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Remarks on some class of continuous mappings of λ-dendroids

by

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A metric compact continuum is said to be a dendroid if it is hereditarily unicoherent and arcwise connected. It follows that it is hereditarily decomposable (see [2], (47), p. 239). A hereditarily unicoherent and hereditarily decomposable continuum is called a *λ-dendroid*. Note that every subcontinuum of a λ-dendroid is also a λ-dendroid.

It is proved in [4], Corollary 2, p. 29 that for every λ-dendroid X there exists a unique decomposition $\mathfrak{D}$ of X (called the canonical decomposition):

(1) $$X = \bigcup \{S_d \mid d \in \Delta(X)\}$$

such that

(i) $\mathfrak{D}$ is upper semicontinuous,

(ii) the elements S_d of $\mathfrak{D}$ are continua,

(iii) the hyperspace $\Delta(X)$ of $\mathfrak{D}$ is a dendroid,

(iv) $\mathfrak{D}$ is the finest possible decomposition among all decompositions satisfying (i), (ii) and (iii).

The elements S_d of $\mathfrak{D}$ are called strata of X. The monotone mapping φ of X onto $\Delta(X)$ defined by

(2) $$\varphi^{-1}(d) = S_d \quad \text{for } d \in \Delta(X)$$

is called canonical.

Let X and Y be λ-dendroids, φ and ψ their canonical mappings onto dendroids $\Delta(X)$ and $\Delta(Y)$ respectively. Continuous mappings of X into Y will be considered in this paper such that they take every stratum of X into a stratum of Y. Denote the class of all such mappings by $\mathfrak{C}$. Thus, by definition, a mapping

$$f\colon X \to Y$$

of X into Y belongs to $\mathfrak{C}$ if and only if for every point $d \in \Delta(X)$ there exists a point $d' \in \Delta(Y)$ such that

(3) $$f(\varphi^{-1}(d)) \subset \psi^{-1}(d') .$$

The following property can be immediately seen from the above definition:

Property 1. *If X, Y and Z are λ-dendroids, if f_1 and f_2 are continuous mappings of X into Y and Y into Z respectively, f_1 and f_2 both being in $\mathcal{C}$, then $f = f_2 f_1$ maps X into Z and $f \in \mathcal{C}$.*

Observe now that the decomposition $\mathfrak{D}$ of a λ-dendroid X into its strata, or the canonical mapping φ of X onto $\Delta(X)$, defines an equivalence relation on X: two points of X are in the relation if they belong to the same stratum of X, or—in other words—if they are mapped in the same point of $\Delta(X)$ under φ. Thus it follows from Theorem 7.7 in [7], p. 17 and from the definition of the class $\mathcal{C}$ of mappings that for every mapping $f\colon X \to Y$ in $\mathcal{C}$ there exists one and only one mapping g (called the mapping induced by f) of $\Delta(X)$ into $\Delta(Y)$ such that

(4) $$g\big(\varphi(x)\big) = \psi\big(f(x)\big) \quad \text{for } x \in X\,,$$

i.e. that the following diagram commutes:

(5) $$\begin{array}{ccc} X & \xrightarrow{\ f\ } & Y \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\psi} \\ \Delta(X) & \xrightarrow[\ g\ ]{} & \Delta(Y) \end{array}$$

Property 2. *If a continuous mapping f of a λ-dendroid X into a λ-dendroid Y belongs to $\mathcal{C}$, then the induced mapping g of $\Delta(X)$ into $\Delta(Y)$ is continuous. Conversely, if f is continuous and if there exists a mapping g such that the diagram* (5) *commutes, then f belongs to $\mathcal{C}$ and g is the mapping induced by f.*

In fact, the first part of the property follows from Theorem 4.3 in [7], p. 126. The second one is a consequence of the previously quoted Theorem 7.7 ([7], p. 17).

Property 3. *If f is a monotone mapping of a λ-dendroid X onto a continuum Y, then Y is a λ-dendroid and $f \in \mathcal{C}$.*

Indeed, the first part of the property is obvious (besides, it is a particular conclusion from a more general theorem which says that a confluent image of a λ-dendroid is a λ-dendroid (see [3], Theorem XIV, p. 217 and Theorem V, p. 214). To prove the second part, that $f \in \mathcal{C}$, observe that the superposition ψf is a monotone mapping of X into $\Delta(Y)$ and it implies that the mapping g of $\Delta(X)$ into $\Delta(Y)$ exists and satisfies (4) by Theorem 7 in [4], p. 29. Therefore $f \in \mathcal{C}$ by Property 2.

We conclude also from Theorem 7 ([4], p. 29) that g is monotone if f is. The inverse is not true: g may be monotone and f need not, as it can be seen by the following example.

Let X be the closure of the set of all points (x, y) in the plane for which

(6) $$y = \sin\frac{\pi}{2x} \quad \text{and} \quad 0 < |x| \leqslant 1\,,$$

i.e. the closure of the graph of the function defined by (6); let Y be the closure of the graph of the same function for $0 < x \leqslant 1$, and f—the mapping of X onto Y defined by

$$f\big((x, y)\big) = \begin{cases} (0, y) & \text{if } x < 0\,, \\ (x, y) & \text{if } x \geqslant 0\,, \end{cases}$$

i.e. for the left half of X the mapping f is the projection parallel to x-axis onto the straight segment $[-1, 1]$ of y-axis, and for the right half of X f is the identity. The canonical mappings φ and ψ are simply projections parallel to y-axis and the hyperspaces $\Delta(X)$ and $\Delta(Y)$ are the segments $[-1, 1]$ and $[0, 1]$ of x-axis respectively. Thus $g\colon \Delta(X) \to \Delta(Y)$ is defined by

$$g\big((x, 0)\big) = \begin{cases} (0, 0) & \text{if } x < 0\,, \\ (x, 0) & \text{if } x \geqslant 0\,, \end{cases}$$

and we see that g is monotone while f is not.

If X is arcwise connected (i.e. a dendroid), then also Y is, as a continuos image of X under f, and the canonical mappings φ and ψ are identities (see [4], (2.25), p. 22). In this case we may put $X = \Delta(X)$ and $Y = \Delta(Y)$, so that g can be taken simply as f. Therefore Property 2 implies

Property 4. *The class $\mathcal{C}$ contains all continuous mappings of dendroids into dendroids.*

The hypothesis on arcwise connectedness of X is essential in the above property because the projection of the continuum X in the previous example onto the limit segment, i.e. the mapping defined by

$$f\big((x, y)\big) = (0, y) \quad \text{for } (x, y) \in X$$

is an interior mapping of the irreducible continuum X onto an arc and does not belong to $\mathcal{C}$.

Recall that a λ-dendroid X is said to be monostratiform if it consists of only one stratum, i.e. if the hyperspace $\Delta(X)$ of the canonical decomposition $\mathfrak{D}$ is just a point (see [6], p. 933; an example of a monostratiform λ-dendroid is given in [5]). It follows immediately from the definitions that if a λ-dendroid Y is monostratiform, then an arbitrary continuous mapping of a λ-dendroid into Y belongs to $\mathcal{C}$. So we have

Property 5. *The class $\mathcal{C}$ contains all continuous mappings of λ-dendroids into monostratiform ones.*

Similarly, if X is a monostratiform λ-dendroid and if a surjection f from $\mathcal{C}$ maps X onto a λ-dendroid Y, then Y must be monostratiform. Hence

Property 6. *Monostratiformity of λ-dendroids is an invariant under mappings belonging to $\mathcal{C}$.*

In particular, we have by Property 3

Property 7. *Monostratiformity of λ-dendroids is an invariant under monotone mappings.*

Dr. J. B. Fugate has asked the following question, an answer to which is still unknown: is monostratiformity of λ-dendroids an invariant under confluent or interior mappings?

Recall further, that a λ-dendroid X is said to be hereditarily stratified if it contains no non-trivial monostratiform subcontinuum, i. e. such that the only monostratiform λ-dendroids contained in X are points (see [6], p. 933). One can ask if the property of λ-dendroids "to be hereditarily stratified" is an invariant under mappings belonging to the class $\mathcal{C}$. The answer is negative: there exist a hereditarily stratified λ-dendroid Z and a monotone mapping f of Z onto the monostratiform λ-dendroid X described in [5]. Namely Z is created from X simply by replacing every separating point of X by an arc. It can be done in the following manner. As we know from [5], (37), p. 81, the set S of all separating points of X is countable. Let $\{s_n\}$ be a sequence of all points of S. Replace each $s_n \in S$ by an arc L_n with

$$\lim_{n\to\infty} \delta(L_n) = 0 .$$

It is easy to see that the continuum Z obtained in this way is a hereditarily stratified λ-dendroid. Define the mapping f of Z onto X by

$$f^{-1}(x) = \begin{cases} L_n, & \text{if } x = s_n, \\ \text{a point}, & \text{if } x \in X \setminus S. \end{cases}$$

Thus f maps monotonously Z onto X which is monostratiform.

The following lemma generalizes Lemma 2 in [6], p. 932 from monotone mappings to mappings belonging to $\mathcal{C}$. The proof is very similar to the proof of that Lemma 2 given in [6].

Lemma. *For every mapping $h \in \mathcal{C}$ of a λ-dendroid X onto $h(X) \subset X$ there exists a stratum of X which contains its image.*

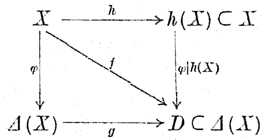
Proof. Let X be a λ-dendroid, φ its canonical mapping onto a dendroid $\Delta(X)$, h—a mapping of X belonging to the class $\mathcal{C}$ such that $h(X) \subset X$ and finally let $D = \varphi(h(X)) \subset \Delta(X)$. It follows from Lemma 1 in [6], p. 932 that $\varphi|h(X)$ is monotone, thus it is in $\mathcal{C}$ by Property 3. Putting

$$f = (\varphi|h(X))h \tag{7}$$

we see that $f \in \mathcal{C}$ by Property 1. Since D is a dendroid as a subcontinuum of the dendroid $\Delta(X)$ ([2], (49), p. 240), it can be taken as its own decomposition space of the canonical decomposition, with the identity ψ as the canonical mapping. For simplicity, ψ will be omitted in the further considerations. Applying Property 2 we see that a continuous mapping g exists of $\Delta(X)$ onto D with

$$g(\varphi(x)) = f(x) \quad \text{for every } x \in X, \tag{8}$$

i.e. such that the following diagram is commutative:



in which all mappings are continuous, onto, and belong to $\mathcal{C}$.

Since dendroids have fixed point property with respect to all continuous mappings (see [1], Theorem 2, p. 17), hence there exists a point $d_0 \in \Delta(X)$ such that $g(d_0) = d_0$. Consider an arbitrary point x of the stratum $S_{d_0} = \varphi^{-1}(d_0)$. Thus $\varphi(x) = d_0$ and, by (8), $g(d_0) = f(x)$, whence $d_0 = f(x)$, i.e. $d_0 = (\varphi|h(X))(h(x))$ by (7). It implies that

$$h(x) \in (\varphi|h(X))^{-1}(d_0) = \varphi^{-1}(d_0) \cap h(X) \subset \varphi^{-1}(d_0) = S_{d_0} .$$

So we have proved $h(x) \in S_{d_0}$ and thereby $h(S_{d_0}) \subset S_{d_0}$ which completes the proof.

Consider now a subclass $\mathcal{C}_h$ of the class $\mathcal{C}$ defined as follows: a mapping f of a λ-dendroid X into a λ-dendroid Y is in $\mathcal{C}_h$ if and only if f belongs to $\mathcal{C}$ hereditarily. In other words, $f \in \mathcal{C}_h$ if and only if for every subcontinuum K of X the partial mapping $f|K\colon K \to f(K)$ is in $\mathcal{C}$.

Since for every subcontinuum K of a λ-dendroid X and for every monotone mapping f of X the partial mapping $f|K$ is also monotone (see [6], Lemma 1, p. 932), hence Property 3 implies

Property 8. *The class $\mathcal{C}_h$ contains all monotone mappings of λ-dendroids.*

Every subcontinuum of a dendroid being a dendroid (see [2], (49), p. 240), Property 4 implies

Property 9. *The class $\mathcal{C}_h$ contains all continuous mappings of dendroids into dendroids.*

One can ask if it is possible to prove in the same way—using Property 5 instead of Property 4 that $\mathfrak{C}_h$ contains all continuous mappings of λ-dendroids into monostratiform ones. The answer is negative because the monostratiformity is not a hereditary property. Moreover, every λ-dendroid contains a subcontinuum K which is not monostratiform. Namely every λ-dendroid has two terminal points (see [10], Theorem 3.4, p. 192) p and q, so the unique irreducible continuum from p to q can be taken as K because strata of K coincide with tranches of K (see [4], (2.24), p. 22) and K contains infinitely many tranches since it is of type λ (see [4], (1.4), p. 15 and Theorem 1, p. 16). It is easy to find a continuous mapping of the monostratiform λ-dendroid X described in [5] onto itself which is not in $\mathfrak{C}_h$.

THEOREM. *Let X be a λ-dendroid and h a continuous mapping of X into itself, such that $h \in \mathfrak{C}_h$. If*

(9) *every nondegenerated monostratiform λ-dendroid M contained in X such that $h(M) = M$ contains a proper subcontinuum K which goes into itself under h,*

then X has fixed point property with respect to the class $\mathfrak{C}_h$.

Proof. Using Lemma 3 in [6], p. 933 it is sufficient to find a transfinite sequence of continua K_α $(\alpha < \Omega)$ of X such that

(10) $$\beta < \alpha \text{ implies } K_\alpha \subset K_\beta ,$$

(11) if $\beta < \alpha$ and K_β is not a point, then $K_\alpha \neq K_\beta$,

(12) $$h(K_\alpha) \subset K_\alpha \quad \text{for every } \alpha < \Omega .$$

Put

$$K_0 = X$$

and assume we have defined continua K_β satisfying (10), (11) and (12) for all $\beta < \alpha$. Now we shall define K_α.

If $\alpha = \beta+1$, consider two cases. If K_β is not monostratiform, we define K_α to be a stratum S of K_β which contains its image by the Lemma. If K_β is monostratiform, then either $h(K_\beta)$ is a proper subset of K_β or not. If $h(K_\beta) \neq K_\beta$ then we put

$$K_\alpha = h(K_\beta) .$$

If $h(K_\beta) = K_\beta$ then either it is a point—thus a fixed point, or it is nondegenerated. Taking K_β as M in (9) we define

$$K_\alpha = K .$$

It is readily seen that in all these cases K_α satisfies (10)–(12).

Finally, let α be a limit ordinal, $\alpha = \lim\limits_{\beta<\alpha} \beta$. Putting

$$K_\alpha = \bigcap_{\beta<\alpha} K_\beta$$

we see that (10) and (11) are fulfilled by definition. To prove the inclusion in (12) let us observe that

$$h(K_\alpha) = h(\bigcap_{\beta<\alpha} K_\beta) \subset \bigcap_{\beta<\alpha} h(K_\beta) .$$

Since $h(K_\beta) \subset K_\beta$ for $\beta < \alpha$ by the induction hypothesis, we have

$$\bigcap_{\beta<\alpha} h(K_\beta) \subset \bigcap_{\beta<\alpha} K_\beta = K_\alpha ,$$

whence (12) follows. So K_α are well-defined for all ordinals $\alpha < \Omega$ and the proof is finished.

Observe that (9) trivially holds in a particular case when X contains no nondegenerated monostratiform λ-dendroid, i.e. when X is hereditarily stratified. Thus the theorem implies

COROLLARY 1. *Every hereditarily stratified λ-dendroid has fixed point property with respect to the class $\mathfrak{C}_h$.*

Further, it follows from the above corollary and from Property 8 that (see [6], Theorem, p. 934).

COROLLARY 2. *Every hereditarily stratified λ-dendroid has fixed point property with respect to monotone mappings.*

The theorem and Corollary 1 partially generalize some other fixed point theorems proved by several authors—see [1], [8] and [9], especially by Ward, [11], and Young, [12] and [13].

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A fixed point theorem for continua which are hereditarily divisible by points

by

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1. Introduction. It has been conjectured (¹) that a continuum which is hereditarily divisible by points (that is, a continuum each of whose non-degenerate subcontinua has a cutpoint) has the fixed point property. The main result of this paper (²) is a special case of this conjecture. Specifically, it will be shown that if H is a continuum which is hereditarily divisible by points and $\tau(H) \neq \infty$, then H has the fixed point property. By $\tau(H)$ we denote the degree of non-local connectedness of H defined by Charatonik in [1] (³). This result generalizes the well known theorem (see [5] and others) that trees have the fixed point property, since (as is observed below) a continuum H is a tree if and only if H is hereditarily divisible by points and $\tau(H) = 0$. In the course of proving the main theorem we also prove a fixed point theorem (⁴) which is the generalization to the non-metric setting of a theorem of Young [7].

2. Preliminaries. This section is devoted to a number of preliminary results which will be needed in the proof of the fixed point theorem mentioned in the introduction. The main theorems of the section are generalizations of theorems due to Charatonik [1] and Young [7].

2.1. Degree of non-local connectedness. In [1] Charatonik defines the degree of non-local connectedness $\tau(H)$ of a hereditarily unicoherent metric continuum H and proves a number of properties of $\tau(H)$. All of the main results of his paper generalize to the non-metric setting

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(¹) The conjecture is due to Knaster.
(²) Theorem 3.27 below.
(³) Numbers in square brackets refer to the bibliography at the end of the paper.
(⁴) Theorem 2.2.18.