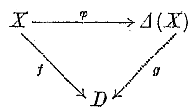
onto a dendroid D there exists one and only one continuous mapping g of $\Delta(X)$ onto D such that the diagram

(4.7) 

commutes, and g is monotone.

In fact, take an arbitrary point $d \in \Delta(X)$. It follows from (4.6) that $f(\varphi^{-1}(d))$ is a point. Denote this point by $g(d)$. If $d = \varphi(x)$, then $g(d) = f(x)$, thus $g(\varphi(x)) = f(x)$ for every $x \in X$, i.e. diagram (4.7) commutes. The mapping φ being continuous and defined on a metric continuum, it is closed (see [4], Theorem 9, p. 104). Since f is continuous, the continuity of g follows from Theorems 1 and 3 in [6], § 13, XV, p. 119. The uniqueness of g follows from the definition. From the definition of g we conclude also that

$$g^{-1}(y) = \varphi(f^{-1}(y)) \quad \text{for every } y \in D.$$

The mapping f being monotone, $f^{-1}(y)$ is a continuum, hence $\varphi(f^{-1}(y))$ is also a continuum. So $g^{-1}(y)$ is, and g is monotone. Therefore we have proved that Corollary 3 leads to Theorem 7. The opposite way is quite obvious.

COROLLARY 4. *If a dendroid D is the hyperspace of an upper semi-continuous decomposition of a λ-dendroid X into continua, then it is a monotone image of the dendroid $\Delta(X)$.*

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Irreducibly generated algebras

by

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The study of functions involves, in a most fundamental way, the study of the composition of functions. If Δ is a set of elements, then any mapping of Δ^p (i.e., of the pth product $\Delta \times \ldots \times \Delta$) into Δ is a *p-place function* over Δ. The composite $F_0(F_1, \ldots, F_p)$ of any $p+1$ p-place functions $F_0, F_1, \ldots, F_p$ is again a p-place function defined in the usual manner:

$$F_0(F_1, \ldots, F_p)(x_1, \ldots, x_p) = F_0(F_1(x_1, \ldots, x_p), \ldots, F_p(x_1, \ldots, x_p))$$

for $(x_1, \ldots, x_p)$ in Δ^p. From here it easily follows that composition satisfies the superassociative law (cf. [2]), namely that

$$(1) \quad (F_0(F_1, \ldots, F_p))(G_1, \ldots, G_p) = F_0(F_1(G_1, \ldots, G_p), \ldots, F_p(G_1, \ldots, G_p))$$

for any p-place functions $F_0, F_1, \ldots, G_p$ over Δ. A set $\mathfrak{S}$ of functions is called an *algebra of functions* if $\mathfrak{S}$ is closed with respect to composition.

Equation (1) serves as a point of departure for the study of a more abstract algebraic structure. Let $\mathfrak{S}$ be a set of elements with a $(p+1)$-ary operation, i.e., an operation which associates with each $(p+1)$-tuple of elements $S_0, S_1, \ldots, S_p$ of $\mathfrak{S}$ an element of $\mathfrak{S}$ denoted by $S_0(S_1, \ldots, S_p)$. If the superassociative law is valid in $\mathfrak{S}$, then $\mathfrak{S}$ will be called a *p-place Menger algebra* and its operation will be called composition. Clearly any algebra of functions is a Menger algebra. That the converse is true was shown by Dicker (cf. [1]) — for any Menger algebra $\mathfrak{S}$ there exists a set Δ such that $\mathfrak{S}$ is isomorphic to an algebra of functions over Δ.

The structure of Menger algebras in general have been studied in [1]–[4]. This paper, however, deals with a particular type of Menger algebra. The Menger algebra $\mathfrak{S}$ is said to be *irreducibly generated* if each subset of $\mathfrak{S}$ is also an algebra, that is, is closed with respect to composition. Therefore, for elements $S_0, S_1, \ldots, S_p$ in $\mathfrak{S}$, the composite $S_0(S_1, \ldots, S_p)$ must be one of the elements $S_0, S_1, \ldots, S_p$ since the set $\{S_0, S_1, \ldots, S_p\}$ forms an algebra. An element S_0 of $\mathfrak{S}$ is *constant* if $S_0(S_1, \ldots, S_p) = S_0$ for each sequence $(S_1, \ldots, S_p)$ of elements from $\mathfrak{S}$; S_0 is called a *k-th place selector* relative to a subset $\mathfrak{J}$ of $\mathfrak{S}$, if $S_0(T_1, \ldots, T_p) = T_k$ for each sequence $(T_1, \ldots, T_p)$ from $\mathfrak{J}$.

As an abbreviation we write $F_0(F_1, \ldots, F_p)(G_1, \ldots, G_p)$ to mean $\big(F_0(F_1, \ldots, F_p)\big)(G_1, \ldots, G_p)$.

THEOREM 1. *If an irreducibly generated algebra $\mathfrak{S}$ contains at least three constant elements, then each nonconstant element S in $\mathfrak{S}$ is a selector relative to the algebra consisting of S and all constant elements in $\mathfrak{S}$.*

Let $\mathfrak{C}$ be the set of constant elements of $\mathfrak{S}$ (containing at least three elements) and let $\mathfrak{C}_S = \{S\} \cup \mathfrak{C}$ for each element S of $\mathfrak{S}$. We proceed by induction on the place number p of $\mathfrak{S}$.

The case $p = 1$ does not require the assumption that $\mathfrak{C}$ contains at least three elements. Let S be a nonconstant element of $\mathfrak{S}$ and M any constant element. Since $\{S, M\}$ is an algebra, $S(M) = M$ or $S(M) = S$. If $S(M) = S$, then $S(T) = S(M)(T) = S\big(M(T)\big) = S(M) = S$ for every element T of $\mathfrak{S}$—that is, S is a constant element. By assumption S is nonconstant and $S(M) = S$ is therefore impossible and hence $S(M) = M$ for every element M of $\mathfrak{C}$. Furthermore, since $\{S\}$ is an algebra $S(S) = S$ and hence S is a first place selector relative to $\mathfrak{C}_S$.

For $p = 2$, since $\{S, M\}$ is an algebra, $S(M, M) = M$ or $S(M, M) = S$. Now $S(M, M) = S$ implies $S(T, U) = S(M, M)(T, U) = S\big(M(T, U), M(T, U)\big) = S(M, M) = S$ for each pair of elements (T, U) from $\mathfrak{S}$. But since S is nonconstant $S(M, M) = S$ is impossible—that is, $S(M, M) = M$ for every element M in $\mathfrak{C}$. Let A be in $\mathfrak{C}$. $S(S, A) = S$ or $S(S, A) = A$. In the first case we will show that S is a first place selector relative to $\mathfrak{C}_S$; in the second case that S is a second place selector relative to $\mathfrak{C}_S$.

Suppose first that

(2) $$S(S, A) = S\,.$$

Then

(3) $$S(M, S) = M \quad \text{for each constant element } M\,.$$

For from (2), $M = S(M, M) = S(S, A)(M, M) = S(M, A) = S(M, S)(A, A)$. And for $M \neq A$, $M = S(M, S)(A, A)$ implies $S(M, S) = M$. Also $S(A, S) = A$. For suppose, on the contrary, that $S(A, S) = S$. Then, similarly as above, $M = S(M, M) = S(A, S)(M, A) = S(A, M) = S(S, M)(A, A)$. And for $M \neq A$, $S(S, M) = M$. Since $\mathfrak{C}$ contains at least three elements, let B and C be distinct constant elements different from A. Then $S(B, C) = S(S, C)(B, B) = C(B, B) = C$. And from (3), $S(B, C) = S(B, S)(C, C) = B(C, C) = B$. Clearly this is impossible since $B \neq C$. Hence also $S(A, S) = A$ and (3) follows.

Equations (2) and (3) imply

(4) $$S(S, M) = S \quad \text{for each constant element } M\,.$$

If $M = A$, then (4) reduces to (2). So suppose $M \neq A$. If, on the contrary, $S(S, M) = M$, then $S(S, M)(A, A) = M(A, A) = M$. But from (3), $S(S, M)(A, A) = S(A, M) = S(A, S)(M, M) = A(M, M) = A$. Since $M \neq A$, $S(S, M) \neq M$ — that is, $S(S, M) = S$ and (4) follows.

If M and N are constant elements, then from (3), $S(M, S)(N, N) = S(M, N) = M(N, N) = M$, whence

(5) $$S(M, N) = M \quad \text{for each pair of constant elements } (M, N)\,.$$

Equations (3), (4), and (5) together imply that S is a first place selector relative to $\mathfrak{C}_S$.

By similar reasoning, it can be shown that S is a second place selector relative to $\mathfrak{C}_S$ if $S(S, A) = A$.

Suppose now that the assertion of Theorem 1 is true for any $(p-1)$-place Menger algebra satisfying its conditions and let $\mathfrak{S}$ be an irreducibly generated p-place Menger algebra (where $p \geqslant 3$) containing at least three constant elements. For any nonconstant element S of $\mathfrak{S}$, we show first that there exists a p-tuple $(H_1, \ldots, H_p)$ such that for some k $(1 \leqslant k \leqslant p)$ $H_k = S$ and for $i = 1, \ldots, k-1, k+1, \ldots, p$, $H_i = A$ for some constant element A and such that $S(H_1, \ldots, H_p) = H_k = S$. Having shown this, we then prove that S is a kth place selector relative to $\mathfrak{C}_S$. (We remark that since S is nonconstant $S(M, \ldots, M) = M$ for any constant element M).

Suppose, on the contrary, that $S(S, M, \ldots, M) = S(M, S, M, \ldots, M) = S(M, \ldots, M, S) = M$ for every constant element M; or briefly, we write

(6) $$S\{S, M, \ldots, M\} = M \quad \text{for every constant element } M\,,$$

where $S\{T_1, \ldots, T_p\} = T_0$ means that for any permutation $\pi(T_1, \ldots, T_p)$ of the sequence $(T_1, \ldots, T_p)$, $S\big(\pi(T_1, \ldots, T_p)\big) = T_0$.

If A and B are distinct constant elements, then

(7) $$S\{A, B, S, \ldots, S\} = S\,.$$

For suppose there is a permutation $\pi(A, B, S, \ldots, S)$ of the sequence $(A, B, S, \ldots, S)$ such that $S\big(\pi(A, B, S, \ldots, S)\big) \neq S$. Without loss of generality we may suppose that $\pi(A, B, S, \ldots, S) = (A, B, S, \ldots, S)$. Then $S(A, B, S, \ldots, S)$ is either A or B. But $S(A, B, S, \ldots, S) = B$ implies $S(A, B, S, \ldots, S)(A, \ldots, A) = S(A, B, A, \ldots, A) = B(A, \ldots, A) = B$ which is contrary to (6) since $S(A, B, A, \ldots, A) = S(A, S, A, \ldots, A)(B, \ldots, B) = A(B, \ldots, B) = A$. Similarly the assumption that $S(A, B, S, \ldots, S) = A$ leads to a contradiction and (7) follows.

From (6) and (7) we have

(8) $$S\{S, A, B, \ldots, B\} = B\,.$$

For suppose there exists a permutation, say $(S, A, B, \ldots, B)$, such that $S(S, A, B, \ldots, B) \neq B$. If $S(S, A, B, \ldots, B) = A$ then $S(B, A, B, \ldots, B)$

$= S(S, A, B, \ldots, B)(B, \ldots, B) = A(B, \ldots, B) = A$ which is contrary to (6). So suppose $S(S, A, B, \ldots, B) = S$ and let C be a constant element distinct from A and B. Then $S(C, A, B, \ldots, B) = S(S, A, B, \ldots, B)(C, \ldots, C) = S(C, \ldots, C) = C$. But from (7), $S(C, A, B, \ldots, B) = S(C, A, S, \ldots, S) \times (B, \ldots, B) = S(B, \ldots, B) = B$. Since $B \neq C$, $S(S, A, B, \ldots, B) \neq S$ and (8) follows.

Now assume that for any two distinct constant elements A and B and for some integer k

$$(9_k) \qquad S\{A, B_1, \ldots, B_k, S, \ldots, S\} = S$$

and

$$(9'_k) \qquad S\{S, B_1, \ldots, B_k, A, \ldots, A\} = A$$

where $B_i = B$ for $i = 1, \ldots, k$. The case $k = 1$ was proved above; assuming (9_k) and $(9'_k)$ we prove (9_{k+1}) and $(9'_{k+1})$.

Suppose there exists a permutation, say $(A, B_1, \ldots, B_{k+1}, S, \ldots, S)$ where $B_i = B$ for $i = 1, \ldots, k+1$ such that $S(A, B_1, \ldots, B_{k+1}, S, \ldots, S) \neq S$. If $S(A, B_1, \ldots, B_{k+1}, S, \ldots, S) = A$, then $S(A, B_1, \ldots, B_{k+1}, S, \ldots, S) \times (B, \ldots, B) = S(A, B_1, \ldots, B_{k+1}, B, \ldots, B) = A$ contradicting (6). If $S(A, B_1, \ldots, B_{k+1}, S, \ldots, S) = B$, then $S(A, B_1, \ldots, B_{k+1}, S, \ldots, S) \times (A, \ldots, A) = S(A, B_1, \ldots, B_{k+1}, A, \ldots, A) = B(A, \ldots, A) = B$, again a contradiction since by $(9'_k)$ $S(A, B_1, \ldots, B_{k+1}, A, \ldots, A) = S(A, B_1, \ldots, B_k, S, A, \ldots, A)(B, \ldots, B) = A(B, \ldots, B) = A$. Hence we must have $S(A, B_1, \ldots, B_{k+1}, S, \ldots, S) = S$.

Now suppose there exists a permutation, say, $(S, B_1, \ldots, B_{k+1}, A, \ldots, A)$ such that $S(S, B_1, \ldots, B_{k+1}, A, \ldots, A) \neq A$. If $S(S, B_1, \ldots, B_{k+1}, A, \ldots, A) = B$, then

$$S(S, B_1, \ldots, B_{k+1}, A, \ldots, A)(A, \ldots, A) = S(A, B_1, \ldots, B_{k+1}, A, \ldots, A) = B(A, \ldots, A) = B.$$

From $(9'_k)$, however, $S(A, S, B_1, \ldots, B_k, A, \ldots, A)(B, \ldots, B) = S(A, B_1, \ldots, B_{k+1}, A, \ldots, A) = A(B, \ldots, B) = A$. So suppose that $S(S, B_1, \ldots, B_{k+1}, A, \ldots, A) = S$ and let C be a constant element different from A and from B. Then $S(S, B_1, \ldots, B_{k+1}, A, \ldots, A)(C, \ldots, C) = S(C, B_1, \ldots, B_{k+1}, A, \ldots, A) = S(C, \ldots, C) = C$.

But from (9_{k+1}) which we have just proved, we have $S(C, B_1, \ldots, B_{k+1}, S, \ldots, S) = S$ and hence $S(C, B_1, \ldots, B_{k+1}, S, \ldots, S)(A, \ldots, A) = S(C, B_1, \ldots, B_{k+1}, A, \ldots, A) = S(A, \ldots, A) = A$, again a contradiction, and one obtains $(9'_{k+1})$. By induction then (9_{p-2}) and $(9'_{p-2})$ follow, that is $S\{A, B, \ldots, B, S\} = S$ and $S\{S, B, \ldots, B, A\} = A$ for every pair of distinct constant elements A and B. This is clearly impossible and hence

there exists a constant element, say A, and a p-tuple which, without loss of generality, we may assume is $(S, A, \ldots, A)$ such that

$$(10) \qquad S(S, A, \ldots, A) = S.$$

From here we prove that S is a first place selector relative to $\mathfrak{C}_S$.

For every p-tuple of elements $(T_0, T_1, \ldots, T_{p-1})$ from $\mathfrak{S}$ and any two integers $a, b \leqslant p$, $a < b$ and $a, b \neq 0$, we define a p-ary operation σ_{ab} by the following:

$$\sigma_{ab}(T_0, T_1, \ldots, T_{p-1}) = T_0(T_1, \ldots, T_a, \ldots, T_a, T_b, \ldots, T_{p-1}).$$

That is, T_a appears in the p-tuple $(T_1, \ldots, T_a, \ldots, T^a, T_b, \ldots, T_{p-1})$ both in the ath and the bth place. It is easy to verify that σ_{ab} is superassociative. Hence denoting by $\mathfrak{S}_{ab}$, the set of elements of $\mathfrak{S}$ with the operation σ_{ab}, we have that $\mathfrak{S}_{ab}$ is a $(p-1)$-place Menger algebra. Since S is nonconstant in $\mathfrak{S}$, and hence $S(A, \ldots, A) = A$, $\sigma_{ab}(S, A, \ldots, A) = S(A, \ldots, A) = A$ and S is nonconstant in $\mathfrak{S}_{ab}$.

Suppose first that $a \neq 1$. Then $\sigma_{ab}(S, S, A, \ldots, A) = S(S, A, \ldots, A) = S$ from (10) and hence S is a first place selector relative to $(\mathfrak{C}_{ab})_S$, where $\mathfrak{C}_{ab}$ is the set of constant elements of $\mathfrak{S}_{ab}$. Clearly $\mathfrak{C} \subseteq \mathfrak{C}_{ab}$ for every a and b. Hence

$$(11) \qquad S(H_1, \ldots, H_p) = H_1$$

for any p-tuple $(H_1, \ldots, H_p)$ from $\mathfrak{C}$ such that $H_a = H_b$ for some integers $a < b \leqslant p$ and $a \neq 1$.

For the case $a = 1$, we first prove that

$$(12) \qquad S\{A, S, B, \ldots, B\} = A.$$

For example, we show $S(A, S, B, \ldots, B) = A$. Since $p \geqslant 3$, $S(A, S, B, \ldots, B)(B, \ldots, B) = S(A, B, \ldots, B) = A$ by Equation (11). But then $S(A, S, B, \ldots, B) = S$ or B is impossible since $S(B, \ldots, B) = B(B, \ldots, B) = B \neq A$. By composing $(A, \ldots, A)$ with the equalities in (12) we also have $S(A, A, B, \ldots, B) = S(A, B, A, B, \ldots, B) = \ldots = S(A, B, \ldots, B, A) = A$. Hence $\sigma_{1b}(S, A, B, \ldots, B) = S(A, B, \ldots, A, B, \ldots, B) = A$ and therefore S is a first place selector relative to $(\mathfrak{C}_{1b})_S$. Or, in other words, Equation (11) holds whenever $H_a = H_b$ for any integers a, b $(1 \leqslant a, b \leqslant p)$.

Now let $(H_1, \ldots, H_p)$ be a p-tuple from $\mathfrak{C}$ such that $H_i \neq H_j$ whenever $i \neq j$ $(i, j = 1, \ldots, p)$. If $H_k = S$ for some $k \leqslant p$, then $S(H_1, \ldots, H_p) = H_1$. If, on the contrary, $S(H_1, \ldots, H_p) = H_j \neq H_1$, then $S(H_1, \ldots, H_p) \times (H_1, \ldots, H_1) = H_j(H_1, \ldots, H_1) = S(H_1, \ldots, H_1, \ldots, H_p) = H_1$ by (11). Since $H_j \neq H_1$, H_j must equal S. But if $S(H_1, \ldots, H_p) = S$, choose m such that $H_m \neq H_1$ and $H_m \neq S$, which is possible since $p \geqslant 3$ and since the $H_1, \ldots, H_p$ are distinct. Then $S(H_1, \ldots, H_p)(H_m, \ldots, H_m) = S(H_m, \ldots, H_m) = H_m = S(H_1, \ldots, H_m, \ldots, H_m, \ldots, H_p) = H_1$, which is clearly impossible.

Thus Equation (11) holds for every p-tuple $(H_1, \ldots, H_p)$ from $\mathfrak{C}_S$ and hence S is a first place selector relative to $\mathfrak{C}_S$. This completes the proof of Theorem 1.

Additional structure of Menger algebras and, in particular, of irreducibly generated Menger algebras may be derived from the following considerations. Elements F and G of a p-place Menger algebra $\mathfrak{S}$ are said to be *equivalent*, written $F \equiv G$, if $F(M_1, \ldots, M_p) = G(M_1, \ldots, M_p)$ for every sequence $(M_1, \ldots, M_p)$ of constant elements in $\mathfrak{S}$. Consequently, if $F_i \equiv G_i$, $i = 0, 1, \ldots, p$, then $F_0(F_1, \ldots, F_p) \equiv G_0(G_1, \ldots, G_p)$. For let $(M_1, \ldots, M_p)$ be a sequence of constant elements in $\mathfrak{S}$. Then $F_0(F_1, \ldots, F_p)(M_1, \ldots, M_p) = F_0\big(F_1(M_1, \ldots, M_p), \ldots, F_p(M_1, \ldots, M_p)\big) = F_0\big(G_1(M_1, \ldots, M_p), \ldots, G_p(M_1, \ldots, M_p)\big)$.

Now $G_i(M_1, \ldots, M_p)$, $i = 1, \ldots, p$, is a constant element, since $G_i(M_i, \ldots, M_p)(S_i, \ldots, S_p) = G_i\big(M_1(S_1, \ldots, S_p), \ldots, M_p(S_1, \ldots, S_p)\big) = G_i(M_1, \ldots, M_p)$.

Hence $\big(G_1(M_1, \ldots, M_p), \ldots, G_p(M_1, \ldots, M_p)\big)$ is a sequence of constant elements in $\mathfrak{S}$ and $F_0\big(G_i(M_1, \ldots, M_p), \ldots, G_p(M_1, \ldots, M_p)\big) = G_0\big(G_1(M_1, \ldots, M_p), \ldots, G_p(M_1, \ldots, M_p)\big) = G_0(G_1, \ldots, G_p)(M_i, \ldots, M_p)$; i.e., $F_0(F_1, \ldots, F_p) \equiv G_0(G_1, \ldots, G_p)$.

Denoting by F^* the class of all elements of $\mathfrak{S}$ equivalent to F, we may therefore define a superassociative operation on the set $\mathfrak{S}^*$, which consists of all equivalence classes of $\mathfrak{S}$, namely,

$$F_0^*(F_1^*, \ldots, F_p^*) = [G_0(G_1, \ldots, G_p)]^*$$

for any elements G_i in F_i^*, $i = 1, \ldots, p$. In particular,

$$F_0^*(F_1^*, \ldots, F_p^*) = [F_0(F_1, \ldots, F_p)]^* . \tag{13}$$

THEOREM 2. *If $\mathfrak{S}$ is a p-place Menger algebra, $\mathfrak{S}^*$ is isomorphic to an algebra of p-place functions over a set whose cardinality is the same as the cardinality of the set of constant elements of $\mathfrak{S}$.*

If A and B are constant elements, then $A^* \neq B^*$ and hence the set $\mathfrak{C}^* = \{A^* |\ A \in \mathfrak{C}\}$ has the same cardinality as $\mathfrak{C}$. If $A_1, \ldots, A_p$ are constant elements, then $F(A_1, \ldots, A_p)$ is constant for any element F. Hence $F^*(A_1^*, \ldots, A_p^*)$ is in $\mathfrak{C}^*$ and we may define a one-to-one mapping α from $\mathfrak{S}^*$ onto a subset of the algebra of p-place functions over $\mathfrak{C}^*$, where αF^* is defined as follows:

$$(\alpha F^*)(A_1^*, \ldots, A_p^*) = F^*(A_1^*, \ldots, A_p^*) .$$

First, α is on-to-one. Let $F^* \neq G^*$ so that there exists a sequence $(H_1, \ldots, H_p)$ of constant elements such that $F(H_1, \ldots, H_p) = A \neq B = G(H_1, \ldots, H_p)$ where A and B are constant elements. Then from (13),

$$(\alpha F^*)(H_1^*, \ldots, H_p^*) = F^*(H_1^*, \ldots, H_p^*) = A^* \neq B^* = G^*(H_1^*, \ldots, H_p^*) = (\alpha G^*)(H_1^*, \ldots, H_p^*) .$$

Hence $\alpha F^* \neq \alpha G^*$. It is easily shown that α is an isomorphism, that is, $\alpha[F_0^*(F_1^*, \ldots, F_p^*)] = \alpha F_0^*(\alpha F_1^*, \ldots, \alpha F_p^*)$ for any elements F_i $(i = 0, \ldots, p)$ in $\mathfrak{S}$.

If the Menger algebra $\mathfrak{S}$ is isomorphic to $\mathfrak{S}^*$, that is, if each equivalence class consists of a single element of $\mathfrak{S}$, then the Dicker result follows. Moreover, if $\mathfrak{S}$ is irreducibly generated and contains at least three constant elements, then, by Theorem 1, each nonconstant element S in $\mathfrak{S}$ is a selector relative to the set of constant element of $\mathfrak{S}$. Hence for for each sequence $(H_1, \ldots, H_p)$ of constant elements, where S is, say a kth place selector relative to $\mathfrak{C}_S$, $\alpha S^*(H_1^*, \ldots, H_p^*) = S^*(H_1^*, \ldots, H_p^*) = [S(H_1, \ldots, H_p)]^* = H_k^*$; that is, αS^* is the kth place selector function. $\mathfrak{S}^*$ therefore has a completely trivial structure, consisting only of selector and constant functions. Furthermore if $(T_0, \ldots, T_p)$ is a sequence from $\mathfrak{S}$ such that $T_0^*, T_1^*, \ldots, T_p^*$ are distinct classes, and T_0 is a kth place selector relative to $\mathfrak{C}_{T_0}$, then $T_0(T_1, \ldots, T_p) = T_k$. For $\alpha T_0^*(\alpha T_1^*, \ldots, \alpha T_p^*) = \alpha T_k^* = \alpha[T_0^*(T_1^*, \ldots, T_p^*)] = \alpha\big([T_0(T_1, \ldots, T_p)]^*\big)$. Since α is one-to-one, $T_k^* = [T_0(T_1, \ldots, T_p)]^*$. Now $T_0(T_1, \ldots, T_p)$ equals one of $T_0, T_1, \ldots, T_p$ since $\mathfrak{S}$ is irreducibly generated and from $T_j^* \neq T_k^*$ for $j \neq k$, it follows that $T_0(T_1, \ldots, T_p) = T_k$. Thus each element S of an irreducibly generated Menger algebra is a selector on a much wider class of p-tuples.

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