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Reçu par la Rédaction le 2. 12. 1968

On the topology of curves I

by

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By a *curve* we mean a 1-dimensional compact connected metric space. Thus all curves are non-degenerate continua. Although a theory of curves had been established at the early stage of set-theoretical topology by Karl Menger and P. S. Urysohn, some set-theoretical aspects of the theory seem to be far from being explored. Among them are various cardinality problems concerning topological structure of curves and their subsets. For non-compact subsets a classification relevant to connectivity properties had been elaborated by A. D. Taĭmanov [5]. A countable ordinal which we call the non-connectivity index of a space (see § 1) indicates the level on which quasi-components become components of a given point. Solving a problem raised by P. S. Novikov (see § 3) we show that there exists a plane G_δ-set whose non-connectivity indexes are arbitrarily high. This is done by constructing a subset of a pseudo-arc, and we use a result of Howard Cook [2] to prove that an uncountable compact bundle of pseudo-arcs is embedable in a pseudo-arc itself (see § 2). On the other hand, it is shown (see § 4) that non-connectivity indexes of a subset of a rational curve are bounded by a countable ordinal. The results of the present paper were partially announced in [4].

§ 1. Non-connectivity indexes. Let us recall that the quasi-component $Q(X, x)$ of a topological space X at a point $x \in X$ is the intersection of all closed-open subsets of X that contain x. We write $Q^0(X, x) = X$, and we use a transfinite induction to define $Q^\alpha(X, x)$ for each ordinal α, namely

$$Q^{\alpha+1}(X, x) = Q\big(Q^\alpha(X, x), x\big)$$

and

$$Q^\lambda(X, x) = \bigcap_{\alpha<\lambda} Q^\alpha(X, x)$$

for limit λ. The set $Q^\alpha(X, x)$ is said to be the *quasi-component of order* α of the space X at the point x. Observe that $Q^\alpha(X, x)$ is a closed subset of X, and therefore the decreasing transfinite sequence

$$Q^0(X, x) \supset Q^1(X, x) \supset \ldots \supset Q^\alpha(X, x) \supset \ldots$$

must have a term $Q^{\alpha}(X, x)$ which is equal to the next term $Q^{\alpha+1}(X, x)$. This means that $Q^{\alpha}(X, x)$ is connected, and $Q^{\alpha}(X, x) = Q^{\beta}(X, x)$ for all ordinals $\beta > \alpha$. We call the ordinal

$$nc(X, x) = \operatorname{Min}\{\alpha\colon\ Q^{\alpha}(X, x) = Q^{\alpha+1}(X, x)\}$$

the *non-connectivity index* of the space X at the point x. Clearly, the quasi-component of order $nc(X, x)$ of X at x coincides with the component of X to which x belongs. All spaces examined here will be separable metric, and thus we shall always have $nc(X, x) < \Omega$ where Ω stands for the least uncountable ordinal.

Our first deal is to construct some examples of spaces with a given prescribed non-connectivity index $\alpha < \Omega$ at a point. Such examples are well-known (see [5], p. 369) but we require an additional property: all of them should be embedable in the pseudo-arc. A curve P is said to be a *pseudo-arc* provided P is hereditarily indecomposable and chainable. A curve C is called *chainable* provided C admits finite open covers whose elements have arbitrarily small diameters and whose nerves are arcs. A theorem of R. H. Bing says that pseudo-arcs are homeomorphic to each other. By a *continuum* we mean a compact connected metric space, and the union of all proper subcontinua of a continuum X which contain a given point is said to be a *composant* of X. A composant of a continuum X is a dense subset of X. If X is an indecomposable continuum, then X has uncountably many composants and they are pairwise disjoint. If $C_0, C_1, \ldots$ are closed subsets of a compact metric space X, we write $C_0 = \operatorname{Lim} C_i$ provided C_i converge to C_0 in the space 2^X of closed subsets of X, the topology in 2^X being generated by the Hausdorff distance (see [3], p. 214).

1.1. *If P is a pseudo-arc and P' is a proper subcontinuum of P, then there exist continua $K^i \subset P \setminus P'$ such that $P' = \operatorname{Lim} K^i$.*

Proof. If P' is degenerate, 1.1 trivially holds. We can assume that P' is non-degenerate. Let U and V be non-empty open subsets of P whose closures are disjoint. Since each composant of P intersects both U and V, we can find an infinite sequence $C_1, C_2, \ldots$ of subcontinua of $P \setminus (U \cup V)$ such that each C_i intersects the boundaries of U and V, and for $i \neq j$ the continua C_i and C_j are contained in different composants of P. The space 2^P being compact, this sequence must have a subsequence C_{k_1}, $C_{k_2}, \ldots$ which converges to a continuum $C_0 \subset P \setminus (U \cup V)$. Clearly, C_0 also intersects the boundaries of U and V, whence C_0 is a non-degenerate proper subcontinuum of P. Thus there exists a homeomorphism h of P onto itself such that $P' = h(C_0)$ (see [1], p. 741).

We claim that C_0 can meet only one of the continua C_{k_i}. Suppose on the contrary that C_0 meets both C_{k_i} and C_{k_j} where $k_i \neq k_j$. Since P

is indecomposable, neither $C_0 \cup C_{k_i}$ nor $C_0 \cup C_{k_j}$ is P. By the same reason, we then have

$$P \neq C_0 \cup C_{k_i} \cup C_{k_j}$$

which implies that C_{k_i} and C_{k_j} are contained in one composant of P contrary to the definition of C_i. Hence there exists a positive integer j_0 such that $C_{k_j} \subset P \setminus C_0$ for $j > j_0$. It follows that the continua

$$K^i = h(C_{k_{j_0+i}}) \qquad (i = 1, 2, \ldots)$$

satisfy the conditions required in 1.1, and 1.1 is proved.

Let $\boldsymbol{B}_j^k(X)$ denote the j-th Borel class ($j = 0, 1, \ldots$), countable-additive when $k = 0$, and countable-multiplicative when $k = 1$, of subsets of X. Thus, for instance, the elements of $\boldsymbol{B}_1^0(X)$ are all F_σ-sets in X, and the elements of $\boldsymbol{B}_1^1(X)$ are all G_δ-sets in X.

1.2. Theorem. *If P is a pseudo-arc and $p \in P$, then for each ordinal $\alpha < \Omega$ there exists a set $Z_\alpha \subset P$ such that*

$$p \in Z_\alpha \in \boldsymbol{B}_1^0(P) \cap \boldsymbol{B}_1^1(P), \qquad nc(Z_\alpha, p) = \alpha.$$

Proof. We construct Z_α inductively for $\alpha < \gamma$ where $\gamma < \Omega$ is a fixed ordinal. Let $d_\alpha > 0$ be a real number for $\alpha < \gamma$, such that $d_\alpha \leqslant \operatorname{diam} P$ and $d_\beta < d_\alpha$ for $\alpha < \beta < \gamma$. The F_σ-G_δ-sets $Z_\alpha \subset P$ will contain p and will satisfy the following conditions.

(i) If $\alpha < \gamma$, then $Q^{\alpha}(Z_\alpha, p) = C_\alpha$ is a continuum with $\operatorname{diam} C_\alpha \geqslant d_\alpha$.

(ii) If $\alpha < \gamma$ and $p \in X \subset C_\alpha$, then $Q^{\alpha}(X \cup (Z_\alpha \setminus C_\alpha), p) = X$.

(iii) If $\alpha < \beta < \gamma$, then $Z_\beta \subset Z_\alpha$, $Z_\alpha \setminus C_\alpha \subset Z_\beta \setminus C_\beta$, and $(C_\alpha \cap Z_\beta) \setminus C_\beta \neq \emptyset$.

We put $Z_0 = P$ and suppose that we are given an ordinal $\beta < \gamma$ such that Z_α are defined for $\alpha < \beta$. To define Z_β, two cases should be distinguished.

Case 1: β is not limit. Then there exists an ordinal α such that $\beta = \alpha + 1$. Since $0 < d_\beta < d_\alpha \leqslant \operatorname{diam} C_\alpha$, the continuum C_α is a pseudo-arc containing p, by (i), and there clearly exists a proper subcontinuum $C_\beta \subset C_\alpha$ containing p with $\operatorname{diam} C_\beta \geqslant d_\beta$. Let $K_\beta^i \subset C_\alpha \setminus C_\beta$ be continua such that $C_\beta = \operatorname{Lim} K_\beta^i$, according to 1.1. We define

$$Z_\beta = (Z_\alpha \setminus C_\alpha) \cup C_\beta \cup \bigcup_{i=1}^{\infty} K_\beta^i$$

and we see that Z_β is an F_σ-G_δ-set because it is obtained from the F_σ-G_δ-set Z_α by subtracting a compact set and then adding another compact set. If $p \in X \subset C_\beta$, we have

$$\begin{aligned} Q^{\beta}(X \cup (Z_\beta \setminus C_\beta), p) &= Q\Big(Q^{\alpha}\big(X \cup \bigcup_{i=1}^{\infty} K_\beta^i \cup (Z_\alpha \setminus C_\alpha), p\big), p\Big) \\ &= Q\big(X \cup \bigcup_{i=1}^{\infty} K_\beta^i, p\big) = X, \end{aligned}$$

by (ii). In particular, $Q^\beta(Z_\beta, p) = C_\beta$. Moreover, it follows from the definition of Z_β that $Z_\beta \subset Z_\alpha$, $Z_\alpha \setminus C_\alpha \subset Z_\beta \setminus C_\beta$, and if $\zeta < \beta$, then $C_\alpha \subset C_\zeta$, by (iii), whence

$$\emptyset \neq \bigcup_{i=1}^{\infty} K_\beta^i = (C_\alpha \cap Z_\beta)\setminus C_\beta \subset (C_\zeta \cap Z_\beta)\setminus C_\beta .$$

Case 2: β is limit. We define

$$Z_\beta = \bigcap_{\alpha<\beta} Z_\alpha$$

and we see that Z_β is a G_δ-set. In order to prove that Z_β is an F_σ-set, let us observe that, given an ordinal $\alpha < \beta$, we have $Z_\alpha \setminus C_\alpha \subset Z_\xi$ for each $\xi < \beta$. Indeed, it follows from (iii) that $Z_\alpha \subset Z_\xi$ or $Z_\alpha \setminus C_\alpha \subset Z_\xi \setminus C_\xi$ provided $\xi \leqslant \alpha$ or $\alpha < \xi$, respectively. Thus $Z_\alpha \setminus C_\alpha \subset Z_\beta$ for $\alpha < \beta$. Condition (iii) also guarantees that f $\alpha < \xi < \beta$, then $C_\xi \subset Z_\xi \subset Z_\alpha$ and $Z_\alpha \setminus C_\alpha \subset Z_\alpha \setminus C_\xi$ whence $C_\xi \subset C_\alpha$. Thus the continua C_α $(\alpha < \beta)$ form a decreasing sequence of type β, and therefore

$$C_\beta = \bigcap_{\alpha<\beta} C_\alpha$$

is a subcontinuum of Z_β. Since

$$Z_\beta \setminus C_\beta \subset \bigcup_{\alpha<\beta} (Z_\alpha \setminus C_\alpha) ,$$

it now follows that

$$Z_\beta = C_\beta \cup \bigcup_{\alpha<\beta} (Z_\alpha \setminus C_\alpha) ,$$

whence Z_β is an F_σ-set. If $p \in X \subset C_\beta$, we have

$$Q^\beta\big(X \cup (Z_\beta \setminus C_\beta), p\big) \subset Q^\beta\big(X \cup (Z_\alpha \setminus C_\beta), p\big)$$
$$\subset Q^\alpha\big(X \cup (C_\alpha \setminus C_\beta) \cup (Z_\alpha \setminus C_\alpha), p\big) = X \cup (C_\alpha \setminus C_\beta)$$

for $\alpha < \beta$, by (ii). Thus we obtain

$$Q^\beta\big(X \cup (Z_\beta \setminus C_\beta), p\big) \subset \bigcap_{\alpha<\beta} [X \cup (C_\alpha \setminus C_\beta)] = X \cup \bigcap_{\alpha<\beta} (C_\alpha \setminus C_\beta) = X ,$$

and the inverse inclusion also holds because

$$X = Q^\alpha\big(X \cup (Z_\alpha \setminus C_\alpha), p\big) \subset Q^\alpha\big(X \cup (Z_\beta \setminus C_\beta), p\big)$$

for $\alpha < \beta$. In particular, $Q^\beta(Z_\beta, p) = C_\beta$. Moreover, since $\operatorname{diam} C_\alpha \geqslant d_\alpha > d_\beta$, by (i), we get $\operatorname{diam} C_\beta \geqslant d_\beta$, and if $\alpha < \beta$, then

$$\emptyset \neq (C_\alpha \cap Z_{\alpha+1})\setminus C_{\alpha+1} = [C_\alpha \cap (Z_{\alpha+1}\setminus C_{\alpha+1})]\setminus C_{\alpha+1} \subset (C_\alpha \cap Z_\beta)\setminus C_\beta .$$

The sets Z_α being defined for $\alpha < \gamma$, it remains to show that the non-connectivity index of Z_α at p is α. From (i) we get $nc(Z_\alpha, p) \leqslant \alpha$. If $\xi < \alpha$, then $Z_\xi \setminus C_\xi \subset Z_\alpha \subset Z_\xi$, by (iii), and

$$Q^\xi(Z_\alpha, p) = Q^\xi[(C_\xi \cap Z_\alpha) \cup (Z_\xi \setminus C_\xi), p] = C_\xi \cap Z_\alpha ,$$

by (ii). Hence $Q^\xi(Z_\alpha, p) \neq C_\alpha$, by (iii), and

$$Q^\xi(Z_\alpha, p) \neq Q^\alpha(Z_\alpha, p) ,$$

by (i). Consequently, $Q^\xi(Z_\alpha, p) \neq Q^{\xi+1}(Z_\alpha, p)$, and thus $nc(Z_\alpha, p) \geqslant \alpha$.

§ 2. Pseudo-arcs on the plane. A function $\varphi\colon \{1, \ldots, m\} \to \{1, \ldots, n\}$ is said to be a *pattern* provided $|i-j| \leqslant 1$ implies $|\varphi(i)-\varphi(j)| \leqslant 1$ for $i, j = 1, \ldots, m$. Let h be a positive integer. We shall say that a pattern φ is *h-constant* provided for each integer $i = 1, \ldots, m$ there exists an integer $j \leqslant i$ such that $|i-j| \leqslant h$ and $\varphi(i) = \varphi(j+v)$ for $v = 1, \ldots, h$. A continuum homeomorphic with the 2-simplex is said to be a *disk*. By a *disk chain* we shall mean a finite sequence $(D_1, \ldots, D_n)$ of disks D_i on the plane such that $D_i \cap D_j \neq \emptyset$ if and only if $|i-j| \leqslant 1$, and each intersection $D_i \cap D_j$ is either empty or a disk. A simple geometrical argument shows that if $(D_1, \ldots, D_n)$ is a disk chain and $\varepsilon > 0$, then there exists a positive integer h with the following property: for each h-constant pattern $\varphi\colon \{1, \ldots, m\} \to \{1, \ldots, n\}$ a disk chain $(D'_1, \ldots, D'_m)$ can be constructed such that

$$D'_i \subset \operatorname{Int} D_{\varphi(i)} , \qquad \operatorname{diam} D'_i < \varepsilon$$

for $i = 1, \ldots, m$. We point out that m here can be arbitrary, but if φ is h-constant, then $h \leqslant m$.

2.1. Theorem. *If C is a chainable curve, then the product $C \times T$ of C by the Cantor ternary set T is embedable in the plane.*

Proof. An embedding will be defined by means of some chains $\boldsymbol{C}(k)$ and disk chains $\boldsymbol{D}(t_1, \ldots, t_k)$ where $t_i = 0$ or 2 for $i = 1, \ldots, k$. We shall construct these chains and disk chains by induction on $k = 1, 2, \ldots$ Each chain $\boldsymbol{C}(k)$ will be a finite sequence of open sets in C whose union is C; such chains composed of sets with arbitrarily small diameters exist by the assumption that C is chainable. For a fixed positive integer k, all 2^k disk chains $\boldsymbol{D}(t_1, \ldots, t_k)$ will consist of the same number of disks, equal to the number of terms in $\boldsymbol{C}(k)$.

Let $\boldsymbol{C}(1) = (C)$ and $\boldsymbol{D}(0) = (D^0)$, $\boldsymbol{D}(2) = (D^2)$ where D^0, D^2 are disjoint disks on the plane with diameters less than 1. Suppose that $\boldsymbol{C}(k-1)$ and $\boldsymbol{D}(t_1, \ldots, t_{k-1})$ are already given where $k \geqslant 2$ is a fixed integer; we are going to construct $\boldsymbol{C}(k)$ and $\boldsymbol{D}(t_1, \ldots, t_k)$. Denoting

$$\boldsymbol{C}(k-1) = (C_1, \ldots, C_n)$$

we see that the boundaries of C_i and C_j are disjoint if $|i-j| = 1$. It follows that if h is a positive integer and $(C'_1, \ldots, C'_m)$ is a chain of open sets in C whose union is C and whose diameters are sufficiently small, then there exists an h-constant pattern $\varphi\colon \{1, \ldots, m\} \to \{1, \ldots, n\}$ such that the closure of C'_i is contained in $C_{\varphi(i)}$ for $i = 1, \ldots, m$. Let us choose h large enough, so that for each disk chain

$$\boldsymbol{D}(t_1, \ldots, t_{k-1}) = (D_1, \ldots, D_n)$$

there exists a disk chain $(D'_1, \ldots, D'_m)$ such that D'_i is contained in the interior of $D_{\varphi(i)}$ and $\operatorname{diam} D'_i < 1/k$ for $i = 1, \ldots, m$. We define $\boldsymbol{C}(k)$ be the chain $(C'_1, \ldots, C'_m)$ just described with the additional requirement that $\operatorname{diam} C'_i < 1/k$ for $i = 1, \ldots, m$. We then split the disk chain $(D'_1, \ldots, D'_m)$ into two disk chains

$$\boldsymbol{D}(t_1, \ldots, t_{k-1}, 0) = (D^0_1, \ldots, D^0_m),$$

$$\boldsymbol{D}(t_1, \ldots, t_{k-1}, 2) = (D^2_1, \ldots, D^2_m)$$

such that

$$D^0_i \cup D^2_i \subset D'_i, \qquad D^0_i \cap D^2_j = \emptyset$$

for $i, j = 1, \ldots, m$. Hence $D^u_i \subset \operatorname{Int} D_{\varphi(i)}$ for $i = 1, \ldots, m$ and $u = 0, 2$. This completes the construction of $\boldsymbol{C}(k)$ and $\boldsymbol{D}(t_1, \ldots, t_k)$.

Now, we can define an embedding f of $C \times T$ into the plane. If $(c, t) \in C \times T$, we write

$$t = (0, t_1 t_2 \ldots)_3$$

where $t_i = 0$ or 2 for $i = 1, 2, \ldots$ The point c belongs to either only one or two adjacent terms C'_j of the chain $\boldsymbol{C}(k)$. Let $F_k(c, t)$ be the union of corresponding terms $D^{t_k}_j$ in the disk chain $\boldsymbol{D}(t_1, \ldots, t_k)$. Thus $\operatorname{diam} F_k(c, t) < 2/k$, and $c \in C'_j \subset C_{\varphi(j)}$ implies

$$D^{t_k}_j \subset D_{\varphi(j)} \subset F_{k-1}(c, t),$$

whence $F_k(c, t) \subset F_{k-1}(c, t)$ for $k = 2, 3, \ldots$ We see that the intersection of all compact sets $F_k(c, t)$ $(k = 1, 2, \ldots)$ is a point which we designate to be $f(c, t)$. If (c, t) and (c', t') are two points of $C \times T$ sufficiently close to each other, we have $t_i = t'_i$ for $i = 1, \ldots, k$, and there is a term of $\boldsymbol{C}(k)$ containing both c and c'. Therefore the sets $F_k(c, t)$ and $F_k(c', t')$ have a point in common, and thus

$$\operatorname{diam}[F_k(c, t) \cup F_k(c', t')] < 4/k$$

whence the distance from $f(c, t)$ to $f(c', t')$ is less than $4/k$. This yields the continuity of f. To see that f is 1-1, let us assume $(c, t) \neq (c', t')$. If $t \neq t'$, there exists an index k such that $t_k \neq t'_k$, and it follows from

$$F_k(c, t) \cap F_k(c', t') \subset \bigcup_{i,j} (D^{t_k}_i \cap D^{t'_k}_j) = \emptyset$$

that $f(c, t) \neq f(c', t')$. If $t = t'$, we have $c \neq c'$, and since diameters of the sets C'_i from $\boldsymbol{C}(k)$ are less than $1/k$, there exists an index k such that no adjacent terms of $\boldsymbol{C}(k)$ contain c and c'. Hence no adjacent terms $D^{t_k}_i$ and $D^{t_k}_j$ of $\boldsymbol{D}(t_1, \ldots, t_k)$ are contained in $F_k(c, t)$ and $F_k(c', t)$, respectively. It follows that $F_k(c, t) \cap F_k(c', t') = \emptyset$ again.

2.2. *If P is a pseudo-arc, then $P \times T$ is embedable in P.*

Proof. By 2.1, there is an embedding f of $P \times T$ into the plane. Since the components of $f(P \times T)$ are the pseudo-arcs $f(P \times \{t\})$, the set $f(P \times T)$ is a subset of a pseudo-arc (see [2], p. 17).

§ 3. Componentwise universal sets. Given any collection $\boldsymbol{E}$ of subsets of a space X, we say that a set U is *componentwise universal* in $\boldsymbol{E}$ provided $U \in \boldsymbol{E}$ and there exists a closed subset Y of X such that $U \subset Y$ and for each set $E \in \boldsymbol{E}$ there exists a component C of Y such that E is homeomorphic with $C \cap U$.

3.1. *If P is a pseudo-arc, then there exists a componentwise universal set in each Borel class $\boldsymbol{B}^k_j(P)$ $(j > 0)$.*

Proof. It is known that there exists a set $U' \in \boldsymbol{B}^k_j(P \times T)$ such that for each set $B \in \boldsymbol{B}^k_j(P)$ there exists a number $t \in T$ such that

$$B \times \{t\} = (P \times \{t\}) \cap U'$$

(see [3], pp. 368–371). Let $U = f(U')$ and $Y = f(P \times T)$ where f is an embedding of $P \times T$ into P, by 2.2. Since $j > 0$, we have $U \in \boldsymbol{B}^k_j(P)$. Moreover, the set $f(P \times \{t\})$ being a component of Y for $t \in T$, it follows that U is componentwise universal in $\boldsymbol{B}^k_j(P)$.

3.2. *If X is a compact metric space containing a pseudo-arc and U is a componentwise universal set in a Borel class $\boldsymbol{B}^k_j(X)$ $(j > 0)$, then*

$$\operatorname{Sup}\{nc(U, u)\colon u \in U\} = \Omega.$$

Proof. Let Y be a closed subset of X such that $U \subset Y$ and the intersections of components of Y with U represent elements of $\boldsymbol{B}^k_j(X)$. Since X contains a pseudo-arc, it follows from 1.2 that for each ordinal $\alpha < \Omega$ there exists a component C_α of Y and a point $p_\alpha \in C_\alpha \cap U$ such that $nc(C_\alpha \cap U, p_\alpha) = \alpha$. Then we have

$$Q(C_\alpha \cap U, p_\alpha) \subset Q(U, p_\alpha) \subset C_\alpha \cap U$$

and consequently

$$Q^{i+1}(C_\alpha \cap U, p_\alpha) \subset Q^{i+1}(U, p_\alpha) \subset Q^{i}(C_\alpha \cap U, p_\alpha)$$

for $i = 0, 1, \ldots$ Hence $Q^{\omega}(U, p_a) = Q^{\omega}(C_a \cap U, p_a)$ where ω is the least infinite ordinal. We conclude that if $\omega < a$, then $nc(U, p_a) = a$ which completes the proof of 3.2.

Remark. According to 3.1 and 3.2, the pseudo-arc contains a G_δ-set, as well as an F_σ-set, which have uncountably many non-connectivity indexes. Since the pseudo-arc is a plane curve, this answers a question of P. S. Novikov (see [5], p. 370).

§ 4. Subsets of rational curves. We are now trying to find conditions which guarantee that all non-connectivity indexes of a space are less than a countable ordinal. A curve C is called *rational* provided C admits an open basis whose elements have countable boundaries. The pseudo-arc is an example of a curve which is not rational.

4.1. THEOREM. *If X is a separable metric space such that*

$$\operatorname{Sup}\{nc(X, x):\ x \in X\} = \Omega$$

and $A \subset X$ is a countable set, then

$$\operatorname{Sup}\{nc(Y, x):\ Y \subset X \setminus A,\ x \in Y\} = \Omega .$$

Proof. Let $\{G_1, G_2, \ldots\}$ be a countable open basis in X. Let $x_a \in X$ be a point such that

$$nc(X, x_a) \geqslant a + a$$

for $a < \Omega$. In what follows we assume that $a > 0$ whence $a + a > a$. Then

$$Y_a = Q^{a+1}(X, x_a)$$

is a closed subset of X and $Q^a(X, x_a) \setminus Y_a \neq \emptyset$ for $a < \Omega$. Consequently, there exists a positive integer $j(a)$ such that

$$G_{j(a)} \cap Q^a(X, x_a) \neq \emptyset , \qquad G_{j(a)} \cap Y_a = \emptyset$$

for $a < \Omega$. We take an integer k such that $j(a) = k$ for $a \in \mathfrak{A}$ where $\mathfrak{A}$ is an uncountable set of countable ordinals. Suppose $a, \beta \in \mathfrak{A}$ and $a < \beta$. Then the set G_k meets $Q^{\beta}(X, x_\beta)$ and G_k is disjoint with the set Y_a. Thus $Q^{\beta}(X, x_\beta)$ cannot be contained in Y_a. Since $a+1 \leqslant \beta$, it follows that

$$Y_a \neq Q^{a+1}(X, x_\beta) ,$$

and $Y_a \cap Q^{a+1}(X, x_\beta) = \emptyset$ because two quasi-components of the same order are either equal or disjoint. Moreover, we have

$$Y_\beta = Q^{\beta+1}(X, x_\beta) \subset Q^{a+1}(X, x_\beta)$$

whence $Y_a \cap Y_\beta = \emptyset$. We see that $\{Y_a:\ a \in \mathfrak{A}\}$ is an uncountable collection of pairwise disjoint subsets of X. Since A is countable, there exists an uncountable set $\mathfrak{A}' \subset \mathfrak{A}$ such that $Y_a \subset X \setminus A$ for $a \in \mathfrak{A}'$. But $\omega \leqslant \gamma$ implies

$$Q^{\gamma}(Y_a, x_a) = Q^{a+\gamma}(X, x_a) ,$$

and therefore $nc(Y_a, x_a) \geqslant a$ if $\omega \leqslant a \in \mathfrak{A}'$. The set $\mathfrak{A}'$ being uncountable, we infer that the least upper bound of the ordinals $nc(Y_a, x_a)$, where $a \in \mathfrak{A}'$, is Ω.

4.2. *If C is a rational curve and $X \subset C$, then*

$$\operatorname{Sup}\{nc(X, x):\ x \in X\} < \Omega .$$

Proof. Since C is rational, there exists a countable set $A \subset C$ such that $C \setminus A$ is 0-dimensional. Then every non-degenerate set $Y \subset X \setminus A$ has $nc(Y, y) = 1$ for $y \in Y$, and 4.2 follows from 4.1.

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Reçu par la Rédaction le 16. 12. 1968