A characterization of smoothness in dendroids

by

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Introduction. In this paper we present a condition equivalent to smoothness for dendroids. In particular, we show that a dendroid is smooth if and only if it is contractible by a retracting homotopy in such a way that some point remains fixed throughout the homotopy. The paper is divided into three sections. In the first we prove the above mentioned theorem. The second section is devoted to applying this result to obtain a slightly stronger result for fans, answering a question raised by Charatonik in [3]. The third section is devoted to some remarks concerning the possibility of extending our results to arbitrary hereditarily unicoherent continua. It will be seen that this generalization is in fact vacuous, since any hereditarily unicoherent continuum satisfying the hypotheses of the proposed theorem must already be a dendroid.

1. A condition equivalent to smoothness in dendroids.

DEFINITION 1.1. A metric space is said to be a *continuum* if it is compact and connected.

DEFINITION 1.2. A continuum is said to be *hereditarily unicoherent* if the intersection of any two of its subcontinua is connected.

In [2], p. 187 Charatonik gives the following characterization of hereditary unicoherence.

LEMMA 1.3. *A continuum H is hereditarily unicoherent if and only if given any set $X \subset H$, there exists a unique subcontinuum $I(X)$ of H which is irreducible with respect to containing X.*

DEFINITION 1.4. A continuum is said to be a *dendroid* if it is hereditarily unicoherent and arcwise connected.

Remark 1.5. It follows from 1.3 that if D is a dendroid and $x, y \in D$ $(x \neq y)$, then there is a unique arc in D whose endpoints are x and y.

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NOTATION 1.6. If D is a dendroid and $x, y \in D$, we let xy denote the unique arc in D whose endpoints are x and y.

DEFINITION 1.7. A dendroid D is said to be *smooth* if there exists some point $p \in D$ such that given any sequence a_n in D with $\lim_{n\to\infty} a_n = a$, it follows that $\operatorname{Lim}_{n\to\infty} a_n p = ap$ (for the definition of Li, Ls and Lim, see [8], pp. 335–339).

We now recall a theorem proved by Charatonik in [3], p. 7.

THEOREM 1.8. *If D is a dendroid and a_n and b_n are sequences in D such that $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$, then $\operatorname{Li}_{n\to\infty} a_n b_n$ is a continuum and $ab \subset \operatorname{Li}_{n\to\infty} a_n b_n$.*

DEFINITION 1.9. A relation $\leqslant$ on a set X is said to be a *partial order* if it is reflexive, antisymetric and transitive.

DEFINITION 1.10. A partial order $\leqslant$ on a topological space X is said to be *continuous* if $\leqslant$ is closed as a subset of $X \times X$.

DEFINITION 1.11. Let D be a dendroid and let $p \in D$. We then define a partial order $\leqslant_p$ on D as follows: If $x, y \in D$, we set $x \leqslant_p y$ if and only if $x \in yT$.

The following lemma is a corollary of theorem 1 in [7]:

LEMMA 1.12. *A dendroid D is smooth if and only if there exists a point $p \in D$ such that the partial order $\leqslant_p$ is continuous.*

For the remainder of the paper we let I denote the unit interval $[0, 1]$ of real numbers.

DEFINITION 1.13. A homotopy $h\colon X \times I \to X$ on a topological space X is called a *retracting homotopy* if for every $t \in I$ the map $h_t\colon X \to X$, given by $h_t(x) = h(x, t)$, is a retraction.

THEOREM 1.14. *If a dendroid D admits a homotopy $h\colon D \times I \to D$ satisfying*

(i) *h is a retracting homotopy,*

(ii) *h contracts D to a point $p \in D$,*

(iii) *$h(p, t) = p$ for every $t \in I$,*

then D is smooth.

Proof. Suppose that D, h and p are as above. We will show that the partial order $\leqslant_p$ is continuous. Let x_n and y_n be sequences in D with $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} y_n = y$ and $x_n \leqslant_p y_n$ for every $n = 1, 2, 3, \ldots$ We wish to show that $x \leqslant_p y$. Now $xy \cap xp$ is connected since D is hereditarily unicoherent. In fact $xy \cap xp$ is an arc (it is a subcontinuum of the arc xp).

Thus it is a closed subinterval of the totally ordered (in the sense of $\leqslant_p$) arc xp and so it must have a minimal element. So let

(1) $$q = \min(xy \cap xp) \quad \text{(in the sense of } \leqslant_p).$$

By theorem 1.8 $xy \subset \operatorname{Li}_{n\to\infty} x_n y_n$. Therefore, since $q \in xy$ we can find points $q_n \in x_n y_n$ such that $\lim_{n\to\infty} q_n = q$. Since $q_n \in x_n y_n$ and $x_n \leqslant_p y_n$ for each n, it is not difficult to see that $x_n \leqslant_p q_n$ for each n, i.e. $x_n \in q_n p$ for each n. Now note that for each n $h(\{q_n\} \times I)$ is a subcontinuum of D containing q_n and p ($h(q_n, 0) = q_n$ and $h(q_n, 1) = p$). Therefore, since D is hereditarily unicoherent we must have $q_n p \subset h(\{q_n\} \times I)$ for each n (see lemma 1.3, recalling that $q_n p = I(\{q_n, p\})$). Thus $x_n \in h(\{q_n\} \times I)$ for each n, i.e. for each n there exists $t_n \in I$ such that $h(q_n, t_n) = x_n$. Passing to a subsequence if necessary, we may assume that $\lim_{n\to\infty} t_n = t$. The continuity of h then yields

(2) $$h(q, t) = \lim_{n\to\infty} h(q_n, t_n) = \lim_{n\to\infty} x_n = x.$$

Now $h_t(D)$ is a subcontinuum of D containing x and p ($h(p, t) = p$ by hypothesis). Therefore, arguing as above, we must have $xp \subset h_t(D)$. But $q \in xp$. Therefore, since h is presumed to be a retracting homotopy, we can infer that $h(q, t) = q$. (2) then yields

(3) $$q = x.$$

Therefore x is minimal in $xy \cap xp$ in the sense of $\leqslant_p$. This is easily seen to imply that $x \leqslant_p y$, completing the proof. ∎

Before proving the converse to the above theorem, we recall a theorem proved by Carruth in [1].

THEOREM 1.15. *If X is a compact metric space and $\leqslant$ is a continuous partial order on X, then there is an equivalent metric r for X satisfying*

(∗) *if $x \leqslant y \leqslant z$ in X and $x \neq y \neq z$, then $r(x, y) < r(x, z)$.*

In fact, r can be constructed so that

(∗∗) *if $x \leqslant y \leqslant z$ in X, then $r(x, z) = r(x, y) + r(y, z)$.*

THEOREM 1.16. *If D is a smooth dendroid, then D admits a homotopy $h\colon D \times I \to D$ satisfying conditions* (i)–(iii) of theorem 1.14.

Proof. Let D be a smooth dendroid and let $p \in D$ be such that $\leqslant_p$ is a continuous partial order. Let ϱ be a metric on D satisfying condition (∗∗) of theorem 1.15. Since D is compact, we may also assume

(1) $$\max\{\varrho(x, p)\colon x \in D\} = 1.$$

CLAIM 1. *Given any $x \in D$ and any real number r such that $0 \leqslant r \leqslant \varrho(x, p)$, there is a unique $y \in D$ with $y \leqslant_p x$ and $\varrho(y, p) = r$.*

Proof of claim. Let x and r be as above. Since $\{y \in D: y \leqslant_p x\} = xp$ is connected, there must be at least one $y \leqslant_p x$ such that $\varrho(y, p) = r$. Suppose now that y and z both have this property. Since $\{y \in D: y \leqslant_p x\} = xp$ is totally ordered, we must have either $y \leqslant_p z$ or $z \leqslant_p y$. We suppose (without loss of generality) that $y \leqslant_p z$. Then since $p \leqslant_p y \leqslant_p z$, our hypothesis concerning ϱ implies that

(2) $$r = \varrho(p, z) = \varrho(p, y) + \varrho(y, z) = r + \varrho(y, z).$$

Therefore we must have $\varrho(y, z) = 0$, i.e. $y = z$.

We now define a homotopy $h: D \times I \to D$ as follows:

$$h(x, t) = \begin{cases} \text{that unique } y \leqslant_p x \text{ such that } \varrho(y, p) = t \text{ if } t \leqslant \varrho(x, p), \\ x \text{ if } \varrho(x, p) \leqslant t. \end{cases}$$

CLAIM 2. *h is continuous.*

Proof of claim. Since ϱ is continuous, the two sets

$$A = \{(x, t) \in D \times I: t \leqslant \varrho(x, p)\} \quad \text{and} \quad B = \{(x, t) \in D \times I: \varrho(x, p) \leqslant t\}$$

are closed in $D \times I$. Moreover $A \cup B = D \times I$. So if we can show that h is continuous on these two sets, we will be done. Certainly h is continuous on B. Now suppose that (x_n, t_n) is a sequence in A and that $\lim_{n\to\infty} (x_n, t_n) = (x, t)$. Let y be a cluster point of $\{h(x_n, t_n): n = 1, 2, 3, \ldots\}$ Since $h(x_n, t_n) \leqslant_p x_n$ for every n, the fact that $\leqslant_p$ is continuous implies that $y \leqslant_p x$. Since $\varrho(h(x_n, t_n), p) = t_n$ for every n and $\lim_{n\to\infty} t_n = t$, the continuity of ϱ implies that $\varrho(y, p) = t$. But then y is that unique $y \leqslant_p x$ such that $\varrho(y, p) = t$, i.e. $y = h(x, t)$. Since D is compact, this establishes the continuity of h and completes the proof of claim 2.

Suppose $y \in h_t(D)$ for some $t \in I$. Then from the definition of h it is clear that $\varrho(y, p) \leqslant t$, i.e. $(y, t) \in B$. Therefore $h(y, t) = y$. This tells us that h is a retracting homotopy. The fact that $\varrho(p, p) = 0$ implies that $(p, t) \in B$ for every $t \in I$ and hence that $h(p, t) = p$ for every $t \in I$. (1) above implies that $h(x, 1) = x$ for every $x \in D$. Finally, $h(x, 0) = p$ for every x since $0 \leqslant \varrho(x, p)$ for every x (i.e. $(x, 0) \in A$ and p is that unique $y \leqslant_p x$ such that $\varrho(y, p) = 0$). It is now clear that $h': D \times I \to D$ defined by

$$h'(x, t) = h(x, 1-t)$$

is the desired homotopy. ∎

The author does not know whether condition (iii) is necessary in establishing theorem 1.14. That is, suppose that D is a dendroid which admits a homotopy $h: D \times I \to D$ satisfying conditions (i) and (ii) of theorem 1.14. Does it still follow that D is smooth? In the next section we will see that this is indeed the case for fans, thus answering a question raised by Charatonik in [3].

2. An application to fans.

In [3] Charatonik asks the following question: If F is a fan which admits a homotopy $h: F \times I \to F$ satisfying conditions (i) and (ii) of theorem 1.14, does it follow that F is smooth? We will answer this question by showing that if a fan F admits such a homotopy, then F also admits a homotopy satisfying conditions (i)–(iii) of theorem 1.14.

DEFINITION 2.1. A point p in a continuum X is called a *ramification point* of X (in the classical sense) if it is the common endpoint of three (or more) arcs in X whose only common point is p.

DEFINITION 2.2. A topological space is called a *fan* if it is a dendroid with exactly one ramification point, which is called its *top*.

DEFINITION 2.3. A fan F with top p is said to be *smooth* if given a sequence x_n in F such that $\lim_{n\to\infty} x_n = x$, it follows that $\operatorname{Lim}_{n\to\infty} x_n p = xp$ (1).

DEFINITION 2.4. A point e of a continuum X is called an *endpoint* of X (in the classical sense) if given any two arcs A_1 and A_2 in X such that $e \in A_1 \cap A_2$, it follows that $A_1 \cap A_2 \backslash \{e\} \neq \emptyset$.

NOTATION 2.5. If F is a fan, we let $E(F)$ denote $\{e \in F: e$ is an endpoint of $F\}$.

Remark 2.6. It is easy to verify that if F is a fan with top p, then $F = \bigcup \{eT: e \in E(F)\}$.

We now begin some lemmas leading to the main result of this section. For the remainder of the section we make the standing assumption that F is a fan with top p and that $h: F \times I \to F$ is a retracting homotopy which contracts F to some point $q \in F$.

LEMMA 2.7. *There is a smallest number $t \in I$ such that $h_t(F)$ is an arc (or a point).*

Proof. Clearly $\{t \in I: h_t(F)$ is an arc or a point$\} \neq \emptyset$ since $h_1(F) = \{q\}$. So it will suffice to show that this set is closed in I. We proceed by showing that its complement is open in I. Suppose that $h_{t_0}(F)$ is neither an arc nor a point. Then we must have $p \in h_{t_0}(F)$ (the components of $F\backslash\{p\}$ are all of the form $ep\backslash\{p\}$ for some $e \in E(F)$). Further, there must be three distinct points $e_1, e_2, e_3 \in E(F)$ such that $e_i p\backslash\{p\} \cap h_{t_0}(F) \neq \emptyset$ for $i = 1, 2, 3$. Let

(*) $$x_i \in (e_i p\backslash\{p\}) \cap h_{t_0}(F) \quad \text{for } i = 1, 2, 3.$$

(1) The reader will note that definition 2.3 appears to impose a stronger condition than definition 1.7. However, in [4] Charatonik and Eberhart show that the two conditions are equivalent for fans.

Let ϱ be a metric on F and define $\varepsilon = \min\{\varrho(x_i, p): i = 1, 2, 3\}$. By the continuity of h we can find a $\delta > 0$ such $|t - t_0| < \delta$ implies that $\varrho(h(x, t), h(x, t_0)) < \varepsilon$ for every $x \in F$. Note that for every $x \in F$ and for every $t \in I$, there is a path in $h(\{x\} \times [t, t_0])$ from $h(x, t)$ to $h(x, t_0)$. So if $|t - t_0| < \delta$, then for $i = 1, 2, 3$ $h(x_i, t)$ must lie in the path component of $\{y \in F: \varrho(h(x_i, t_0), y) < \varepsilon\}$ containing $h(x_i, t_0)$. But since $x_i \in h_{t_0}(F)$ for $i = 1, 2, 3$ and h is a retracting homotopy, we have $h(x_i, t_0) = x_i$ for $i = 1, 2, 3$. Therefore, by the choice of δ we have

$$(**) \qquad h(x_i, t) \in e_i p \setminus \{p\} \quad \text{for } i = 1, 2, 3 \quad \text{and} \quad |t - t_0| < \delta .$$

Since $h_t(F)$ is connected and contains $h(x_i, t)$ for $i = 1, 2, 3$ and for every t, it follows that $h_t(F)$ cannot be arc if $|t - t_0| < \delta$. This completes the proof. ∎

LEMMA 2.8. *If t_0 is the smallest number in I such that $h_{t_0}(F)$ is an arc or a point, then $p \in h_{t_0}(F)$.*

Proof. As we observed in the proof of 2.7, $p \in h_t(F)$ for every $t \in \{t \in I: h_t(F)$ is neither an arc nor a point$\}$. But clearly t_0 lies in the closure of this set. The continuity of h yields the desired result. ∎

By lemma 2.8, if t_0 is the smallest number in the unit interval such that $h_{t_0}(F)$ is an arc, then $p \in h_{t_0}(F)$. Thus there is a retracting homotopy h' which over the interval $[t_0, 1]$ contracts $h_{t_0}(F)$ to p, i.e. there is a map

$$h': h_{t_0}(F) \times [t_0, 1] \to h_{t_0}(F)$$

such that h'_t is a retraction for every $t \in [t_0, 1]$, $h'_{t_0} = i_{h_{t_0}(F)}$ $(= h_{t_0}|h_{t_0}(F))$, $h'(x, 1) = p$ for every $x \in h_{t_0}(F)$ and $h'(p, t) = p$ for every $t \in [t_0, 1]$ (all of this can be done since $h_{t_0}(F)$ is an arc containing p). Letting h' be as described above, we now define a new homotopy $h'': F \times I \to F$ as follows:

$$h''(x, t) = \begin{cases} h(x, t) & \text{if } t \in [0, t_0], \\ h'(h(x, t_0), t) & \text{if } t \in [t_0, 1]. \end{cases}$$

Observe that with h'' defined as above we have $p \in h''_t(F)$ for every $t \in I$. Certainly h'' is continuous, and the fact the composition of retractions is a retraction implies that h'' is a retracting homotopy. The following lemma is now clear.

LEMMA 2.9. *If a fan F with top p admits a retracting homotopy h: $F \times I \to F$ which contracts F to a point, then F admits a homotopy h'' such that*

(i) *h'' is a retracting homotopy,*

(ii) *h'' contracts F to p,*

(iii) *h'' $(p, t) = p$ for every $t \in I$.*

Lemma 2.9 and theorem 1.14 now imply the following theorem:

THEOREM 2.10. *If a fan F is contractible to a point by a retracting homotopy, then F is smooth.*

It should be remarked that theorem 1.16 is already known for fans. In fact Charatonik and Eberhart have shown that if F is a smooth fan, then F is embeddable in the cone over the cantor set (this follows from [3], theorem 9, p. 27 and [6], Corollary 4). This theorem and theorem 2.10 above suffice to answer all questions raised by Charatonik in [3].

Theorem 2.10 above states that if a fan fails to be smooth, then it also fails to be contractible by a retracting homotopy. The author does not known the answer to the following question: Does there exist a fan which fails to be contractible?

3. Some remarks on smoothness in hereditarily unicoherent continua.

Remark 3.1. Since any contractible Hausdorff space is arcwise connected, if a hereditarily unicoherent continuum is contractible by a retracting homotopy, then it is a dendroid.

DEFINITION 3.2. Let H be a hereditarily unicoherent continuum and let $p \in H$. We then define the relation $\leqslant_p$ on H as follows: we set $x \leqslant_p y$ in H *if* $x \in I(\{p, y\})$ (recall lemma 1.3).

We now state two definitions of smoothness for hereditarily unicoherent continua. Lemma 1.12 above implies that these definitions agree for dendroids.

DEFINITION 3.3. A hereditarily unicoherent continuum is said to be *smooth*$_1$ if there exists a point $p \in H$ such that the relation $\leqslant_p$ is a continuous partial order on H.

DEFINITION 3.4. A hereditarily unicoherent continuum is said to be *smooth*$_2$ if there exists a point $p \in H$ such that given any sequence a_n in H such that $\lim_{n\to\infty} a_n = a$, it follows that $\operatorname{Lim}_{n\to\infty} I(\{a_n, p\}) = I(\{a, p\})$.

Remark 3.5. If a hereditarily unicoherent continuum H is smooth$_1$, then Koch's theorem on the existence of order arcs (see [6]) implies that H is arcwise connected.

There exist hereditarily unicoherent continua which are smooth$_2$ but which fail to be contractible by a retracting homotopy. An easy example is the subset H of the plane E^2 defined by

$$H = \mathrm{Cl}(\{(x, y) \in E^2: y = \sin(1/x) \text{ and } 0 < x \leqslant 1\}).$$

It is easy to show that, setting $p = (1, \sin(1))$, H is smooth$_2$. But H is not even contractible.

The above remarks show that the generalizations of theorems 1.14 and 1.16 into the setting of hereditarily unicoherent continua are either vacuous or false.

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