is paired with a unique element in Y and vice versa, so that F is a function of X onto Y. The hypothesis of Lemma 2.9 is satisfied so that F is a homeomorphism. Also, since $x_i \in H$ implies $F(x_i) = f(x_i)$, it follows that $F|_H = f$.

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More decompositions of E^n which are factors of E^{n+1}

by

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1. Introduction. Let G be a monotone upper semicontinuous decomposition of E^n, G_1 be the collection of nondegenerate elements of G, and assume G_1 is countable. We show in this paper that $E^n/G \times E^1$ is topologically E^{n+1} if either (1) for each element $g \in G_1$, there exists a positive integer n_g such that g is an n_g-frame, or (2) G_1 is a null collection and if $g \in G_1$, there exists a positive integer n_g such that g is an n_g-cell which is flat in E^{n+1}.

Bing proved [5] that the product of the dogbone space [4] and E^1 is E^4. Thus E^4 has non-manifold factors. Using analogous techniques, Andrews and Curtis [1] have shown that the product of E^1 and a decomposition of E^n whose only non-degenerate element is an arc is E^{n+1}. Gillman and Martin [8] announced an extension of the result of Andrews and Curtis by proving case (1) above if each element of G_1 is an arc. Recently Bryant [6] has shown that if D is a k-cell in E^n that is flat in E^{n+1} and G is the decomposition of E^n whose only nondegenerate element is D, then $E^n/G \times E^1$ is topologically E^{n+1}. For a more complete summary of related results, the reader is referred to [2].

In Section 3 we show that if G_1 is countable and each element in G' ($G' = \{g \times w |\ g \in G,\ w \in E^1\}$) that corresponds to a given element of G_1 can be shrunk in a certain way (condition I), then all the elements of G' can be shrunk simultaneously (condition II). Using this we show that if G_1 is countable and G satisfies condition I, then E^n/G is a factor of E^{n+1}. The result for n_g-frames referred to above is proved in Section 4 by first showing, using techniques of Andrews and Curtis [1], that the product of E^1 and a decomposition of E^n whose only nondegenerate element is a k-frame is E^{n+1}. The result is then obtained by noticing that G satisfies condition I. The case for certain null collections of cells is shown in Section 5 by using Bryant's work [6] and condition I.

2. Notation and terminology. The statement that G is an *upper semi-continuous decomposition of* E^n means that (1) G is a collection of sub-

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sets of E^n such that each point of E^n belongs to one and only one set of G and (2) if $g \in G$ and U is an open set in E^n containing g, then there exists an open set V in E^n such that $g \subset V$, $V \subset U$, and V is the union of elements of G. G is *monotone* if and only if each element of G is a compact continuum.

If G is a monotone upper semicontinuous decomposition of E^n, E^n/G denotes the associated decomposition space, P the projection map of E^n onto E^n/G, G_1 the set of nondegenerate elements of G, and G_1^* the union of the elements of G_1. We shall identify E^n and $E^n \times 0$, thus $E^n \subset E^{n+1}$. If $g \subset E^n$ and $\varepsilon > 0$, let $V^n(g, \varepsilon)$ denote the ε-neighborhood of g in E^n, the superscript emphasizing that the space in which the neighborhood is being defined is E^n.

If G is a monotone upper semicontinuous decomposition of E^n, let G' be $\{g \times w \mid g \in G \text{ and } w \in E^1\}$. Then G' is an upper semicontinuous decomposition of E^{n+1} and E^{n+1}/G' is topologically equivalent to $E^n/G \times E^1$. Thus, to show that $E^n/G \times E^1$ is topologically equivalent to E^{n+1}, it suffices to show that E^{n+1}/G' is topologically E^{n+1}. We will think of E^{n+1} as $E^n \times E^1$, and the E^1 coordinate will be referred to as the w-coordinate.

If A is a set, $\operatorname{Int} A$ denotes the interior of A, and $\bar{A}$ or $\operatorname{Cl} A$ denotes the closure of A. We denote the usual metric for E^n by d. A collection η of subsets of E^n is a *null collection* if for each positive number ε, there exists, at most, a finite number of sets of η of diameter more than ε.

3. Conditions I and II. Let G be a monotone upper semi-continuous decomposition of E^n, and consider the following two conditions. The identity map is denoted by id.

CONDITION I. *If $\varepsilon > 0$ and $g \in G_1$, there exists an isotopy $\mu_t^g (t \in [0,1])$ of E^{n+1} onto itself such that*

(0) *if* $w \in E^1$, $\operatorname{diam} \mu_1^g(g \times w) < \varepsilon$,

(1) $\mu_0^g = \text{id}$,

(2) *if* $t \in [0,1]$, $\mu_t^g = \text{id}$ *on* $\big(E^n - V^n(g, \varepsilon)\big) \times E^1$,

(3) *if* $g' \in G$ *and* $w \in E^1$, *then either*

$$\operatorname{diam} \mu_1^g(g' \times w) < \varepsilon\,, \quad \text{or} \quad \mu_1^g(g' \times w) \subset V^{n+1}(g' \times w, \varepsilon)\,,$$

(4) μ_1^g *is uniformly continuous,*

(5) *if* $t \in [0,1]$, μ_t^g *moves no w-coordinate by as much as* ε.

CONDITION II. *If $\varepsilon > 0$ and U is an open subset of E^n containing G_1^*, then there exists an isotopy f_t $(t \in [0,1])$ of E^{n+1} onto itself such that*

(i) $f_0 = \text{id}$,

(ii) *if* $t \in [0,1]$, $f_t = \text{id}$ *on* $(E^n - U) \times E^1$,

(iii) *if* $g \in G$ *and* $w \in E^1$, *then* $\operatorname{diam} f_1(g \times w) < \varepsilon$,

(iv) f_1 *is uniformly continuous,*

(v) *if* $t \in [0,1]$, f_t *moves no w-coordinate by as much as* ε.

Notice that Condition I states that for each element of G_1 a certain type of function exists, where as Condition II allows one to attack all of G_1.

THEOREM 1. *Suppose G is a monotone upper semicontinuous decomposition of E^n such that G_1 is countable. If G satisfies Condition* I, *then G satisfies Condition* II.

Proof. Let ε be a positive number, U an open set in E^n containing G_1^*, and assume each component of U is bounded. We will work with each component D of U which contains elements of G with diameter as much as ε and get an isotopy f_t^D $(t \in [0,1])$ of E^{n+1} onto itself such that $f_t^D = \text{id}$ on $(E^n - D) \times E^1$, if $g \in G$, $g \subset D$, and $w \in E^1$, then $\operatorname{diam} f_1^D(g \times w) < \varepsilon$, and f_t^D satisfies properties i, iv, and v of Condition II. Since each bounded subset of E^n intersects only a finite number of such components, f_t will be taken to be the composite of the isotopies f_t^D.

Thus let D be a component of U and $g_1, g_2, g_3, \ldots$ be the elements of G_1 which are contained in D. We will construct a sequence of isotopies $f_t^0, f_t^1, f_t^2, \ldots$ of E^{n+1} onto itself and a sequence $U_0, U_1, U_2, \ldots$ of open sets in E^n such that

(a) if i is a positive integer and $w \in E^1$,

$$\operatorname{diam} f_1^i(g_i \times w) < \varepsilon\,,$$

(b) $\big(\bigcup_{j \leqslant i} g_j\big) \subset U_i \subset U_{i+1} \subset D$ and $f_1^i = f_1^{i+1} = f_1^{i+2} = \ldots$ on $U_i \times E^1$,

(c) if $w \in E^1$ and i and j are positive integers, then

$$\operatorname{diam} f_1^i(g_j \times w) \leqslant \operatorname{diam} f_1^{i-1}(g_j \times w) + \varepsilon/2^{i+1}\,,$$

(d) $f_t^i = \text{id}$ outside $D \times E^1$,

(e) f_1^i is uniformly continuous,

(f) f_t^i moves no w coordinate by as much as ε.

Then, as in the proof of Theorem 3 of [3], it will follow that there exists a positive integer N such that if j is a positive integer and w is a real number, then $\operatorname{diam} f_1^N(g_j \times w) < \varepsilon$. We will take f_t^D to be f_t^N.

STEP 1. Let f_t^0 be the identity and $U_0 = \varphi$. It follows from Condition I that there exists an isotopy f_t^1 of E^{n+1} onto itself such that

(1a) if $w \in E^1$, $\operatorname{diam} f_1^1(g_1 \times w) < \varepsilon/2^2$,

(1b) $f_0^1 = \text{id}$,

(1c) if $w \in E^1$ and j is a positive integer, then

$$\operatorname{diam} f_1^1(g_j \times w) < \operatorname{diam} f_1^0(g_j \times w) + (\varepsilon/2^2)\,,$$

(1d) $f_t^1 = \text{id}$ outside $D \times E^1$,

(1e) f_1^1 is uniformly continuous,

(1f) f_t^1 moves no w coordinate by as much as $\varepsilon/2^2$.

Let U_1 be an open set in E^n such that $g_1 \subset U_1 \subset D$, and if $g \in G$ and $g \cap \overline{U}_1 \neq \varphi$, then for each w in E^1, diam $f_1^1(g \times w) < \varepsilon$. To see that such an open set U_1 exists, notice that f_1^1 is uniformly continuous. Therefore there exists a positive number δ such that if $d(p_1, p_2) < \delta$, then $d(f_1^1(p_1), f_1^1(p_2)) < \varepsilon/4$. By upper semicontinuity, there is an open set U_1 in E^n such that $g_1 \subset U_1 \subset D$, and if $g \in G$ and $g \cap \overline{U}_1 \neq \varphi$, then $g \subset V^n(g_1, \delta)$. Then if $g \in G$, $g \cap \overline{U}_1 \neq \varphi$, $w \in E^1$, $p_1 \in (g \times w)$, and $p_2 \in (g \times w)$, there exist points q_1 and q_2 of $g_1 \times w$ such that $d(p_1, q_1) < \delta$ and $d(p_2, q_2) < \delta$. Then

$$d(f_1^1(p_1), f_1^1(p_2)) \leqslant d(f_1^1(p_1), f_1^1(q_1)) + d(f_1^1(q_1), f_1^1(q_2)) + d(f_1^1(q_2), f_1^1(p_2)) < \varepsilon/4 + \varepsilon/4 + \varepsilon/4 < \varepsilon .$$

STEP 2. If $g_2 \cap \overline{U}_1 \neq \varphi$, let f_t^2 be f_t^1 and U_2 be an open set in E^n such that $U_1 \cup g_2 \subset U_2 \subset D$, and if $g \in G$, $w \in E^1$, and $g \cap \overline{U}_2 \neq \varphi$ then diam$f_1^2(g \times w) < \varepsilon$.

If $g_2 \cap \overline{U}_1 = \varphi$, let V_1 be an open set in E^n such that $g_2 \subset V_1 \subset D$, and if $g \in G$ and $g \cap \overline{U}_1 \neq \varphi$, then $g \cap V_1 = \varphi$.

Since f_1^1 is uniformly continuous, there exists a positive number δ_1 less than $\varepsilon/2^3$ such that if $S \subset E^{n+1}$ and diam $S < \delta_1$, then diam$f_1^1(S) < \varepsilon/2^4$.

Let h_t^2 be an isotopy of E^{n+1} onto itself such that

(2a)′ if $w \in E^1$, diam$h_1^2(g_2 \times w) < \delta_1$,

(2b)′ $h_0^2 = \text{id}$,

(2c)′ if j is a positive integer and $w \in E^1$, then

$$\text{either diam}\, h_1^2(g_j \times w) < \delta_1 \text{ or } h_1^2(g_j \times w) \subset V^{n+1}(g_j \times w, \delta_1) ,$$

(2d)′ $h_t^2 = \text{id}$ outside $V_1 \times E^1$,

(2e)′ h_1^2 is uniformly continuous, and

(2f)′ h_t^2 moves no w coordinate as much as δ_1.

Let $f_t^2 = f_1^1 h_t^2$ and U_2 be $U_1 \cup V_1$. Then we have

(2a) if $w \in E^1$, diam$f_1^2(g_2 \times w) < \varepsilon/2^4 < \varepsilon$,

(2b) $f_1^2 = f_1^1$ on $U_1 \times E^1$,

(2c) if $w \in E^1$ and j is a positive integer, then

$$\text{diam} f_1^2(g_j \times w) \leqslant \text{diam} f_1^1(g_j \times w) + (\varepsilon/2^3) ,$$

(2d) $f_t^2 = \text{id}$ outside $D \times E^1$,

(2e) f_1^2 is uniformly continuous, and

(2f) f_t^2 moves no w-coordinate by more than $\varepsilon/2^2 + \varepsilon/2^3$.

It is clear that one can continue inductively, thereby obtaining the sequences $f_t^0, f_t^1, f_t^2, \ldots$ of isotopies and $U_0, U_1, U_2 \ldots$ of open sets in E^n having properties (a)–(f).

We now show that there exists a positive integer N such that if j is a positive integer and $w \in E^1$, then diam$f_1^N(g_j \times w) < \varepsilon$. For suppose that this statement is false. Then for each positive integer i, there exists a positive integer n_i and a real number w_i such that diam$f_1^i(g_{n_i} \times w_i) \geqslant \varepsilon$. Without loss of generality we may assume that $\{g_{n_i}\}$ converges to a compact subset B of $\overline{D}$. Since G is an upper semicontinuous collection, there exists an element g_0 of G such that $B \subset g_0$. We show that g_0 is neither degenerate nor nondegenerate.

Notice that if i and j are positive integers and $w \in E^1$, then

$$\begin{aligned}\text{diam} f_1^i(g_j \times w) &\leqslant \varepsilon/2^{i+1} + \text{diam} f_1^{i-1}(g_j \times w) \\ &\leqslant \varepsilon/2^{i+1} + \varepsilon/2^i + \ldots + \varepsilon/2^2 + \text{diam} f_1^0(g_j \times w) \\ &\leqslant \varepsilon/2 + \text{diam}\,(g_j \times w) .\end{aligned}$$

Thus

$$\varepsilon \leqslant \text{diam} f_1^i(g_{n_i} \times w_i) < \varepsilon/2 + \text{diam}\,(g_{n_i} \times w_i) = \varepsilon/2 + \text{diam}\, g_{n_i}.$$

Therefore $\varepsilon/2 < \text{diam}\, g_{n_i}$, g_0 is not degenerate, and there exists a positive integer k such that $g_0 = g_k$.

Now there exists a neighborhood O of g_k in U_k such that if $g \in G$, $g \subset O$, and $w \in E^1$, then diam $f_1^k(g \times w) < \varepsilon$. Therefore, if i is a positive integer greater than k, $g_{n_i} \not\subset O$ because diam$f_1^i(g_{n_i} \times w_i) \geqslant \varepsilon$ and $f_1^i = f_1^k$ on O. But then $\{g_{n_i}\}$ does not converge to B, a contradiction. Our claim is established. We let f_t^D be f_t^N and f_t be the composite of the f_t^D's.

LEMMA 1. *Suppose that G satisfies the hypothesis of Theorem 1, $\varepsilon > 0$, U is an open set in E^n containing G_1^*, and f_t is an isotopy satisfying condition* II *relative to U and ε. There exists an open set V in E^n and a positive number γ such that*

(1) *$G_1^* \subset V \subset U$ and each component of $P(V)$ is of diameter less than ε (relative to the metric of E^n/G), and*

(2) *if u is a component of V and $0 < (b-a) < \gamma$, then*

$$\text{diam} f_1(u \times [a, b]) < 2\varepsilon .$$

Proof. Since f_t satisfies condition II, f_1 is uniformly continuous. Thus there exists a positive number $\gamma < \varepsilon/4$ such that if x and y belong to E^{n+1} and $d(x, y) < \gamma$, then $d(f_1(x), f_1(y)) < \varepsilon/4$. It follows that if $g \in G$ and $0 < (b-a) < \gamma$, then

$$\text{diam } f_1(V^n(g, \gamma) \times [a, b]) < 2\varepsilon .$$

Since G_1 is countable, there exists an open set V in E^n such that $G_1^* \subset V \subset U$, each component of $P(V)$ is of diameter less than ε (relative to the metric of E^n/G), and if u is a component of V, there is an element g_u of G_1 such that $u \subset V^n(g_u, \gamma)$. The lemma follows.

THEOREM 2. *Suppose G is a monotone upper semicontinuous decomposition of E^n such that G_1 is countable and G satisfies condition* II. *There exists a pseudo isotopy $f(x, t)$ $(x \in E^{n+1}, t \in [0, 1])$ of E^{n+1} onto itself such that*

(1) $f(x, 0) = x$,

(2) *if* $0 \leqslant t < 1$, $f(x, t)$ *is a homeomorphism of* E^{n+1} *onto itself, and*

(3) $f(x, 1)$ *takes* E^{n+1} *onto itself and each element of* G' *onto a distinct point.*

Proof. For each positive integer i, let $\varepsilon_i = 1/2^i$. There exists an isotopy $f(x, t)$ $(x \in E^{n+1}, 0 \leqslant t \leqslant 1/2)$ such that $f(x, 1/2)$ is uniformly continuous, $f(x, 0) = x$, if $g \in G$ and $w \in E^1$, $\operatorname{diam} f(g \times w, 1/2) < \varepsilon_1$, and $f(E^n \times w, 1/2) \subset E^n \times [w-\varepsilon_1, w+\varepsilon_1]$.

By Lemma 1, there exists an open set V_2 and a positive number γ_2 such that $G_1^* \subset V_2$, each component of $P(V_2)$ is of diameter less than ε_1, and if u is a component of V_2 and $0 < (b-a) < \gamma_2$, then $\operatorname{diam} f(u \times [a, b], 1/2) < 2\varepsilon_1$.

Let δ_2 be a positive number such that if $S \subset E^{n+1}$ and $\operatorname{diam} S < \delta_2$, then $\operatorname{diam} f(S, 1/2) < \varepsilon_2$. It follows, by condition II, that there exists an isotopy $h(x, t)$ $(x \in E^{n+1}, 1/2 \leqslant t \leqslant 2/3)$ such that $h(x, 1/2) = x$, $h(x, 2/3)$ is uniformly continuous, $h(x, t) = x$ on $(E^n - V_2) \times E^1$, if $g \in G$ and $w \in E^1$, $\operatorname{diam} h(g \times w, 2/3) < \delta_2$, and $h(x, t)$ moves no point in the w direction more than $\min\{\gamma_2/2, \varepsilon_2\}$.

If $1/2 \leqslant t \leqslant 2/3$, let $f(x, t) = f(h(x, t), 1/2)$. Then $f(x, 2/3)$ is uniformly continuous.

(1') if $x \notin V_2 \times E^1$ and $1/2 \leqslant t \leqslant 2/3$, $f(x, t) = f(x, 1/2)$,

and

(2') if $g \in G$ and $w \in E^1$, $\operatorname{diam} f(g \times w, 2/3) < \varepsilon_2$.

(3') If u is a component of V_2, $1/2 \leqslant t \leqslant 2/3$, and $w \in E^1$, then $h(u \times w, t) \subset u \times [w - \gamma_2/2, w + \gamma_2/2]$ and $\operatorname{diam} f(u \times [w - \gamma_2/2, w + \gamma_2/2], 1/2) < 2\varepsilon_1$. Thus no point moves more than $2\varepsilon_1$ during $f(x, t)$ $(1/2 \leqslant t \leqslant 2/3)$.

(4') If $w \in E^1$, $f(E^n \times w, 2/3) \subset f(E^n \times [w - \varepsilon_2, w + \varepsilon_2], 1/2)$.

Continuing inductively, we define $f(x, t)$ $((i-1)/i \leqslant t \leqslant i/(i+1))$ as follows. There exists an open set V_i and a positive number γ_i such that $G_1^* \subset V_i \subset V_{i-1}$, each component of $P(V_i)$ is of diameter less than ε_i, and if u is a component of V_i and $0 < (b-a) < \gamma_i$, then $\operatorname{diam} f(u \times [a, b], (i-1)/i) < 2\varepsilon_{i-1}$. Let δ_i be a positive number such that if $S \subset E^{n+1}$ and $\operatorname{diam} S < \delta_i$, then $\operatorname{diam} f(S, (i-1)/i) < \varepsilon_i$. By condition II, there exists an isotopy $h(x, t)$ $(x \in E^{n+1}, (i-1)/i \leqslant t \leqslant i/(i+1))$ such that $h(x, (i-1)/i) = x$, $h(x, i/(i+1))$ is uniformly continuous, $h(x, t) = x$ on $(E^n - V_i) \times E^1$, if $g \in G$ and $w \in E^1$, $\operatorname{diam} h(g \times w, i/(i+1)) < \delta_i$, and $h(x, t)$ moves no point in the w direction more than $\min\{\gamma_i/2, \varepsilon_i\}$.

If $(i-1)/i \leqslant t \leqslant i/(i+1)$, let $f(x, t) = f(h(x, t), (i-1)/i)$. Then $f(x, i/(i+1))$ is uniformly continuous,

(1) if $x \notin V_i \times E^1$ and $(i-1)/i \leqslant t \leqslant i/(i+1)$, $f(x, t) = f(x, (i-1)/i)$,

(2) if $g \in G$ and $w \in E^1$, $\operatorname{diam} f(g \times w, i/(i+1)) < \varepsilon_i$,

(3) no point moves more than $2\varepsilon_{i-1}$ during $f(x, t)((i-1)/i \leqslant t \leqslant i/(i+1))$, and

(4) if $w \in E^1$, $f(E^n \times w, i/(i+1)) \subset f(E^n \times [w - \varepsilon_i, w + \varepsilon_i], (i-1)/i)$.

We now have conditions satisfied which are analogous to conditions (1)–(4) in the proof of Theorem 3 of [5], and the proof can be completed using a similar argument.

Since E^{n+1}/G' and $E^n/G \times E^1$ are topologically equivalent, the following theorem is an immediate consequence of Theorems 1 and 2.

THEOREM 3. *Suppose G is a monotone upper semicontinuous decomposition of E^n, G_1 is countable, and G satisfies condition* I. *Then $E^n/G \times E^1$ is topologically E^{n+1}.*

4. Decomposition into frames. In this section it is shown in Theorem 4 that the product of E^1 and a decomposition of E^n whose only nondegenerate element is a k-frame is E^{n+1}. Theorem 5 follows by noticing that the shrinking which took place in the proof of Theorem 4 can be done without stretching certain sets a great deal. Thus condition I is satisfied and we can apply Theorem 3. Much of this section is completely analogous to parts of [1], thus we will be omitting many of the details.

Let k be a positive integer. *A k-frame* α_k is the union of k arcs $A_1, A_2, \ldots,$ and A_k, with a distinguished point p such that 1) if $k = 1$, p is an end point of A_1, and 2) if $k > 1$, p is an end point of each A_i and if $i \neq j$, $A_i \cap A_j = p$. If i is a positive integer, $1 \leqslant i \leqslant k$, let B_i be the arc in E^2 with polar coordinates $0 \leqslant r \leqslant 1$, $\theta = (2\pi/k)i$. The *standard k-frame* β_k is $\bigcup_{i=1}^{k} B_i$. A k-frame α_k in E^n is *tame* in E^n if there is a homeomorphism of E^n onto itself which carries α_k onto β_k.

LEMMA 2. *If α_k is a k-frame in E^n, then there exists a homeomorphism Φ of E^{n+1} onto itself such that $\Phi(\beta_k) = \alpha_k$.*

Proof. If, as in the notation above, $\alpha_k = \bigcup_{i=1}^{k} A_i$, it follows from Lemma 1 of [1] that each A_i is tame in E^{n+1}. Then by Theorem 1 of [7], α_k is tame in E^{n+1}. Thus the desired homeomorphism Φ exists.

Construction of neighborhoods of α_k. Assuming the notation of Lemma 2, we begin constructing certain neighborhoods of α_k

by first describing nice neighborhoods of β_k. These neighborhoods of β_k will be the union of a collection of $(n+1)$-cells as suggested in Figure 1 below. The homeomorphism Φ will map these neighborhoods onto neighborhoods (in E^{n+1}) of α_k.

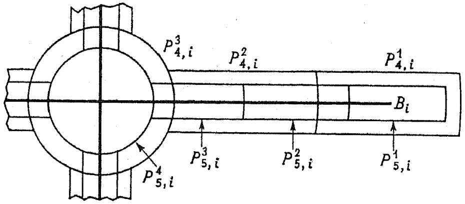

Fig. 1

If i, j, and s are integers such that $1 \leqslant i \leqslant k$, $j \geqslant 4$, and $1 \leqslant s \leqslant (j-2)$, let $\varepsilon_j = (j+2)/j(j-1)$, $\delta_j = (\varepsilon_j - 1/j)\sin(\pi/2k)$,

$$P^s_{j,i} = \big(V^2(B_i, \delta_j) \cap \{(r, \theta) |\ (j-s-1)\varepsilon_j - (1/j) \leqslant r \leqslant (j-s)\varepsilon_j - (1/j)\}\big) \times [-\delta_j, \delta_j]^{n-1},$$

and

$$P^{(j-1)}_{j,i} = V^2\big((0,0), \varepsilon_j - 1/j\big) \times [-\delta_j, \delta_j]^{n-1}.$$

Notice that $P^{j-1}_{j,i} = P^{j-1}_{j,w}$. Let $P_j = \bigcup \{P^s_{j,i} \mid 1 \leqslant i \leqslant k, 1 \leqslant s \leqslant (j-1)\}$, $Q_j = \Phi(P_j)$, $Q^s_{j,i} = \Phi(P^s_{j,i})$, $R_j = E^n \cap Q_j$, $R^s_{j,i} = E^n \cap Q^s_{j,i}$, $C_j = \{P^s_{j,i} \mid 1 \leqslant i \leqslant k, 1 \leqslant s \leqslant (j-1)\}$, $D_j = \{Q^s_{j,i} \mid 1 \leqslant i \leqslant k, 1 \leqslant s \leqslant (j-1)\}$, and $E_j = R^s_{j,i} \mid 1 \leqslant i \leqslant k, 1 \leqslant s \leqslant (j-1)\}$.

We will want these neighborhoods to have properties analogous to properties (i)–(iii) of [1] in order to allow us to build the appropriate $(n+1)$-cells. Recall that the construction of certain $(n+1)$-cells was accomplished in [1] by a push in the direction of the $(n+1)$-st axis. In particular, we can choose a subsequence $\{C_i\}$ of the C_j such that

(i) for each i and each $d \in D_i$, $(\operatorname{diam} d) < 1/i$,

(ii) for each i and each element d of D_{i+1}, there exist two adjacent elements e_1 and e_2 of E_i such that $d \subset (e_1 \cup e_2) \times E^1$, and

(iii) for each i and each element e of E_i,

a) there is an element d of D_{i+1} such that $d \subset e \times E^1$, and

b) if $\{e_1, e_2, \ldots, e_s = e\}$ is a chain of elements of E_i from one of the noncut points p of α_k to e, and $\{d_1, d_2, \ldots, d_r = d\}$ is the chain of elements of D_{i+1} from p to d, then $(\bigcup_{i=1}^{r} d_i) \subset (\bigcup_{i=1}^{s} e_i) \times E^1$.

Now let $T_i = R_{2i}$. Then $\{T_i\}$ will be the sequence of neighborhoods of α_k for which we can construct isotopies and apply Theorem 1 of [1].

Building certain $(n+1)$-cells. We now have hypothesis and notation satisfied so that Theorem 2 and Corollary 1 of [1] follow. Note also that if s is a positive integer, $\varepsilon > 0$, $a < b$, and E is the $(n+1)$-cell such that

$$(T_{s+1} \times [a, b]) \subset E \subset T_s \times [a-\varepsilon, b+\varepsilon],$$

then condition (iii) above implies that if i and j are positive integers such that $1 \leqslant j \leqslant k$, $1 \leqslant i < (2s-1)$, there exists two $(n+1)$-cells U_1 and U_2 such that $U_1 \cap U_2 = \operatorname{Bd} U_1 \cap \operatorname{Bd} U_2$ is an n-cell, $U_1 \cup U_2 = E$, $U_1 \subset (\bigcup_{q=1}^{i} R^q_{2s,j}) \times E^1$, and $U_2 \subset (T_s - \bigcup_{q=1}^{i-1} R^q_{2s,j}) \times E^1$.

Lemma 3. *Let $T_s \in \{T_i\}$. There exists an isotopy μ_t $(t \in [0,1])$ of E^{n+1} onto itself such that* (1) $\mu_0 = \mathrm{id}$, (2) μ_1 *is uniformly continuous*, (3) $\mu_t = \mathrm{id}$ *outside* $T_s \times E^1$, *and* (4) *if* $w \in E^1$, *there exist four elements* e_1, e_2, e_3, *and* e_4 *of* E_{2s} *such that* $\bigcup_{i=1}^{4} e_i$ *is connected and* $\mu_1(T_{s+(4s-7)} \times w) \subset (\bigcup_{i=1}^{4} e_i) \times [w-3(2s-1)k, w+3(2s-1)k]$.

Proof. In an effort to make things easier to follow, we shall go through the proof in detail for the case $k = 3$, i.e., α_k is a 3-frame or triod. The general case can be proved in a completely analogous fashion.

Recall that $T_s = R_{2s}$. Let m be $(2s-1)$. Our isotopy is built using the techniques of [1] and [5], and differs only in the sense that we first attempt to push everything in towards $R^m_{2s,1} \times E^1$ by the map h. We then adjust h to push things to "small size" in the sets on which h was fixed. It follows from Lemma 2 of [1] that our final map can be assumed to be an isotopy.

Step 1. Pushing in various levels of $A_1 \times w$. There exists a sequence of $(n+1)$-cells $K_{1,1}, K_{1,2}, \ldots$, and $K_{1,m-1}$ such that

$$\begin{aligned} T_s \times [0, 9m] &\supset K_{1,1} \supset \operatorname{Int} K_{1,1} \supset T_{s+1} \times [1, 9m-1] \\ &\supset K_{1,2} \supset \operatorname{Int} K_{1,2} \supset T_{s+2} \times [2, 9m-2] \ldots \\ &\supset K_{1,m-1} \supset \operatorname{Int} K_{1,m-1} \supset T_{s+(m-1)} \times [m-1, 9m-(m-1)]. \end{aligned}$$

There exists a homeomorphism $h_{1,1}$ of $E^n \times [0, 9m]$ onto itself such that $h_{1,1} = \mathrm{id}$ outside $K_{1,1}$, $h_{1,1} = \mathrm{id}$ on $\big(T_s - (R^1_{2s,1} \cup R^2_{2s,1})\big) \times [0, 9m]$, and

$$h_{1,1}\Big(\big(T_{s+1} \cap (R^1_{2s,1} \cup R^2_{2s,1})\big) \times [1, 9m-1]\Big) \subset R^2_{2s,1} \times [0, 9m].$$

Notice that $h_{1,1}$ may stretch $L_1 = \big(T_{s+1} \cap (R^1_{2s,1} \cup R^2_{2s,1})\big) \times ([0,1] \cup [9m-1, 9m])$ into $(R^1_{2s,1} \cup R^2_{2s,1}) \times [0, 9m]$, but future $h_{1,i}$'s, $1 < i$

$\leqslant (m-1)$, will leave $h_{1,1}(L_1)$ fixed. Also, if $1 \leqslant i \leqslant (m-1)$, $h_{1,i}$ will leave $\big(T_{s+1}-(R^1_{2s,1} \cup R^2_{2s,1})\big) \times ([0,1] \cup [9m-1, 9m])$ fixed.

Suppose $1 \leqslant i < (m-1)$, and $h_{1,1}, h_{1,2}, \ldots,$ and $h_{1,i-1}$ have been chosen. There exists a homeomorphism $h_{1,i}$ of $E^n \times [0, 9m)]$ onto itself such that $h_{1,i} = \mathrm{id}$ outside $h_{1,i-1}h_{1,i-2} \ldots h_{1,1}(K_{1,i})$, $h_{1,i} = \mathrm{id}$ on $\big(T_s - (\bigcup_{j=1}^{i+1} R^j_{2s,1})\big) \times [0, 9m]$, and

$$h_{1,i}h_{1,i-1} \ldots h_{1,1}\Big(\big(T_{s+i} \cap (\bigcup_{j=1}^{i+1} R^j_{2s,1})\big) \times [i, 9m-i]\Big) \subset R^{i+1}_{2s,1} \times [i-1, 9m-i+1].$$

Again, $h_{1,i}h_{1,i-1} \ldots h_{1,1}$ may stretch $L_i = \big(T_{s+i} \cap (\bigcup_{j=1}^{i+1} R^j_{2s,1})\big) \times ([i-1, i] \cup \cup [9m-i, 9m-i+1])$ into $(R^i_{2s,1} \cup R^{i+1}_{2s,1}) \times [0, 9m]$, but future $h_{1,j}$'s leave $h_{1,i}h_{1,i-1} \ldots h_{1,1}(L_i)$ fixed.

Continuing, we obtain the composite $h_{1,m-1}h_{1,m-2} \ldots h_{1,1}$. We extend this composite map vertically obtaining the homeomorphism h_1 of $E^n \times E^1$ onto itself. That is, if $q \in E^n \times E^1$, let q_n be the E^n coordinate of q and q_1 be the E^1 coordinate of q. Then if $(a, b) \in E^n \times [0, 9m]$ and r is an integer, let $h_1(a, b+r9m) = \big(h_{1,m-1}h_{1,m-2} \ldots h_{1,1}(a, b)_n, h_{1,m-1}h_{1,m-2} \ldots h_{1,1}(a, b)_1 + +r9m\big)$. h_1 copies $h_{1,m-1}h_{1,m-2} \ldots h_{1,1}$ on each block $E^n \times [r\,9m, (r+1)9m]$.

STEP 2. Pushing in various levels of $A_2 \times w$. There exist $(n+1)$-cells $K_{2,1}, K_{2,2}, \ldots,$ and $K_{2,m-1}$ such that

$$T_s \times [3m, 12m] \supset K_{2,1} \supset \mathrm{Int}\,K_{2,1} \supset T_{s+1} \times [3m+1, 12m-1] \supset \ldots$$
$$\supset K_{2,m-1} \supset \mathrm{Int}\,K_{2,m-1} \supset T_{s+(m-1)} \times [3m+(m-1), 12m-(m-1)].$$

Now, relative to the cells $K_{2,i}$ and the chain elements $R^i_{2s,2}$, define the functions $h_{2,i}$ in a completely analogous way as we did $h_{1,i}$. That is, if $1 \leqslant i < (m-1)$ and $h_{2,1}, h_{2,2}, \ldots,$ and $h_{2,i-1}$ have been chosen, let $h_{2,i}$ be a homeomorphism of $E^n \times [3m, 12m]$ onto itself such that $h_{2,i} = \mathrm{id}$ outside $h_{2,i-1}h_{2,i-2} \ldots h_{2,1}(K_{2,i})$, $h_{2,i} = \mathrm{id}$ on $(T_s - \bigcup_{j=1}^{i+1} R^j_{2s,2}) \times [3m, 12m]$, and

$$h_{2,i}h_{2,i-1} \ldots h_{2,1}\Big(\big(T_s \cap (\bigcup_{j=1}^{i+1} R^j_{2s,2})\big) \times [3m+i, 12m-i]\Big)$$
$$\subset R^{i+1}_{2s,2} \times [3m+(i-1), 12m-(i-1)].$$

We thus obtain the composite map $h_{2,m-1}h_{2,m-2} \ldots h_{2,1}$ which is a homeomorphism of $E^n \times [3m, 12m]$ onto itself. Extend this homeomorphism vertically obtaining the homeomorphism h_2 of $E^n \times E^1$ onto itself.

STEP 3. Pushing in various levels of $A_3 \times w$. There exists a sequence $K_{3,1}, K_{3,2}, \ldots,$ and $K_{3,m-1}$ of $(n+1)$-cells such that

$$T_s \times [6m, 15m] \supset K_{3,1} \supset \mathrm{Int}\,K_{3,1} \supset T_{s+1} \times [6m+1, 15m-1] \supset \ldots$$
$$\supset K_{3,m-1} \supset \mathrm{Int}\,K_{3,m-1} \supset T_{s+(m-1)} \times [6m+(m-1), 15m-(m-1)].$$

Now, relative to the cells $K_{3,i}$ and the chain elements $R^i_{2s,3}$, define the functions $h_{3,1}$ on $E^n \times [6m, 15m]$, and extend the composite vertically obtaining the homeomorphism h_3 of $E^n \times E^1$ onto itself.

Let h be the composite $h_1h_2h_3$. Let

$$W = \bigcup_{j=0}^{\infty} \big([3mj-(m-1), 3mj+(m-1)] \cup [-3mj-(m-1), -3mj+(m-1)]\big).$$

Notice that if $w \in E^1 - W$, $h(a_k \times w)$ is already "small" in the sense that $h(a_k \times w) \subset \big(R^m_{2s,1} \cup (\bigcup_{j=1}^{3} R^{m-1}_{2s,j})\big) \times [w-9m, w+9m]$. However, for $w \in W$, we can not make such a statement. In particular, $h(a_k \times 0)$ may be very "long" as h is fixed on $\big(a_k \cap (\bigcup_{j=1}^{m-1} R^j_{2s,1})\big) \times 0$. It remains to adjust h on parts of $E^n \times W$.

STEP 4. Adjusting h on components of $T_s \times W$. We shall describe how to adjust h on the subset $T_s \times [9m-(m-1), 9m+(m-1)]$ of $T_s \times W$.

There exists a sequence $K_{1,m}, K_{1,m+1}, \ldots,$ and $K_{1,2m-5}$ of $(n+1)$-cells such that

$$(\mathrm{Int}\,K_{3,m-1} \cap \mathrm{Int}\,K_{2,m-1}) \supset T_{s+(m-1)} \times [9m-(m-3), 9m+(m-3)]$$
$$\supset K_{1,m} \supset \mathrm{Int}\,K_{1,m} \supset T_{s+m} \times [9m-(m-4), 9m+(m-4)] \supset \ldots$$
$$\supset K_{1,2m-5} \supset \mathrm{Int}\,K_{1,2m-5} \supset T_{s+2m-5} \times [9m-1, 9m+1].$$

There exists a homeomorphism $h_{1,m}$ of $h\big(T_s \times [9m-(m-1), 9m+ +(m-1)]\big)$ onto itself such that $h_{1,m} = \mathrm{id}$ outside $h(K_{1,m})$, $h_{1,m} = \mathrm{id}$ on $h\Big(\big(T_s \cap (\bigcup_{j=1}^{m-2} R^j_{2s,1})\big) \times [9m-(m-1), 9m+(m-1)]\Big)$, and

$$h_{1,m}h\Big(\big(T_{s+m} \cap (T_s - \bigcup_{j=1}^{m-2} R^j_{2s,1})\big) \times [9m-(m-4), 9m+(m-4)]\Big)$$
$$\subset R^{m-1}_{2s,1} \times [9m-(m-3), 9m+(m-3)].$$

If $m < m+i \leqslant 2m-5$ and $h_{1,m}, h_{1,m+1}, \ldots,$ and $h_{1,m+i-1}$ have been defined, let $h_{1,m+i}$ be a homeomorphism of $h\big(T_s \times [9m-(m-1), 9m+ +(m-1)]\big)$ onto itself such that $h_{1,m+i} = \mathrm{id}$ outside $h_{1,m+i-1} \ldots h_{1,m}h(K_{1,i})$, $h_{1,m+i} = \mathrm{id}$ on

$$h_{1,m+i-1} \ldots h_{1,m}h\Big(\big(T_s \cap (\bigcup_{j=1}^{m-i-2} R^j_{2s,1})\big) \times [9m-(m-1), 9m+(m-1)]\Big),$$

and

$$h_{1,m+i}h_{1,m+i-1}\dots h_{1,m}h\Big(\big(T_{s+m+i}\cap(T_s-\bigcup_{j=1}^{m-i-2}R^j_{2s,1})\big)$$
$$\times[9m-(m-i-4),\,9m+(m-i-4)]\Big)$$
$$\subset R^{m-i+1}_{2s,1}\times[9m-(m-i-3),9m+(m-i-3)]\,.$$

The new map $h_{1,2m-5}\dots h_{1,m}h$ is the adjusted h on the subset $T_s\times\times[9m-(m-1),\,9m+(m-1)]$ of $T_s\times W$. Now for each component Q of W, adjust h on $T_s\times Q$ in a similar way, and let μ_1 be the resulting homeomorphism. Since μ_1 can be assumed to be the final stage of an isotopy μ_t such that $\mu_0=\mathrm{id}$, and μ_1 satisfies the conclusion of Lemma 3, this establishes the lemma for the special case $k=3$. The general case follows in much the same way; there are simply more "directions" in which one must push after pushing "almost everything" towards $R^m_{2s,1}\times E^1$.

THEOREM 4. *Suppose G is a decomposition of E^n whose only non-degenerate element is a k-frame. Then $E^n/G\times E^1$ is topologically E^{n+1}.*

Proof. This follows as in Corollary 4 and Theorem 1 of [1].

The following lemma is a technical lemma pointing out what happens to certain subsets of E^{n+1} under the homeomorphism h_1 described in the proof of Lemma 3. Analogous considerations for the maps of which the final μ_1 is the composite point out why we can require G to satisfy Condition I in the case when G satisfies the hypothesis of Theorem 5.

Continue to assume the above notation. If i is an integer, $0\leqslant i\leqslant m$, let N_i be $T_{s+i}\times[i,\,9m-i]$.

LEMMA 4. *If $0\leqslant i+2\leqslant m$ and $Q\subset(N_i-N_{i+2})$, then*

$$h_1(Q)\subset\big((R^{i+1}_{2s,1}\cup R^{i+2}_{2s,1})\times[0,\,9m]\big)\cup\Big(Q\cap\big((T_s\bigcup_{j=1}^{i+2}R^j_{2s,1})\times[0,\,9m]\big)\Big).$$

Proof. $N_{i+2}=T_{s+i+2}\times[i+2,\,9-(i+2)]\subset K_{1,i+2}$, and each of $h_{1,i+2},h_{1,i+3},\dots,$ and $h_{1,m-1}$ is the identity outside $h_{1,i+1}\dots h_{1,1}(K_{1,i+2})$. Since $Q\cap N_{i+2}=\varphi$, it follows that $h_1|\,Q=h_{1,i+1}h_{1,i}\dots h_{1,1}|\,Q$.

Let

$$Q'=Q\cap\Big(\big(T_{s+i}\cap(\bigcup_{j=1}^{i+2}R^j_{2s,1})\big)\times[i,\,9m-i]\Big)$$

and

$$Q''=Q\cap\big((T_{s+i}-\bigcup_{j=1}^{i+2}R^j_{2s,1})\times[i,\,9m-i]\big)\,.$$

$h_{1,i+1}h_{1,i}\dots h_{1,1}|\,Q''=\mathrm{id}$. Thus we need only check the result of $h_{1,i+1}h_{1,i}\dots h_{1,1}$ on Q'.

$h_{1,i+1}=\mathrm{id}$ outside $h_{1,i}h_{1,i-1}\dots h_{1,1}(K_{1,i+1})$, $h_{1,i+1}=\mathrm{id}$ on $(T_s-\bigcup_{j=1}^{i+2}R^j_{2s,1})\times[0,\,9m]$, and $K_{1,i+1}\subset N_i$. Therefore

$$h_{1,i+1}h_{1,i}\dots h_{1,1}\Big(\big(T_{s+i}\cap(\bigcup_{j=1}^{i+2}R^j_{2s,1})\big)\times[i,\,9m-i]\Big)$$
$$=h_{1,i}\dots h_{1,1}\Big(\big(T_{s+i}\cap(\bigcup_{j=1}^{i+2}R^j_{2s,1})\big)\times[i,\,9m-i]\Big)\subset(R^{i+1}_{2s,1}\cup R^{i+2}_{2s,1})\times[0,\,9m]\,.$$

Thus $h_1(Q)=h_1(Q'\cup Q'')\subset\big((R^{i+1}_{2s,1}\cup R^{i+2}_{2s,1})\times[0,\,9m]\big)\cup Q''$. This establishes the lemma.

We remark here that if ε is a positive number, Q is connected, and the diameter of the set $(R^{i+1}_{2s,1}\cup R^{i+2}_{2s,1})\times[0,\,9m]$ is less than ε, then in addition to the conclusion of Lemma 4 we would have that either $\operatorname{diam}h_1(Q)<\varepsilon$ or $h_1(Q)\subset V^{n+1}(Q,\varepsilon)$, depending upon whether $Q''=\varphi$ or $Q''\neq\varphi$.

THEOREM 5. *Suppose G is a monotone upper semicontinuous decomposition of E^n, G_1 is countable, and if $g\in G_1$, there exists a positive integer k_g such that g is a k_g-frame. Then $E^n/G\times E^1$ is topologically E^{n+1}.*

Proof. We wish to show that G satisfies condition I. The result will then follow from Theorem 3. Thus let $\varepsilon>0$ and $g\in G_1$. There exists a positive integer k such that g is a k-frame a_k.

Now let $\{T_i\}$ be the sequence of neighborhoods constructed above with the additional property (iv) that if $g'\in G$ and $g'\cap T_{i+1}\neq\varphi$, then $g'\subset T_i$. That this property can be assumed follows from the fact that G is an upper semicontinuous collection.

There exists a positive integer s such that $T_s\subset V^n(a_k,\varepsilon)$, and if i and j are integers such that $1\leqslant i\leqslant k$, $1\leqslant j\leqslant 2s-1$, then $(\operatorname{diam}R^j_{2s,i})<(\varepsilon/8)$. In addition, assume the w-scale has been adjusted so that $\operatorname{diam}[0,\,3km]<\varepsilon/8$. Then the isotopy μ_t of Lemma 3 has properties (0)—(5) of Condition I.

To see that property (3) of Condition I is satisfied, let $g'\in G$ such that $g'\cap T_{s+1}\neq\varphi$, and let $w\in[0,\,3km]$. Then by property (iv) above, there exists a positive integer n such that $g'\subset(T_{s+n}-T_{s+n+2})$.

It follows that either $g'\times w\subset T_{s+m-1}\times[m-1,\,3km-(m-1)]$, or there exists a positive integer j such that $0\leqslant j<(m-1)$ and $g'\times w\subset(T_{s+j}\times[j,\,3km-j])-\big(T_{s+j+2}\times[j+2,\,3km-(j+2)]\big)$. Then Lemma 4 indicates what happens to $g'\times w$ under the function h_1, which would be the first function of which the μ_1 of Lemma 3 is the composite.

Similar considerations for $g'\in G$, $w\in E^1$, and the functions of which μ_1 is the composite can be given to show that either $\mu_1(g'\times w)\subset V^{n+1}(g'\times w,\varepsilon)$ or $\operatorname{diam}\mu_1(g'\times w)<\varepsilon$.

An alternative way of describing the image of $g'\times w$ under μ_1 is the following. Let O_1 be $\{\mu_1(p)|p\in g'\times w,\,\mu_1(p)\neq p\}$ and O_2 be $\{\mu_1(p)|p$

$\in g' \times w$; $\mu_1(p) = p\}$. Then $\mu_1(g' \times w) = O_1 \cup O_2$. By considerations similar to those of Lemma 4, it can be shown that the components of O_1 have small diameter. Thus, if $O_2 \neq \varphi$, $\mu_1(g' \times w) \subset V^{n+1}(g' \times w, \varepsilon)$, and if $O_2 = \varphi$, $\operatorname{diam} \mu_1(g' \times w) < \varepsilon$. This completes the proof of Theorem 5.

5. The case for certain null collections of cells. In this section we show that if G is a monotone upper semicontinuous decomposition of E^n, G_1 is a null collection, and if $g \in G_1$, there exists a positive integer k_g such that g is a k_g-cell in E^n which is flat in E^{n+1}, then $E^n/G \times E^1$ is topologically E^{n+1}. The proof consists of noticing that with a slight adjustment of Bryant's techniques of proof in [6], we are able to show that G satisfies Condition I. Our result then follows from Theorem 3.

Let $I^k = \{(x_1, x_2, \dots, x_{n+1}) \in E^{n+1} |\ 0 \leqslant x_i \leqslant 1$, for $1 \leqslant i \leqslant k$, and $x_i = 0$, for $i > k\}$. A k-cell in E^n is the image of I^k under an embedding $f\colon I^k \to E^n \times 0 \subset E^{n+1}$. A k-cell D in E^{n+1} is *flat* in E^{n+1} if and only if there exists a homeomorphism F of E^{n+1} onto itself such that $F(D) = I^k$.

Given integers $1 \leqslant m \leqslant k \leqslant n$, consider $I^k = I^{k-m} \times I^m$, $I^0 = \{0\}$, and let $H'(n, k, m)$ denote the following statement:

Let $f\colon I^k \to E^n \times 0 \subset E^n \times E^1 = E^{n+1}$ be an embedding and $\eta = \{g_1, g_2, g_3, \dots\}$ be a null collection of mutually disjoint continua such that if j is a positive integer, $g_j \subset E^n - f(I^k)$. Then for each $\varepsilon > 0$, there exists an isotopy $h_t (t \in [0, 1])$ of E^{n+1} such that

(1) $h_0 = \mathrm{id}$,

(2) if $t \in [0, 1]$, $h_t = \mathrm{id}$ outside $V^{n+1}(f(I^k) \times E^1, \varepsilon)$,

(3) h_1 is uniformly continuous,

(4) if $t \in [0, 1]$, h_t changes E^1 coordinates less than ε,

(5) if $x \in I^{k-m}$ and $w \in E^1$, then $\operatorname{diam} h_1(f(x \times I^m) \times w) < \varepsilon$,

(6) if $w \in E^1$, there exists $y_w \in I^m$ such that for each $x \in I^{k-m}$,
$$h_1(f(x \times I^m) \times w) \subset V^{n+1}(f(x, y_w) \times w, \varepsilon), \quad \text{and}$$

(7) if $g \in \eta$ and $w \in E^1$, then either $h_1 |\ g \times w = \mathrm{id}$ or
$$\operatorname{diam} h_1(g \times w) < \varepsilon.$$

Notice that our statement $H'(n, k, m)$ differs from Bryant's [6] statement $H(n, k, m)$ only in that we require an additional property relative to the null collection η. Since the following discussion is analogous to that of [6], we shall include only a brief outline.

If m and p are positive integers, let A be $\{(a_1, a_2, \dots, a_k) |\ a_i$ is an integer, if $1 \leqslant i < k$, $1 \leqslant a_i \leqslant p$, and $1 \leqslant a_k \leqslant 2m\}$. Let
$$I_a = \left[\frac{a_1-1}{p}, \frac{a_1}{p}\right] \times \dots \times \left[\frac{a_{k-1}-1}{p}, \frac{a_{k-1}}{p}\right] \times \left[\frac{a_k-1}{2m}, \frac{a_k}{2m}\right].$$
Then $\{I_a |\ a \in A\}$ is a subdivision of I^k into rectangular cubes.

Let $f\colon I^k \to E^n \times 0 \subset E^n \times E^1 = E^{n+1}$ be an embedding. We shall assume through out the remainder of this section that $f(I^k)$ is flat in E^{n+1}. Thus let g be a homeomorphism of E^{n+1} onto itself such that $g|\ I^k = f$.

If $\varepsilon > 0$, let $\{P_a^\varepsilon |\ a \in A\}$ be a covering of I^k in E^{n+1} of $(n+1)$-cubes P_a^ε such that

(1) each P_a^ε is a product of closed intervals,

(2) $P^\varepsilon = \bigcup \{P_a^\varepsilon |\ a \in A\}$ is the $(n+1)$-cell $\operatorname{Cl}(V^{n+1}(I^k, \varepsilon))$,

(3) $P_a^\varepsilon \cap P_b^\varepsilon$ is a face of each,

(4) $P_a^\varepsilon \cap I^k = I_a$, and

(5) if $0 < \varepsilon' < \varepsilon$, then $P_a^{\varepsilon'} \subset P_a^\varepsilon$.

Let N_0 and N_0' be compact neighborhoods of $f(I^k)$ in E^n and δ_0, δ_0' and ε_0 be positive numbers such that
$$N_0' \times [-\delta_0', \delta_0'] \subset \operatorname{Int} g(P_0^\varepsilon) \subset g(P_0^\varepsilon) \subset \operatorname{Int}(N_0 \times [-\delta_0, \delta_0]).$$

Let ψ be a homeomorphism of $E^{n+1} = E^n \times E^1$ onto itself that changes only E^1 coordinates such that $\psi(N_0 \times [-\delta_0, \delta_0]) = N_0 \times [0, 2m-1]$ and $\psi(N_0' \times [-\delta_0', \delta_0']) = N_0' \times [1, 2m-2]$.

Let $Q^{\varepsilon_0} = \psi g(P^{\varepsilon_0})$; if $a \in A$, $Q_a^{\varepsilon_0} = \psi g(P_a^{\varepsilon_0})$; if $i = 1, 2, \dots, m-1$, let $Q_r^{\varepsilon_0} = \bigcup \{Q_a^{\varepsilon_0} |\ a_k \geqslant r\}$; and for $a' \in A' = \{(a_1', a_2', \dots, a_{k-1}') |\ 1 \leqslant a_i' \leqslant p$, a_i' an integer$\}$, let $R_{a'}^{\varepsilon_0}$ be $\cup \{Q_a^{\varepsilon_0} |\ a_i = a_i'$ for $1 \leqslant i < k\}$. Let γ be max $\{\operatorname{diam}((Q_{2i}^{\varepsilon_0} - Q_{2i+3}^{\varepsilon_0}) \cap R_{a'}^{\varepsilon_0}) |\ a' \in A'$ and $0 \leqslant i \leqslant (m-2)\}$, with $Q_0^{\varepsilon_0} = Q^{\varepsilon_0}$. Let η be a collection satisfying the hypothesis of $H'(n, k, m)$.

THEOREM 6. *There exist an isotopy $h_t (t \in [0, 1])$ of E^{n+1} onto itself and a sequence $N_1, N_2, \dots, N_m$ of compact neighborhoods of $f(I^k)$ in E^n such that $N_1 \subset N_0$ and $N_{i+1} \subset \operatorname{Int} N_i$;*

$$h_0 = \mathrm{id},$$
$$h_t |\ E^{n+1} - (N_1 \times [0, 2m-1]) = \mathrm{id},$$
$$h_t |\ Q_{2m}^{\varepsilon_0} = \mathrm{id},$$
$$h_t |\ Q_{2m-2}^{\varepsilon_0} \cap (E^{n+1} - (N_{m-1} \times [m-2, m+1])) = \mathrm{id},$$
$$\vdots$$
$$h_t |\ Q_4^{\varepsilon_0} \cap (E^{n+1} - (N_2 \times [1, 2m-2])) = \mathrm{id};$$
$$h_1(f(I^k) \times [m-1, m]) \subset Q_{2m-2}^{\varepsilon_0},$$
$$h_1(f(I^k) \times [m-2, m+1]) \subset Q_{2m-4}^{\varepsilon_0},$$
$$\vdots$$
$$h_1(f(I^k) \times [1, 2m-2]) \subset Q_2^{\varepsilon_0};$$

if $a' \in A'$, $h_t(R_{a'}^{\varepsilon_0}) = R_{a'}^{\varepsilon_0}$; *and if* $w \in [0, 2m-1]$ *and* j *is a positive integer, then either* $h_1| \; g_j \times w = \mathrm{id}$ *or* $\operatorname{diam} h_1(g_j \times w) < 3\gamma$.

Proof. Let N_1 be a compact neighborhood of $f(I^k)$ in E^n such that $N_1 \subset N_0'$ and if $g_j \in \eta$ and $g_j \cap N_1 \neq \varphi$, then $\operatorname{diam} g_j < \gamma$. If $1 \leqslant i \leqslant m-1$, let X_i be $f(I^k) \times [i, 2m-1-i]$. Let ε_1 and δ_1 be positive numbers and N_1 be a compact neighborhood of $f(I^k)$ in E^n such that $\varepsilon_1 < \varepsilon_0$ and

$$N_1' \times [-\delta_1, \delta_1] \subset \operatorname{Int} g(P^{\varepsilon_1}) \subset g(P^{\varepsilon_1}) \subset \operatorname{Int}(N_1 \times [-\delta_0', \delta_0']) .$$

Now adjust ψ on $\operatorname{Int} g(P^{\varepsilon_0})$ so that $\psi(N_1' \times [-\delta_1, \delta_1]) = N_1' \times [1, 2m-2]$. Thus $X_1 \subset \operatorname{Int} g(P^{\varepsilon_1})$, and if $g_j \in \eta$ and $g_j \cap g(P^{\varepsilon_1}) \neq \varphi$, then $(\operatorname{diam} g_j) < \gamma$. Assume definitions of Q^{ε_1}, $Q_a^{\varepsilon_1}$, $Q_r^{\varepsilon_1}$, and $R_{a'}^{\varepsilon_1}$ for $\psi g(P^{\varepsilon_1})$ analogous to those defined above for $\psi g(P^{\varepsilon_0})$. Notice that in each case we have

$$Q_a^{\varepsilon_1} \subset Q_a^{\varepsilon_0},\ Q_r^{\varepsilon_1} \subset Q_r^{\varepsilon_0}, \quad \text{and } R_{a'}^{\varepsilon_1} \subset R_{a'}^{\varepsilon_0} .$$

There exists an isotopy $h_t^1 (t \in [0, 1])$ of E^{n+1} onto itself such that $h_0^1 = \mathrm{id}$, $h_1^1(X_1) \subset \operatorname{Int} Q_2^{\varepsilon_1}$, and if $t \in [0, 1]$ and $a' \in A'$, then $h_t^1 = \mathrm{id}$ on $(E^{n+1} - Q^{\varepsilon_1}) \cup Q_3^{\varepsilon_1}$ and $h_t^1(R_{a'}^{\varepsilon_1}) = R_{a'}^{\varepsilon_1}$. Notice that if $x \in E^{n+1}$, then $d(x, h_t^1(x)) < \gamma$. Thus if h_1^1 is not the identity on some $g_j \times w$, then $\operatorname{diam} h_1^1(g_j \times w) < 3\gamma$.

Let N_2 be a closed neighborhood of $f(I^k)$ in E^n such that $N_2 \subset N_1'$ and if $g_j \in \eta$, $g_j \cap N_2 \neq \varphi$, and $w \in E^1$, then $\operatorname{diam} h_1^1(g_j \times w) < \eta$. Let ε_2 and δ_2 be positive numbers and N_2' be a compact neighborhood of $f(I^k)$ in E^n such that $\varepsilon_2 < \varepsilon_1$ and

$$N_2' \times [\delta_2, \delta_2] \subset \operatorname{Int} g(P^{\varepsilon_2}) \subset P^{\varepsilon_2} \subset \operatorname{Int} N_2 \times [-\delta_1, \delta_1] .$$

Again assume ψ has been adjusted on $\operatorname{Int} g(P^{\varepsilon_1})$ so that $\psi(N_2' \times [-\delta_2, \delta_2]) = N_2' \times [2, 2m-3]$, define Q^{ε_2}, $Q_a^{\varepsilon_2}$, $Q_r^{\varepsilon_2}$, and $R_{a'}^{\varepsilon_2}$, and assume that Q^{ε_2} has been chosen so that $h_1^1(Q^{\varepsilon_2}) \subset Q_2^{\varepsilon_1}$.

There exists an isotopy $h_t^2 (t \in [0, 1])$ of E^{n+1} onto itself such that $h_0^2 = h_1^1$, $h_1^2 h_1^1(X_2) \subset Q_4^{\varepsilon_2}$, and if $t \in [0, 1]$ and $a' \in A'$, then $h_t^2 = h_1^1$ on $\big(E^{n+1} - h_1^1(Q^{\varepsilon_2})\big) \cup Q_5^{\varepsilon_2}$ and $h_t^2(R_{a'}^{\varepsilon_2}) = R_{a'}^{\varepsilon_2}$. Notice that if $h_1^2(y) \neq h_1^1(y)$, then $d\big(h_1^2 h_1^1(y), h_1^1(y)\big) \leqslant \gamma$. In particular, if $g_j \in \eta$ and $w \in E^1$, then $h_1^2 = h_1^1$ on $g_j \times w$ or $\operatorname{diam} h_1^2 h_1^1(g_j \times w) < 3\gamma$. Thus if $g_j \in \eta$, either $h_1^2 h_1^1| \; g_j \times w$ is the identity or $\operatorname{diam} h_1^2 h_1^1(g_j \times w) < 3\gamma$.

One continues inductively in this manner, obtaining the final isotopy $h_t (t \in [0, 1])$ as a composite of isotopies.

As mentioned above, the following lemmas follow as in [6]. Lemmas 5 and 6 imply that under the hypothesis of Theorem 6, Condition I is satisfied. Then Theorem 6 follows from Theorem 3.

LEMMA 5. *If* n *and* k *are positive integers,* $1 \leqslant k \leqslant n$, *and* $f\colon I^k \to E^n \times 0 \subset E^{n+1}$ *is an embedding such that* $f(I^k)$ *is flat in* E^{n+1}, *then* $H'(n, k, 1)$ *is true.*

LEMMA 6. *If* n *and* k *are positive integers,* $1 \leqslant k \leqslant n$, *then* $H'(n, k, 1)$ *implies* $H'(n, k, k)$.

THEOREM 7. *If* G *is an upper semicontinuous decomposition of* E^n, G_1 *is a null collection, and if* $g \in G_1$, *there exists a positive integer* k_g *such that* g *is a* k_g*-cell in* E^n *which is flat in* E^{n+1}, *then* $E^n/G \times E^1$ *is topologically* E^{n+1}.

COROLLARY 1. *If* G *is an upper semicontinuous decomposition of* E^3, G_1 *is a null collection, and if* $g \in G_1$, *there is an integer* $i_g \in \{1, 2, 3\}$ *such that* g *is an* i_g*-cell, then* $E^3/G \times E^1$ *is topologically* E^4.

Proof. Cells of dimension 1, 2 or 3 in E^3 are flat in E^4. See [1] and [9].

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