# On extending homeomorphisms on zero-dimensional spaces *

by

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**1. Introduction and notation.** A metric space is zero-dimensional if for each $p \in X$ and each $\varepsilon > 0$, there exists an open and closed set $U$ such that $p \in U$, and $\operatorname{diam} U < \varepsilon$. In this paper we shall be concerned primarily with zero-dimensional separable metric spaces. We define a 0-space to be a non-null non-degenerate zero-dimensional separable metric space.

In considering the problem of extending homeomorphisms on topological spaces, a basic question is whether or not each single point homeomorphism is extendable, i.e. for $p, q \in X$ whether or not there exists a homeomorphism $F$ of $X$ onto $X$ such that $F(p) = q$. A space having the above property is said to be *homogeneous*. If the space is not homogeneous, then, inevitably, special cases arise in formulating general homeomorphism extension theorems. Therefore, in what follows, we shall be primarily concerned with 0-spaces which are homogeneous and, indeed, which we show to satisfy a much stronger homogeneity property.

A topological space is said to be *perfect* if every point is a limit point of the space. Under the requirement that a space $X$ be homogeneous, it is clear that if $X$ has one limit point, then $X$ is perfect; also if $X$ is locally compact at one point, then $X$ is locally compact.

We define a separable metric space $X$ to be an *absolute* $G_\delta$ if $X$ is a $G_\delta$ subset of a compact metric space. Under the condition of separability, this definition is equivalent to the standard one as found in Kelley [1]. It is known that any such absolute $G_\delta$ space is homeomorphic to a complete metric space; thus such a space is called topologically complete. However, the fact that an absolute $G_\delta$ separable metric space is metrizable as a complete metric space is not used in this paper. Indeed, for the special types of zero-dimensional spaces, this result follows as a corollary. Further, we note that if a space $X$ is countable and perfect, then $X$ is not locally

* The results of this paper are contained in the author's Master's Thesis written at Louisiana State University.

compact and not topologically complete. Also, either compactness or local compactness implies topological completeness.

The preceding remarks may be considered to provide a proof of the following theorem which gives a classification of homogeneous 0-spaces.

THEOREM 1.1. *If $X$ is a homogeneous 0-space, then exactly one of the following is true:*

(1) *$X$ is discrete;*

(2) *$X$ is perfect and compact;*

(3) *$X$ is perfect and locally compact, but not compact;*

(4) *$X$ is topologically complete (an absolute $G_\delta$) and is nowhere locally compact;*

(5) *$X$ is countable and perfect;*

(6) *$X$ is not countable and not topologically complete.*

In what follows we shall be primarily concerned with 0-spaces of types (1) to (5), but without the assumption of homogeneity. In fact we prove a theorem of which homogeneity of all 0-spaces of types (1) to (5) is a trivial corollary.

As examples of 0-spaces of types (1) to (5), we have:

(1) Any non-null, non-degenerate subspace of the space of integers;

(2) The Cantor Set;

(3) The Cantor Set minus a point;

(4) The space of irrationals on the line;

(5) The space of rationals on the line.

The methods developed in this paper are, in general, only applicable to 0-spaces of types (1) to (5). However, the product of the rationals with the irrationals and the product of the rationals with the Cantor Set, both with the product topology, supply us with two examples of homogeneous, uncountable and non-topologically complete 0-spaces of type (6). Note that these two examples of 0-spaces of type (6) are not homeomorphic to each other since the latter is the countable union of compact sets, while the former is not.

Since we wish to derive, as corollaries to the main theorems, that certain 0-spaces are homeomorphic, we state all theorems as if we are dealing with two distinct 0-spaces or two copies of the same space.

DEFINITION. Two 0-spaces $X$ and $Y$ are said to be *compatible* if $X$ and $Y$ are both of the same type (1) through (5) as given in the conclusion of Theorem 1.1. In this definition neither $X$ nor $Y$ is assumed to be homogeneous.

It is a consequence of simple limit point considerations as to whether or not a homeomorphism on an arbitrary set can be homeomorphically extended to the closure of the set. Hence in what follows we consider the domains of the given homeomorphisms to be closed.



The principal result of this paper is the following homeomorphism extension theorem. The proof involves a number of lemmas and theorems to be given later.

THEOREM 1.2. *If $X$ and $Y$ are compatible 0-spaces, if $H$ and $K$ are closed proper subsets of $X$ and $Y$ respectively, and if $f$ is a homeomorphism carrying $H$ onto $K$ then $f$ can be extended to a homeomorphism $F$ of $X$ onto $Y$ if and only if* (A) *$f$ carries $\overline{X\setminus H}\cap H$ onto $\overline{Y\setminus K}\cap K$ and* (B) *$\overline{X\setminus H}$ and $\overline{Y\setminus K}$ are homeomorphic. Moreover, condition* (A) *is unnecessary for 0-spaces of type* (1) *and condition* (B) *is unnecessary for 0-spaces of types* (2), (4), *and* (5).

For $X$ and $Y$ of type (1), the proof of the theorem on the basis of condition (B) above is obvious. It depends only on the cardinality of $\overline{X\setminus H}$ and the cardinality of $\overline{Y\setminus K}$. Since under a homeomorphism of $X$ onto $Y$ open sets are carried onto open sets, it is immediate that for all other types, condition (A) is a necessary condition for $f$ to be extendable. The proof of sufficiency for types (2) through (5) reduces to the following theorem by making three observations. First, if $f|_{\overline{X\setminus H}\cap H}$ can be extended to $\overline{X\setminus H}$, then $f$ can be extended to $X$. Secondly, $\overline{X\setminus H}\cap H$ and $\overline{Y\setminus K}\cap K$ are closed and nowhere dense in $\overline{X\setminus H}$ and $\overline{Y\setminus K}$ respectively. Thirdly, $\overline{X\setminus H}$ and $\overline{Y\setminus K}$ are compatible 0-spaces. In the case where $X$ and $Y$ are of type (3), condition (B) of the theorem insures us that $\overline{X\setminus H}$ and $\overline{Y\setminus K}$ are compatible, both being of type (2) or both being of type (3).

THEOREM 1.3. *If $X$ and $Y$ are compatible 0-spaces of type* (2), (3), (4), *or* (5), *if $H$ and $K$ are closed nowhere dense (possibly null) subsets of $X$ and $Y$ respectively, and if $f$ is a homeomorphism carrying $H$ onto $K$, then $f$ has a homeomorphic extension $F$ of $X$ onto $Y$.*

In Section 2, we give lemmas which are generally applicable to all 0-spaces under consideration. In Section 3, we prove the special cases of Theorem 1.3 on the basis of the lemmas given in Section 2.

The particular case of Theorem 1.3 where $X$ and $Y$ are compact and perfect has already been proved by Knaster and Reichbach [2]. However, this case is included here as a special case of the methods developed. As a special case of Theorem 1.3, where $H$ and $K$ are regarded as null, we have the well known characterization of certain types of 0-spaces; that is, *every pair of non-discrete compatible 0-spaces are homeomorphic.* See, for example, [3] and [4, p. 287] for theorems on 0-spaces of types 4 and 5 respectively.

It is obvious that cardinality alone determines when two discrete 0-spaces are homeomorphic. Thus, any 0-space of type (1) to (5) is homeomorphic to one of the examples of 0-spaces as cited earlier.

At this point we give special definitions and notation which will be used in this paper. A *clopen* set refers to a set which is both open and closed, and by a *clopen cover* of a space, we mean a cover consisting of non-null clopen sets. Also, a cover will be called *disjoint* if its elements are pairwise disjoint. If $U$ is a subset of a metric space $X$ where the distance, $d$, between any two points of $U$ is bounded, then diam $U = \sup\{d(x,y)\colon x, y \in U\}$, and if $\mathcal{U}$ is a collection of subsets of $X$ with bounded diameters, then mesh $\mathcal{U} = \sup\{\text{diam}\, U\colon U \in \mathcal{U}\}$. If $\mathcal{U}$ and $\mathcal{M}$ are covers of a space $X$, then $\mathcal{U}$ is a *refinement* of $\mathcal{M}$ if for each element $U \in \mathcal{U}$, there exists an element $V \in \mathcal{M}$ such that $U \subset V$. If $\mathcal{U}$ is a disjoint cover of a space $X$, then for $p \in X$, $U(p)$ will refer to the element of $\mathcal{U}$ containing $p$. If a cover $\mathcal{U}$ is asserted to exist, we say it is *arbitrarily finite* if there exists $N \geqslant 1$ such that for each $n \geqslant N$, $\mathcal{U}$ can be constructed in such a manner as to consist of exactly $n$ elements. If $X$ is a metric space and $p \in X$, then $S_\varepsilon(p) = \{x \in X\colon d(x,p) < \varepsilon\}$. $X$ is defined to be a *0-subset* of $Y$ if $X \subset Y$ and $X$ is separable as a space and zero-dimensional in itself.

**2. Lemmas.** In this section we state lemmas on which the proofs of the theorems in Section 3 will be based. Lemmas 2.1 through 2.5 are either known in forms similar to the ones stated in this paper or the techniques used in proving them are standard topological techniques. Thus, the proofs of these lemmas are omitted. The lemmas are given in their natural order although only Lemma 2.2 and Lemma 2.5 are actually quoted in later proofs.

LEMMA 2.1. *If $X$ is a 0-subset of a metric space $Y$ and $F$ is a closed subset of $Y$ such that $X \cap F = \emptyset$, then for each $\varepsilon > 0$, there exists a countable clopen basis $\mathcal{B}$ of $X$ such that mesh $\mathcal{B} \leqslant \varepsilon$ and if $B \in \mathcal{B}$, then there exists an open set $B' \subset Y$ such that $B = B' \cap X$ and $\overline{B'} \cap F = \emptyset$.*

LEMMA 2.2. *If $X$ is a non-compact [compact and perfect] 0-subset of a metric space $Y$ and $F$ is a closed subset of $Y$ such that $X \cap F = \emptyset$, then for each $\varepsilon > 0$, there exists a disjoint clopen cover $\mathcal{M}$ of $X$ such that:*

1) *mesh $\mathcal{M} \leqslant \varepsilon$,*

2) *$\mathcal{M}$ is countably infinite [arbitrarily finite],*

3) *If $M \in \mathcal{M}$, then $M = M' \cap X$ where $M'$ is open in $Y$ and $\overline{M'} \cap F = \emptyset$.*

LEMMA 2.3. *If $A$ and $B$ are disjoint closed subsets of a 0-space $X$, then there exists a clopen set $U$ such that $A \subset U$ and $U \cap B = \emptyset$.*

LEMMA 2.4. *Let $X$ be a 0-space and let $H$ be a closed nowhere dense subset of $X$. If $\mathcal{U}' = \{U'_i\}_{i=1}^{a}$, $a = \infty$ ($a$ is finite), is a disjoint clopen, relative to $H$, cover of $H$ then there exists a disjoint clopen cover $\mathcal{U} = \{U_i\}_{i=0}^{a}$ of $X$ where for each $i \geqslant 1$, $U'_i = U_i \cap H$.*



LEMMA 2.5. *If in addition to the hypothesis of Lemma 2.4, $\mathcal{M} = \{M_i\}_{i=1}^{\beta}$ is a disjoint clopen cover of $X$ and $\mathcal{U}'$ is a refinement of $\mathcal{M}$, then we can require that if $U'_i \subset M_j$, then $U_i \subset M_j$ and for each $j \geqslant 1$, $M_j \cap U_0 \neq \emptyset$.*

While Lemmas 2.1 to 2.5 were concerned with only a single space and one subset of it, Lemmas 2.6 to 2.9 are stated for two spaces and a subset of each.

LEMMA 2.6. *Let $X$ and $Y$ be non-compact (compact and perfect) 0-spaces, and let $H$ and $K$ be closed nowhere dense subsets of $X$ and $Y$ respectively. If $f$ is a homeomorphism of $H$ onto $K$, then for each $\varepsilon > 0$, there exist disjoint clopen covers $\mathcal{A}$ and $\mathcal{B}$ of $X$ and $Y$ respectively and a one-to-one onto function $h\colon \mathcal{A} \to \mathcal{B}$ such that:*

1) *mesh $\mathcal{A} \leqslant \varepsilon$ and mesh $\mathcal{B} \leqslant \varepsilon$,*

2) *for each $p \in H$, $h(A(p)) = B(f(p))$.*

Proof. By Lemma 2.2, there exist disjoint clopen covers $\mathcal{A}'$ and $\mathcal{B}'$ of $X$ and $Y$ respectively such that mesh $\mathcal{A}' \leqslant \varepsilon$, mesh $\mathcal{B}' \leqslant \varepsilon$ and $\mathcal{A}'$ and $\mathcal{B}'$ are countably infinite (arbitrarily finite). Thus let $\mathcal{A}' = \{A'_i\}_{i=1}^{a}$ and $\mathcal{B}' = \{B'_i\}_{i=1}^{a}$ where $a = \infty$ ($a$ is finite). For each $i \geqslant 1$ such that $A'_i \cap H \neq \emptyset$, $f(A_i \cap H)$ is a clopen set in $K$ since $A'_i \cap H$ is clopen in $H$ and $f$ is a homeomorphism of $H$ onto $K$. Let $\{B'_{ij}\colon j = 1, \ldots, \beta_i\} = \{f(A'_i \cap H) \cap B'_k\colon k \geqslant 1$ and $f(A'_i \cap H) \cap B'_k \neq \emptyset\}$. For each $j \geqslant 1$, let $A'_{i,j} = f^{-1}(B'_{i,j})$. $\{A'_{i,j}\colon i, j \geqslant 1\}$ is a disjoint clopen, relative to $H$, cover of $H$ and $\{B'_{i,j}\colon i, j \geqslant 1\}$ is a disjoint clopen, relative to $K$, cover of $K$. Hence, by Lemma 2.5 there exist disjoint clopen covers $\mathcal{A}_0 = \{A_0, A_{i,j}\colon i, j \geqslant 1\}$ and $\mathcal{B}_0 = \{B_0, B_{i,j}\colon i, j \geqslant 1\}$ of $X$ and $Y$ respectively such that:

1) for each pair $i, j \geqslant 1$, $A'_{i,j} = A_{i,j} \cap H$, $B'_{i,j} = B_{i,j} \cap K$;

2) if $A'_{i,j} \subset A'_m$ and $B'_{i,j} \subset B'_n$, then $A_{i,j} \subset A'_m$ and $B_{i,j} \subset B'_n$;

3) for each $i \geqslant 1$, $A'_i \cap A_0 \neq \emptyset$ and $B'_i \cap B_0 \neq \emptyset$.

Let $\mathcal{A}_1 = \{A_{i,j}\colon i, j \geqslant 1\}$, $\mathcal{B}_1 = \{B_{i,j}\colon i, j \geqslant 1\}$ and define $h\colon \mathcal{A}_1 \to \mathcal{B}_1$ by $h(A_{i,j}) = B_{i,j}$. Let $\mathcal{A}_2 = \{A'_i \cap A_0\colon i \geqslant 1\}$, $\mathcal{B}_2 = \{B'_i \cap B_0\colon i \geqslant 1\}$ and define $h\colon \mathcal{A}_2 \to \mathcal{B}_2$ in any arbitrary one-to-one and onto fashion. Then $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$, $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$ and $h\colon \mathcal{A} \to \mathcal{B}$ satisfy the properties as stated in the lemma.

LEMMA. 2.7. *If in addition to the hypothesis of Lemma 2.6, we have that $F$ and $Y$ are subsets of a metric space $Y'$ with $F$ closed and $F \cap Y = \emptyset$, then the conclusion of Lemma 2.6 can be strengthened to assert, additionally, that for each $B \in \mathcal{B}$, there is an open set $N$ in $Y'$ such that $B = N \cap Y$ and $\overline{N} \cap F = \emptyset$.*

Proof. We refer to the proof of Lemma 2.6. By using the fuller strength of Lemma 2.2, we can require that for each $B' \in \mathcal{B}'$, there exists an open set $N'$ in $Y'$ such that $B' = N' \cap Y$ and $\overline{N'} \cap F = \emptyset$. Since for each $B \in \mathcal{B}$, there exists a $B' \in \mathcal{B}'$ such that $B \subset B'$, we can find an open set $N$ in $Y'$ such that $B = N \cap Y$ and $N \subset N'$. Hence, if $\overline{N}' \cap F = \emptyset$, then $\overline{N} \cap F = \emptyset$.

In setting up the apparatus we shall need for the extension of homeomorphisms on the 0-spaces under consideration, the following lemma will play an important role. Since this is the case, it will be stated in full, and although the statement is long, the proof esentially involves no more than the inductive use of Lemmas 2.6 and 2.7.

LEMMA 2.8. *Let $X$ be a nowhere locally compact (compact and perfect) 0-space, and let $Y$ be a nowhere locally compact (compact and perfect) 0-subset of a metric space $Y'$. Let $\{F_n\}_{n\geqslant 1}$ be a collection of closed subsets of $Y'$ where for each $n \geqslant 1$, $Y \cap F_n = \emptyset$. Let $H$ and $K$ be closed nowhere dense subsets of $X$ and $Y$ respectively, and let $f$ be a homeomorphism of $H$ onto $K$. Then there exist sequences $(\mathcal{A}_n)_{n\geqslant 1}$ and $(\mathcal{B}_n)_{n\geqslant 1}$ of disjoint clopen covers of $X$ and $Y$ respectively and a sequence $(h_n)_{n\geqslant 1}$ where for each $n \geqslant 1$, $h_n$ is a one-to-one function of $\mathcal{A}_n$ onto $\mathcal{B}_n$ such that:*

1) *mesh $\mathcal{A}_n \to 0$, mesh $\mathcal{B}_n \to 0$;*

2) *$\mathcal{A}_{n+1}$ and $\mathcal{B}_{n+1}$ are refinements of $\mathcal{A}_n$ and $\mathcal{B}_n$ respectively;*

3) *for each $n \geqslant 1$, $A_{n+1} \subset A_n$ if and only if $h_{n+1}(A_{n+1}) \subset h_n(A_n)$;*

4) *for each $p \in H$ and each $n \geqslant 1$, $h_n(A_n(p)) = B_n(f(p))$;*

5) *if $B_n \in \mathcal{B}_n$, then there exists an open set $B'_n$ in $Y'$ such that $B_n = B'_n \cap Y$ and $\overline{B'_n} \cap F_n = \emptyset$.*

Proof. The hypothesis of Lemma 2.7 is satisfied, hence there exist disjoint clopen covers $\mathcal{A}_1$ and $\mathcal{B}_1$ of $X$ and $Y$ respectively and a one-to-one function $h_1$ of $\mathcal{A}_1$ onto $\mathcal{B}_1$ such that mesh $\mathcal{A}_1 \leqslant 1$, mesh $\mathcal{B}_1 \leqslant 1$ and for each $p \in H$, $h(A_1(p)) = B_1(f(p))$. Since $X$ and $Y$ are nowhere locally compact (compact and perfect), each clopen subset is non-compact (compact and perfect), hence we may reapply Lemma 2.7 to each $A_1 \in \mathcal{A}_1$ and $h_1(A_1) \in \mathcal{B}_1$. In this way, using induction, we may construct sequences of disjoint clopen covers $(\mathcal{A}_n)$ and $(\mathcal{B}_n)$ of $X$ and $Y$ respectively and a sequence of one-to-one functions $(h_n)$ all having the desired properties.

The following lemma gives conditions for two 0-spaces to be homeomorphic and actually a method for constructing the homeomorphism. Once the lemma is stated properly, the proof is merely an easy verification that the function defined is a homeomorphism.

LEMMA 2.9. *If $X$ and $Y$ are 0-spaces and there exist sequences $\mathcal{A} = (\mathcal{A}_n)_{n\geqslant 1}$ and $\mathcal{B} = (\mathcal{B}_n)_{n\geqslant 1}$ of disjoint clopen covers of $X$ and $Y$ respectively and a sequence $(h_n)_{n\geqslant 1}$ having the following properties:*

1) *mesh $\mathcal{A}_n \to 0$ and mesh $\mathcal{B}_n \to 0$;*

2) *$\mathcal{A}_{n+1}$ and $\mathcal{B}_{n+1}$ are refinements of $\mathcal{A}_n$ and $\mathcal{B}_n$ respectively;*



3) *for each $n \geqslant 1$, $h_n$ is a one-to-one function of $\mathcal{A}_n$ onto $\mathcal{B}_n$ such that for any sequence $(A_n)_{n\geqslant 1}$ where, for each $n$, $A_n \in \mathcal{A}_n$ and $A_{n+1} \subset A_n$, then $\bigcap_{n\geqslant 1} A_n \neq \emptyset$ if and only if $\bigcap_{n\geqslant 1} h_n(A_n) \neq \emptyset$;*

*then $X$ and $Y$ are homeomorphic under the function $h$ induced by the sequence $(h_n)_{n\geqslant 1}$.*

Proof. For each $x \in X$, $x \in \bigcap_{n\geqslant 1} A_n(x)$, where $A_n(x) \in \mathcal{A}_n$. Mesh $\mathcal{A}_n \to 0$ implies $\{x\} = \bigcap_{n\geqslant 1} A_n(x)$. By property 3, $\bigcap_{n\geqslant 1} h_n(A_n(x)) \neq \emptyset$. Define $h\colon X \to Y$ by $h(x) = \bigcap_{n\geqslant 1} h_n(A_n(x))$. Since mesh $\mathcal{B}_n \to 0$, $h$ is a function of $X$ into $Y$. For each $y \in Y$, $\{y\} = \bigcap_{n\geqslant 1} B_n(y)$, $B_n(y) \in \mathcal{B}_n$ and hence by property 3, $\bigcap_{n\geqslant 1} h_n^{-1}(B_n(y)) \neq \emptyset$; let $\{x\} = \bigcap_{n\geqslant 1} h_n^{-1}(B_n(y))$. Thus by definition of $h$, $h(x) = y$. Hence $h$ is onto. Let $x, x' \in X$, $x \neq x'$, $\{x\} = \bigcap_{n\geqslant 1} A_n(x)$, $\{x'\} = \bigcap_{n\geqslant 1} A_n(x')$. Then there exists $k \geqslant 1$ such that $A_k(x) \neq A_k(x')$ so that $h_k(A_k(x)) \neq h_k(A_k(x'))$ and $h_k(A_k(x)) \cap h_k(A_k(x')) = \emptyset$. Thus $h(x) \neq h(x')$ so that $h$ is one-to-one. For each $x \in X$ and for each $V$, open in $Y$, such that $h(x) \in V$, there exists $k \geqslant 1$ such that $h_k(A_k(x)) \subset V$. Hence $h[A_k(x)] \subset V$ and therefore $h$ is continuous. Similarly, $h$ is open. Thus $h$ is a homeomorphism of $X$ onto $Y$.

**3. Theorems.** In this section we prove Theorem 1.3. We do this by proving four theorems each of which is the special case of Theorem 1.3 for the type of 0-space under consideration.

THEOREM 3.1. (Knaster–Reichbach). *Let $X$ and $Y$ be compact, perfect 0-spaces, and let $H$ and $K$ be closed nowhere dense subsets of $X$ and $Y$ respectively. Let $f$ be a homeomorphism of $H$ onto $K$. Then there exists a homeomorphism $F$ of $X$ onto $Y$ such that $F|_H = f$.*

Proof. By Lemma 2.8, there exist sequences $(\mathcal{A}_n)_{n\geqslant 1}$ and $(\mathcal{B}_n)_{n\geqslant 1}$ of disjoint clopen covers of $X$ and $Y$ respectively and a sequence $(h_n)_{n\geqslant 1}$, where for each $n \geqslant 1$, $h_n$ is a one-to-one function of $\mathcal{A}_n$ onto $\mathcal{B}_n$ such that:

1) mesh $\mathcal{A}_n \to 0$, mesh $\mathcal{B}_n \to 0$;

2) $\mathcal{A}_{n+1}$ and $\mathcal{B}_{n+1}$ are refinements of $\mathcal{A}_n$ and $\mathcal{B}_n$ respectively;

3) for each $n \geqslant 1$, $A_{n+1} \subset A_n$ if and only if $h_{n+1}(A_{n+1}) \subset h_n(A_n)$;

4) for each $p \in H$ and each $n \geqslant 1$, $h_n(A_n(p)) = B_n(f(p))$.

If $\{A_n\}_{n\geqslant 1}$ is a collection of sets such that for each $n \geqslant 1$, $A_n \in \mathcal{A}_n$, $A_{n+1} \subset A_n$, then $\bigcap_{n\geqslant 1} A_n \neq \emptyset$ since $X$ is a compact metric space. The same is true for a collection $\{B_n\}_{n\geqslant 1}$ where for each $n \geqslant 1$, $B_n \in \mathcal{B}_n$ and $B_{n+1} \subset B_n$. Thus the hypothesis of Lemma 2.9 is satisfied and the induced function $F$

defined by $F(x) = \bigcap_{n\geqslant 1} h_n(A_n(x))$ is a homeomorphism of $X$ onto $Y$. Also if $x \in H$, then $F(x) = \bigcap_{n\geqslant 1} h_n(A_n(x)) = \bigcap_{n\geqslant 1} B_n(f(x)) = f(x)$, so that $F|_H = f$.

THEOREM 3.2. *Let $X$ and $Y$ be perfect and locally compact but not compact 0-spaces and let $H$ and $K$ be closed nowhere dense subsets of $X$ and $Y$ respectively. Let $f$ be a homeomorphism of $H$ onto $K$. Then there exists a homeomorphism $F$ of $X$ onto $Y$ such that $F|_H = f$.*

Proof. Let $X'$ be the one point compactification of $X$ and let $Y'$ be the one point compactification of $Y$. Then $X'$ and $Y'$ are compact perfect 0-spaces and $H' = H \cup \{\infty\}$ and $K' = K \cup \{\infty\}$ are closed nowhere dense subsets of $X'$ and $Y'$ respectively. Also, since $f$ is a homeomorphism of $H$ onto $K$, $H$ is closed in $X'$ if and only if $K$ is closed in $Y'$. Thus $f'\colon H' \to K'$ defined by $f'|_H = f$ and $f'(\infty) = \infty$ is a homeomorphism of $H'$ onto $K'$. Thus, by Theorem 3.1, there exists a homeomorphism $F'$ of $X'$ onto $Y'$ such that $F'|_{H'} = f'$. Now define $F$ by $F = F'|_X$, then $F$ is a homeomorphism of $X$ onto $Y$ such that $F|_H = f$.

THEOREM 3.3. *Let $X$ be a complete, nowhere locally compact 0-space and let $Y$ be an absolute $G_\delta$, nowhere locally compact 0-space, and let $H$ and $K$ be closed nowhere dense subsets of $X$ and $Y$ respectively. Let $f$ be a homeomorphism of $H$ onto $K$. Then there exists a homeomorphism $F$ of $X$ onto $Y$ such that $F|_H = f$.*

Proof. Since $Y$ is an absolute $G_\delta$ metric space, there exists a compact metric space $Y'$ and a collection $\{F_n\}_{n\geqslant 1}$ of closed subsets of $Y'$ such that $Y = \bigcap_{n\geqslant 1} Y' \setminus F_n$. The hypothesis of Lemma 2.8 is satisfied, thus we can find sequences $\mathcal{A} = (\mathcal{A}_n)_{n\geqslant 1}$ and $\mathcal{B} = (\mathcal{B}_n)_{n\geqslant 1}$ of disjoint clopen covers of $X$ and $Y$ respectively and a sequence $(h_n)_{n\geqslant 1}$ where for each $n \geqslant 1$, $h_n$ is a one-to-one function of $\mathcal{A}_n$ onto $\mathcal{B}_n$ such that:

1) mesh $\mathcal{A}_n \to 0$, mesh $\mathcal{B}_n \to 0$;

2) $\mathcal{A}_{n+1}$ and $\mathcal{B}_{n+1}$ are refinements of $\mathcal{A}_n$ and $\mathcal{B}_n$ respectively;

3) for each $n \geqslant 1$, $A_{n+1} \subset A_n$ if and only if $h_{n+1}(A_{n+1}) \subset h_n(A_n)$;

4) for each $p \in H$ and each $n \geqslant 1$, $h_n(A_n(p)) = B_n(f(p))$;

5) for each $B_n \in \mathcal{B}_n$, there exists an open set $B_n'$ in $Y'$ such that $B_n = B_n' \cap Y$ and $\overline{B_n'} \cap F_n = \emptyset$.

If $\{A_n\}_{n\geqslant 1}$ is a collection of sets where for each $n \geqslant 1$, $A_n \in \mathcal{A}_n$ and $A_{n+1} \subset A_n$, then $\bigcap_{n\geqslant 1} A_n \neq \emptyset$ since $X$ is a complete metric space and each $A_n$ is closed. For each $n \geqslant 1$, $h_n(A_n) = h_n(A_n)' \cap Y$ where $h_n(A_n)'$ is open in $Y'$ and $\overline{h_n(A_n)'} \cap F = \emptyset$. Also $h_{n+1}(A_{n+1}) \subset h_n(A_n)$. Thus $\bigcap_{n\geqslant 1} \overline{h_n(A_n)'} \neq \emptyset$ since $Y'$ is a compact metric space. Let $y \in \bigcap_{n\geqslant 1} \overline{h_n(A_n)'}$. For each $n \geqslant 1$, $\overline{h_n(A_n)'} \cap F_n = \emptyset$. Thus $y \notin \bigcup_{n\geqslant 1} F_n$ and hence $y \in Y$ since $Y = \bigcap_{n\geqslant 1} Y' \setminus F_n$. Therefore, $y = \bigcap_{n\geqslant 1} h_n(A_n)$ since $\overline{h_n(A_n)'} \cap Y = h_n(A_n)$ and mesh $\mathcal{B}_n \to 0$. Define $F\colon X \to Y$ by $F(x) = \bigcap_{n\geqslant 1} h_n(A_n(x))$. By Lemma 2.9, $F$ is a homeomorphism of $X$ onto $Y$. Also, since for each $p \in H$, and each $n \geqslant 1$, $h_n(A_n(p)) = B_n(f(p))$, $F(p) = \bigcap_{n\geqslant 1} h_n(A_n(p)) = \bigcap_{n\geqslant 1} B_n(f(p)) = f(p)$. Thus $F|_H = f$.



COROLLARY 3.1. *If $X$ is a complete, nowhere locally compact 0-space and $Y$ is an absolute $G_\delta$, nowhere locally compact 0-space, then $X$ and $Y$ are homeomorphic.*

THEOREM 3.4. *Let $X$ and $Y$ be countable perfect metric spaces and let $H$ and $K$ be closed nowhere dense subsets of $X$ and $Y$ respectively. Let $f$ be a homeomorphism of $H$ onto $K$. Then there exists a homeomorphism $F$ of $X$ onto $Y$ such that $F|_H = f$.*

Proof. Since every countable perfect metric space is nowhere locally compact and zero-dimensional, by Lemma 2.8 there exist sequences $\mathcal{A} = (\mathcal{A}_n)_{n\geqslant 1}$ and $\mathcal{B} = (\mathcal{B}_n)_{n\geqslant 1}$ of disjoint clopen covers of $X$ and $Y$ respectively and a sequence $(h_n)_{n\geqslant 1}$ where for each $n \geqslant 1$, $h_n$ is a one-to-one function of $\mathcal{A}_n$ onto $\mathcal{B}_n$ having the following properties:

1) mesh $\mathcal{A}_n \to 0$, mesh $\mathcal{B}_n \to 0$;

2) $\mathcal{A}_{n+1}$ and $\mathcal{B}_{n+1}$ are refinements of $\mathcal{A}_n$ and $\mathcal{B}_n$ respectively;

3) for each $n \geqslant 1$, $A_{n+1} \subset A_n$ if and only if $h_{n+1}(A_{n+1}) \subset h_n(A_n)$;

4) for each $p \in H$ and each $n \geqslant 1$, $h_n(A_n(p)) = B_n(f(p))$.

Since $X$ and $Y$ are countable, we can write $X = \{x_i\}_{i\geqslant 1}$, $Y = \{y_i\}_{i\geqslant 1}$. Now, $x_1 = \bigcap_{n\geqslant 1} A_n(x_1)$. If $x_1 \in H$, then $f(x_1) = \bigcap_{n\geqslant 1} B_n(f(x_1))$; so define $F(x_1) = f(x_1)$. There exists $k_0 \geqslant 1$ such that $f(x_1) = y_{k_0} \in Y$. Relabel $y_{k_0}$ as $y_1$ and $y_1$ as $y_{k_0}$. If $x_1 \notin H$, just define $F(x_1) = y_1$.

Now suppose that $k$ is an odd positive integer and for all $i < k$, $F(x_i)$ has been defined. Since for all $i < k$, $x_i \neq x_k$ and $x_i = \bigcap_{n\geqslant 1} A_n(x_i)$, $x_k = \bigcap_{n\geqslant 1} A_n(x_k)$, there exists $n_0 \geqslant 1$ such that for all $i < k$, $A_{n_0}(x_i) \neq A_{n_0}(x_k)$. Also if $x_k \in H$ and $x_k \in A_n(x_i)$ for $n < n_0$ and $i < k$, then $f(x_k) \in h_n(A_n(x_i))$. Otherwise, choose $y_m \in h_{n_0}(A_{n_0}(x_k))$ such that if $x_k \in A_n(x_i)$ for $i < k$, then $y_m \in h_n(A_n(x_i))$. Depending on the case, relabel $f(x_k)$ or $y_m$ as $y_k$ and vice versa. Define $F(x_k) = y_k$.

Now if $k$ is even, we consider $y_k$ and reverse the previous procedure to obtain an $x_k$ such that if $B_n(y_k) \neq B_n(y_i)$ for $i < k$, then $A_n(x_k) \neq A_n(x_i)$, and if $y \in B_n(y_i)$ for $i < k$, then $x_k \in A_n(x_i)$. Also, if $y_k \in K$, then $x_k = f^{-1}(y_k)$. Define $F(x_k) = y_k$. By this procedure, each element in $X$, at some stage, is paired with a unique element in $Y$ and vice versa, so that $F$ is a function of $X$ onto $Y$. The hypothesis of Lemma 2.9 is satisfied so that $F$ is a homeomorphism. Also, since $x_i \in H$ implies $F(x_i) = f(x_i)$, it follows that $F|_H = f$.

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# More decompositions of $E^n$ which are factors of $E^{n+1}$

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**1. Introduction.** Let $G$ be a monotone upper semicontinuous decomposition of $E^n$, $G_1$ be the collection of nondegenerate elements of G, and assume $G_1$ is countable. We show in this paper that $E^n/G \times E^1$ is topologically $E^{n+1}$ if either (1) for each element $g \in G_1$, there exists a positive integer $n_g$ such that $g$ is an $n_g$-frame, or (2) $G_1$ is a null collection and if $g \in G_1$, there exists a positive integer $n_g$ such that $g$ is an $n_g$-cell which is flat in $E^{n+1}$.

Bing proved [5] that the product of the dogbone space [4] and $E^1$ is $E^4$. Thus $E^4$ has non-manifold factors. Using analogous techniques, Andrews and Curtis [1] have shown that the product of $E^1$ and a decomposition of $E^n$ whose only non-degenerate element is an arc is $E^{n+1}$. Gillman and Martin [8] announced an extension of the result of Andrews and Curtis by proving case (1) above if each element of $G_1$ is an arc. Recently Bryant [6] has shown that if $D$ is a $k$-cell in $E^n$ that is flat in $E^{n+1}$ and $G$ is the decomposition of $E^n$ whose only nondegenerate element is $D$, then $E^n/G \times E^1$ is topologically $E^{n+1}$. For a more complete summary of related results, the reader is referred to [2].

In Section 3 we show that if $G_1$ is countable and each element in $G'$ ($G' = \{g \times w \mid g \in G, w \in E^1\}$) that corresponds to a given element of $G_1$ can be shrunk in a certain way (condition I), then all the elements of $G'$ can be shrunk simultaneously (condition II). Using this we show that if $G_1$ is countable and $G$ satisfies condition $I$, then $E^n/G$ is a factor of $E^{n+1}$. The result for $n_g$-frames referred to above is proved in Section 4 by first showing, using techniques of Andrews and Curtis [1], that the product of $E^1$ and a decomposition of $E^n$ whose only nondegenerate element is a $k$-frame is $E^{n+1}$. The result is then obtained by noticing that $G$ satisfies condition I. The case for certain null collections of cells is shown in Section 5 by using Bryant's work [6] and condition I.

**2. Notation and terminology.** The statement that $G$ is an *upper semi-continuous decomposition of* $E^n$ means that (1) $G$ is a collection of sub-

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