T}^*(abc) ,$$

whence we conclude that

(10) $ab \cup \mathcal{T}^*(abc)$ is a continuum.

A point p of a connected set A is said to be a *separating* point of A if $A \setminus (p)$ is not connected. So we see by (5) and (6) that

(11) every point a_i for $i > 1$ as well as every point b_i is a separating point of the continuum $ab \cup \mathcal{T}^*(abc)$.

Now we shall define for every $n = 1, 2, \ldots$ a countable family $\mathcal{S}_n$ of straight line segments and a countable family $\mathcal{T}_n$ of oriented triangles. Namely we put

(12) $$\mathcal{S}_1 = \{ab\}, \quad \mathcal{T}_1 = \mathcal{T}(abc)$$

and

(13) $$\mathcal{S}_{n+1} = \mathcal{S}_n \cup \{a'b' | T(a'b'c') \in \mathcal{T}_n\} ,$$

(14) $$\mathcal{T}_{n+1} = \bigcup \{\mathcal{T}(a'b'c') | T(a'b'c') \in \mathcal{T}_n\} .$$

We also agree that $\mathcal{T}_0 = \{T(abc)\}$.

Observe that for an arbitrary sequence of oriented triangles $T^n \in \mathcal{T}_n$ we have by (7) and (14)

(15) $$\text{if } T^n \in \mathcal{T}_n, \text{ then } \lim_{n\to\infty} \delta(T^n) = 0 .$$

The following properties of the above families $\mathcal{S}_n$ and $\mathcal{T}_n$ are readily seen:

(16) $\mathcal{S}_n^*$ is a continuum,

(17) $\mathcal{S}_n^* \cup \mathcal{T}_n^*$ is a continuum,

(18) $$\mathcal{S}_{n+1}^* \cup \mathcal{T}_{n+1}^* \subset \mathcal{S}_n^* \cup \mathcal{T}_n^* .$$

The λ-dendroid X is defined by the formula

(19) $$X = \bigcap_{n=1}^{\infty} (\mathcal{S}_n^* \cup \mathcal{T}_n^*) .$$

It is a continuum as the common part of the decreasing sequence of continua by (17) and (18).

Properties. It follows from (19) by construction that every triangle $T \in \mathcal{T}_n$ for any natural n contains a homeomorphic image of X:

(20) if $T \in \mathcal{T}_n$ for some n, then $X \cap T$ is homeomorphic to X.

Further, we see — also by construction — that

(21) for every point x of X and for every neighbourhood U of x there is a natural n, sufficiently great, such that U contains some triangle $T \in \mathcal{T}_n$.

Note that if a continuum K is the common part of a decreasing sequence of continua K_n, then every separating point of any K_n is a separating point of K. Since the common vertex of every two succesive triangles $T_i, T_{i+1} \in \mathcal{T}_1$ is a separating point of the continuum $\mathcal{S}_1^* \cup \mathcal{T}_1^* = ab \cup \mathcal{T}^*(abc)$ by (11), hence it is a separating point of X. Thus (20) implies that

(22) if a point s is a common vertex of two triangles $T', T'' \in \mathcal{T}_n$ $(n = 1, 2, \ldots)$, then s is a separating point of X.

Let S denote the set of all such points s. This means that s is in S if and only if s is a common vertex of some two triangles T' and T'' belonging to $\mathcal{T}_n$ for any natural n. Hence

(23) every point of S is a separating point of X

by (22), and we conclude from (20) and (21) that S is dense in X:

(24) $$\overline{S} = X .$$

The families $\mathcal{T}_n$ being countable for each n, the set S is countable. We shall show below that no point of $X \setminus S$ separates X, i.e. that S consists of all separating points of X.

It follows from definitions (12)–(14) of the families $\mathcal{S}_n$ and $\mathcal{T}_n$ and from (19) that $\mathcal{S}_n^*$ is in X for every n. Thereby

(25) $$\bigcup_{n=1}^{\infty} \mathcal{S}_n^* \subset X .$$

According to (12) and (13) the union $\bigcup_{n=1}^{\infty} \mathcal{S}_n^*$ consists of the side ab of $T(abc)$ and of the sides $a'b'$ of oriented triangles $T(a'b'c') \in \mathcal{T}_{n'}$, $n = 1, 2, \ldots$ Since a common vertex of any two oriented triangles $T' = T(a'b'c')$ and $T'' = T(a''b''c'')$ both belonging to the same $\mathcal{T}_n$ is just the common end point of the sides $a'b'$ and $a''b''$ of these triangles, hence, by definition of S, we have

$$(26) \qquad S \subset \bigcup_{n=1}^{\infty} S_n^*$$

which implies by (24) and (25) that

$$(27) \qquad X = \overline{\bigcup_{n=1}^{\infty} S_n^*}\,.$$

Note that in every triangle $T(a'b'c') \in \mathcal{T}_n$ its side $a'b'$ is a continuum of convergence of the sequence of sides $a''b''$ of triangles $T(a''b''c'') \in \mathcal{T}(a''b''c'') \subset \mathcal{T}_{n+1}$ by (9) and (20). So, no interior point of the side $a'b'$ separates X. Thus

(28) if a separating point of X is in $\bigcup_{n=1}^{\infty} S_n^*$, it is in S.

Now put

$$(29) \qquad E = \overline{\bigcup_{n=1}^{\infty} S_n^*} \setminus \bigcup_{n=1}^{\infty} S_n^*$$

i.e. by (27)

$$(30) \qquad E = X \setminus \bigcup_{n=1}^{\infty} S_n^*\,.$$

Let t be a point of E. Thus t is a common point of some decreasing sequence of triangles T^n such that $T^n \in \mathcal{T}_n$ and $t \in \operatorname{Int} T^n$ for every n:

$$(31) \qquad t = \bigcap_{n=1}^{\infty} T^n\,.$$

Now let $x \neq t$ be a point of X, and let ε be a positive number, less than the distance from x to t. Therefore by (15) there is a natural m such that $t \in \operatorname{Int} T^m$, $x \in X \setminus T^m$ and $\delta(T^m) < \varepsilon$. The sequence of triangles T^n to which t belongs being decreasing, we have $t \in T^{m+1} \subset T^m$ and $T^{m+1} \in \mathcal{T}_{m+1}$. So T^{m+1} must be a term of the sequence $\mathcal{T}(a'b'c')$ of triangles, where $T^m = T(a'b'c')$. Let T_i^{m+1} $(i = 1, 2, \ldots)$ denote the ith term of this sequence, where indices i are placed in the same manner like it was done by (4) for the sequence $\mathcal{T}(abc)$. If j denote the index of T^{m+1} in the sequence $\mathcal{T}(a'b'c')$, i.e. if

$$T^{m+1} = T_j^{m+1},$$

then the common vertex s of T_j^{m+1} and T_{j+1}^{m+1} is a separating point of X according to (22) because both these triangles belong to $\mathcal{T}_{m+1}$. We see that s separates X into two sets M and N such that $x \in M$ and $t \in N$:

$$X \setminus (s) = M \cup N\,.$$

But $N \cup (s) \subset \bigcup_{i=1}^{j} T_i^{m+1} \subset T^m$, whence $\delta(N) < \varepsilon$. It shows that t is an end point of X in the sense of Menger-Urysohn, i.e. that

$$(32) \qquad \operatorname{ord}_t X = 1$$

(see e.g. [8], § 46, I, p. 200), whence we conclude that

(33) X is locally connected at t

by [8], § 46, IV, 1, p. 209. Obviously X is not locally connected at any point of the union $\bigcup_{n=1}^{\infty} S_n^*$, whence

(34) E is the set of all points at which X is locally connected.

Further, it is readily seen that no point of $X \setminus E$ is an end point of X. So by (32) we have

(35) E is the set of all end points of X,

whence

$$(36) \qquad \dim E = 0$$

by [8], § 46, V, 2, p. 217.

It is immediately seen by (20) that every triangle $T \in \mathcal{T}_n$ for some n contains a point of E. Thus (21) implies that E is dense in X.

Remark that (30) gives

$$E = \bigcap_{n=1}^{\infty} (X \setminus S_n^*)\,,$$

whence we see by (16) that E is a G_δ-set.

Let us come back now to separating points of X. Since no end point of a continuum is a separating point (see [8], § 46, V, 1, p. 217), hence the set of all separating points of X and the set E are disjoint by (35). It implies by (30) that all separating points of X are in the union $\bigcup_{n=1}^{\infty} S_n^*$ therefore by (23) and (28) the set of all separating points of X is just S.

So we have the following statement concerning separating points of X:

(37) The set of all separating points of X is dense and countable. It consists of common vertices of any two triangles T', T'', both being in the same $\mathcal{T}_n$, where $n = 1, 2, \ldots$

To prove the hereditary decomposability of X consider two different points x and y of X. Since they are different, hence the sequence of triangles

$$T(abc) = T^0 \supset T^1 \supset T^2 \supset \ldots \supset T^m \supset \ldots$$

such that x and y are both in each T^m and $T^m \in \mathcal{T}_n$ for $n = 1, 2, \ldots$ has the last element, say T^m. Observe that there is no irreducible continuum from x to y which intersects the complementary of T^m: every such a continuum must lie entirely in T^m. The set $X \cap T^m$ being homeomorphic to X by (20), we may assume without loss of generality that $m = 0$, i.e. that no triangle of the sequence $\mathcal{T}(abc)$ contains both x and y.

If x and y are both in ab, then the unique irreducible continuum from x to y is an arc.

If x is in ab and y is not, then y belongs to some triangle $T_k \in \mathcal{T}(abc)$ and we have countably many points, common vertices of every two succesive triangles T_i, T_{i+1} of $\mathcal{T}(abc)$ for $i > k$, which separate x from y. Thus an irreducible continuum joining x and y is separated by these points.

Finally if neither x nor y is in ab, then they are in different triangles of $\mathcal{T}(abc)$. Let $x \in T_j$, $y \in T_k$ where $T_j, T_k \in \mathcal{T}(abc)$. We may assume $j < k$ (the opposite case, $j > k$, is quite similar). Thus every common vertex of two succesive triangles T_i and T_{i+1}, where $i = j, j+1, \ldots, k-1$, is a separating point of X and separates x from y, whence an irreducible continuum from x to y must be separated by such a point.

Therefore we conclude that every irreducible subcontinuum of X is separated by a point, whence the hereditary decomposability of X follows (see [8], § 43, V, 1, p. 145).

Before proving the hereditary unicoherence of X, firstly we recall some known notions and a theorem concerning continua lying in the plane E^2, and secondly, we observe some properties of the construction in $T(abc)$.

A continuum $C \subset E^2$ is said to *cut* (or to be a *cutting of*) E^2 *between points* a *and* b provided that $a, b \in E^2 \setminus C$ and every continuum which contains a and b intersects C. It is called an *irreducible cutting of* E^2 *between* a *and* b provided that it is a cutting of E^2 between a and b and no proper its subcontinuum cuts E^2 between these points. A continuum $C \subset E^2$ is said to *cut* (or to be a *cutting of*) E^2 if there exist two points such that C cuts E^2 between them.

The following theorem is known:

(38) A hereditarily decomposable plane continuum is hereditarily unicoherent if and only if it does not cut the plane.

Indeed, every cutting of E^2 between two points contains an irreducible cutting between these points (a theorem due to S. Mazurkiewicz; see [7], Theorem I, p. 133). So if we suppose that a hereditarily decomposable and hereditarily unicoherent continuum K cuts E^2 between a and b, then we conclude that K must contain an irreducible cutting L of E^2 between a and b which is decomposable and unicoherent. Therefore for

an arbitrary decomposition of L into two its proper subcontinua L_1 and L_2 we see that neither L_1 nor L_2 cuts E^2 between a and b, and that $L_1 \cap L_2$ is a continuum; hence the union $L_1 \cup L_2 = L$ cannot cut E^2 between a and b according to a theorem due to Z. Janiszewski ([6], Theorem A, p. 48; see also [7], (ii), p. 136), and we get a contradiction.

Invertedly, if a hereditarily decomposable plane continuum K is not hereditarily unicoherent, then it contains a subcontinuum M that can be decomposed into two subcontinua M_1 and M_2 the intersection of which is not a continuum. Thus $M_1 \cup M_2 = M$ is a cutting of E^2 (see [6], Theorem B, p. 55; also [7], (iii), p. 136), and so is K because K is 1-dimensional.

Observe now the following two properties of the construction in the triangle $T(abc)$.

(39) For every point $x_0 \in T(abc) \setminus \overline{\mathcal{T}^*(abc)}$ there exists an arc $x_0 p_0$ such that $p_0 \in ac \cup bc$ and $x_0 p_0 \subset T(abc) \setminus \overline{\mathcal{T}^*(abc)}$.

In fact, if $x_0 \in T(a_{j-1} a_j a_{j+1}) \subset T(abc)$, then we take as p_0 an arbitrary point of $a_{j-1} a_{j+1} \setminus (a_{j-1}) \setminus (a_{j+1})$ (see the figure) and we see that such an arc $x_0 p_0$ does exist.

Similarly, if $T(a'b'c') \in \mathcal{T}(abc)$, then

(40) For every point $p_1 \in (a'c' \cup b'c') \setminus X$ there exists an arc $p_1 p_0$ such that $p_0 \in ac \cup bc$ and $p_1 p_0 \setminus (p_1) \subset T(abc) \setminus \overline{\mathcal{T}^*(abc)}$.

Indeed, if $p_1 \in a'c' \cup b'c' \subset T(a'b'c') \subset T(a_{j-1} a_j a_{j+1})$, then we take as p_0 an arbitrary point of $a_{j-1} a_{j+1} \setminus (a_{j-1}) \setminus (_{j+1})$ and we can easy find such an arc $p_1 p_0$ lying entirely in $T(a_{j-1} a_j a_{j+1})$.

Now, in order to prove the hereditary unicoherence of the continuum X, it is sufficient to prove, according to (38), that X does not cut the plane, i.e. that for any two points x_1 and x_2 in $E^2 \setminus X$ there is a continuum joining x_1 and x_2 and having no common point with X. Let y be a point of $E^2 \setminus T(abc)$. Observe that if K_1 and K_2 are continua such that $x_i \in K_i$, $y \in K_i$ and $K_i \cap X = \emptyset$ for $i = 1$ and 2, then their union $K_1 \cup K_2$ is a continuum K with properties $x_1, x_2 \in K$ and $K \cap X = \emptyset$. Therefore it is enough to show that a point $x \in E^2 \setminus X$ can be joined with y by a continuum which does not intersect X. The existence of such a continuum is obvious if $x \in E^2 \setminus T(abc)$. In the opposite case, if $x \in T(abc) \setminus X$, we see that x is not in $\bigcup_{n=1}^{\infty} S_n^*$ by (25), whence, the sequence of sets $\mathcal{T}_n^*$ being decreasing by (14), we conclude by (19) that there exists a natural m with property

$$x \in \mathcal{T}_m^* \setminus \mathcal{T}_{m+1}^* . \tag{41}$$

Consider the finite sequence of triangles $T^n \in \mathcal{T}_n$ such that

$$x \in T^m \subset T^{m-1} \subset \ldots \subset T^1 \subset T^0 = T(abc) . \tag{42}$$

Putting $T^m = T(uvw)$ we have $\mathcal{C}(uvw) \subset \mathcal{C}_{m+1}$, whence $\mathcal{C}^*(uvw) \subset \mathcal{C}^*_{m+1}$. Thus

$$(43) \qquad x \in T(uvw) \setminus \mathcal{C}^*(uvw)$$

by (42) and (41). The intersection $X \cap T^m$ being homeomorphic to X by (20), we have similarly to (9)

$$(44) \qquad uv = \overline{\mathcal{C}^*(uvw)} \setminus \mathcal{C}^*(uvw) .$$

Since $T^m \in \mathcal{C}_m$, hence we conclude from (13) that $uv \subset S^*_{m+1}$, thus $uv \subset X$ by (25). But the point x is not in X, so it is not in uv, thereby (43) and (44) give

$$x \in T(uvw) \setminus \overline{\mathcal{C}^*(uvw)} .$$

Applying (39) to the triangle $T(uvw)$ in place of $T(abc)$, which is possible by virtue of (20), we deduce that there exists an arc xp_m such that p_m lies in the boundary of T^m and $xp_m \cap X = \emptyset$. Using (20) and applying (40) to triangles $T^{m-1}, T^{m-2}, \ldots, T^0$ we infer that there is a finite sequence of arcs $p_m p_{m-1}, p_{m-1}p_{m-2}, \ldots, p_1 p_0$ every of which is disjoint with X and such that p_i belongs to the boundary of T^i for $i = 0, 1, \ldots, m-1$. So the union $xp_m \cup p_m p_{m-1} \cup \ldots \cup p_1 p_0$ is a continuum which lies in $T(abc) \setminus X$ and which joins the point x with the point $p_0 \in ac \cup bc$. Thus if we join p_0 with y by an arc $p_0 y$ such that $p_0 y \cap T(abc) = (p_0)$, we obtain a continuum

$$xp_m \cup p_m p_{m-1} \cup \ldots \cup p_1 p_0 \cup p_0 y \subset E^2 \setminus X ,$$

and therefore the proof of the hereditary unicoherence of X is finished.

Being hereditarily decomposable and hereditarily unicoherent, X is a λ-dendroid.

Now we shall prove the main property of the λ-dendroid X, that X is monostratiform. Let

$$\phi\colon X \to \Delta(X)$$

be the canonical mapping of X onto the dendroid $\Delta(X)$, i.e. such a mapping that

$$\phi^{-1}(d) = S_d \quad \text{for } d \in \Delta(X) ,$$

where S_d are the elements of the canonical decomposition $\mathfrak{D}$ (see [3], p. 25). We shall show that $\Delta(X)$ reduces to a point. To establish this it is sufficient to verify that S^*_n goes to a point under ϕ for each $n = 1, 2, \ldots$ In fact, if it is so, each S^*_n must go onto the same point, because the sequence of continua S^*_n, $n = 1, 2, \ldots$, is increasing by (13). It implies that the union $\bigcup_{n=1}^{\infty} S^*_n$ is mapped onto the point, thus X is by (27).

So, we should demonstrate that

(45) $\phi(S^*_n)$ is a point for each $n = 1, 2, \ldots$

Recall that if a continuum C is a tranche (in the sense of Kuratowski, see [8], § 43, IV, p. 139) of an irreducible subcontinuum of an arbitrary λ-dendroid and if f is a monotone mapping of this λ-dendroid onto a dendroid, then $f(C)$ is a point (see [3], Theorem 5, p. 26).

Firstly, observe that $S^*_1 = ab$ is a tranche of the irreducible continuum S^*_2. Thus $\phi(S^*_1)$ is a point by the above reason. Denote this point by d:

$$\phi(S^*_1) = d .$$

Secondly, assume that

$$(46) \qquad \phi(S^*_n) = d$$

for some fixed n. The family $\mathcal{C}_{n-1}$ of oriented triangles being obviously countable, let

$$T^1, T^2, \ldots, T^k, \ldots, \quad \text{where } T^k \in \mathcal{C}_{n-1}$$

be an arbitrary sequence of all its elements. Thus

$$(47) \qquad \mathcal{C}_n = \bigcup_{k=1}^{\infty} \mathcal{C}(T^k)$$

by (14) with n instead of $n+1$, where $\mathcal{C}(T^k)$ means the same as $\mathcal{C}(pqr)$ if $T^k = T(pqr)$. Denote by $\mathcal{R}_k$ the family of all straight segments $a'b'$ such that $T(a'b'c') \in \mathcal{C}(T^k)$. Therefore (47) implies that

$$\{a'b' \mid T(a'b'c') \in \mathcal{C}_n\} = \bigcup_{k=1}^{\infty} \mathcal{R}_k .$$

whence

$$(48) \qquad S^*_{n+1} = S^*_n \cup \bigcup_{k=1}^{\infty} \mathcal{R}^*_k$$

by (13). Consider now oriented triangles T^k_i, terms of the sequence $\mathcal{C}(T^k)$, where indices i are defined in the same way as is was done by (4) for triangles T_i of the sequence $\mathcal{C}(abc)$. Let A_i denote the side $a''b''$ of the oriented triangle $T^k_i = T(a''b''c'')$. Hence by definition of $\mathcal{R}_k$ we have

$$(49) \qquad \mathcal{R}^*_k = \bigcup_{i=1}^{\infty} A_i .$$

Applying (5) to the sequence $\mathcal{C}(T^k)$ we see that

(50) A_i and A_{i+1} have an end point in common.

Observe that every segment A_i is a tranche of an irreducible continuum. Namely the set

$$A_i \cup \bigcup \{uv \mid T(uvw) \in \mathcal{C}(T^k_i)\} \subset S^*_{n+2}$$

is an irreducible continuum, homeomorphic to the closure of the graph of the function $y = \sin(1/x)$ for $0 < x \leqslant 1$, and it has A_i as the only tranche different from a point. By the above argument $\phi(A_i)$ is a point. Call it d'_i:

$$\phi(A_i) = d'_i .$$

If $\phi(A_{i+1}) = d'_{i+1}$, then $d'_i = d'_{i+1}$ by (50), and we may omit the indices i and write

$$\phi(A_i) = d' \quad \text{for } i = 1, 2, \ldots$$

Thus by (49)

$$\phi(\mathcal{R}_k^*) = d' .$$

But $\overline{\mathcal{R}_k^*}$ is a continuum as well as $\mathcal{S}_n^*$ is, and we have

$$\mathcal{S}_n^* \frown \overline{\mathcal{R}_k^*} = pq ,$$

where $T(pqr) = T^k$, which implies by (46) that d and d' coincide. So we conclude that

$$\phi(\mathcal{S}_{n+1}^*) = d$$

by (48). Therefore (45) is established.

Remarks. Some modifications of the above construction lead to various kinds of monostratiform λ-dendroids, e.g. to an example of a monostratiform λ-dendroid without separating points. Since X is a plane continuum no subcontinuum of which separates the plane, it is tree-like (see [1], the definition on p. 653 and Theorem 6, p. 656). But H. Cook has recently proved, [5], that every λ-dendroid, not necessarily embeddable in the plane, is tree-like. It can be seen from Corollary 1 in [11], p. 379, that X has the fixed point property. It is not known if all λ-dendroids have this property. For a partial solution see, e.g., [4], where a list of references is given.

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