obvious remarks may be made: a Boolean algebra is atomless if and only if every maximal chain of it is dense-in-itself; a Boolean algebra is $\aleph^-$-complete if and only if every maximal chain in it is $\aleph^-$-complete; a Boolean algebra is superatomic (see [1]) if and only if no maximal chain in it has a dense-in-itself subchain. Nevertheless, the order types of the maximal chains of a Boolean algebra together with the cardinal number of maximal chains of each order type does not completely determine the structure of the algebra: letting $\aleph$ be any cardinal that is the limit of a strictly increasing ω-sequence of infinite cardinals, so that $\aleph < \aleph^{\aleph_0}$, it is easily shown that if B and B' are the Boolean algebras of finite and cofinite subsets of sets of cardinalities $\aleph$ and $\aleph^{\aleph_0}$, respectively, then both B and B' will have maximal chains only of type $\omega+\omega^*$, and both will have exactly $\aleph^{\aleph_0}$ of these.

EXAMPLE. J. Jakubik, in [3], has constructed Boolean algebras having maximal chains of varying length. A method of construction alternate to that presented there is indicated by the following observation: if $\{B_a: a \in A\}$ is a set of atomless Boolean algebras, and B is their Boolean product, and C is a maximal chain in B_a for some $a \in A$, then C is a maximal chain in B.

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On smooth dendroids

by

J. J. Charatonik and Carl Eberhart (Lexington, Ky.)

§ 1. Introduction. Investigating fans, especially very simple ones, called smooth (see [7], p. 7 and [9]) we have observed that the notion of smoothness of fans can be easily extended to smoothness of dendroids. Smooth dendroids are very close to some partially ordered spaces, called generalized trees, which were studied by Ward (see [16], p. 801). He assumed that the considered space is Hausdorff but not necessarily metrizable, and defined a generalized tree as a hereditarily unicoherent continuum which admits a closed, order-dense partial order with unique minimal element. Koch and Krule in [11], p. 679 have replaced the condition "order-dense" by the weaker one, "monotone", and have proved (op. cit. p. 680) the following

THEOREM 1. (Koch and Krule). *Let X be a hereditarily unicoherent continuum, and let $p \in X$. The following are equivalent:*

(1) *$\leqslant_p$ is a monotone, closed partial order on X;*

(2) *there exists a monotone, closed partial order $\leqslant$ on X with unique minimal element p;*

(3) *X is arcwise connected, and for every net $\{x_\gamma\}$ in X it is true that $px_\gamma \to px$ if $x_\gamma \to x$.*

If further (1), (2) *or* (3) *holds, then X is locally connected at p.*

The weak cut point order on X with respect to p, $\leqslant_p$, is defined by $x \leqslant_p y$ if and only if $x \in py$, where py denotes the intersection of all subcontinua of X containing p and y (see e.g. [11], p. 680). If X is a dendroid, then $\leqslant_p$ is a partial order.

This paper contains investigations of smooth dendroids, i.e. metric generalized trees in sense (3) of above Theorem. Some of our theorems are generalizations of known theorems concerning fans, contained in [7].

§ 2. Definitions and preliminary properties. All continua considered in this paper are metric, provided the opposite is not said. The distance from x to y will be denoted by $d(x, y)$. A *dendroid* is a hereditarily unicoherent and arcwise connected continuum. It follows that it must be hereditarily decomposable ([4], (47), p. 239), thus 1-dimensional (loco cit., (48)). If a dendroid has only one ramification point t (op. cit., p. 230), it is called a *fan* with the top t (see [7], p. 6). A fan X with the top t is said to be smooth provided that if a sequence of points a_n of X tends to a limit point a, then the sequence of arcs ta_n is convergent and $\operatorname*{Lim}_{n\to\infty} ta_n = ta$. Generalizing this notion, admit the following

DEFINITION. A dendroid X is said to be *smooth* if there exists a point $p \in X$, called an *initial point* of X, such that for every convergent sequence of points a_n of X the condition

$$\lim_{n\to\infty} a_n = a \tag{2.1}$$

implies that

(2.2) *the sequence of arcs pa_n is convergent*

and

$$\operatorname*{Lim}_{n\to\infty} pa_n = pa\,. \tag{2.3}$$

The set of all points p of X each of them can be taken as an initial point of X will be called the *initial set* of X.

Further, the notion of generalized tree will be used in the sense of [11], i.e. as a Hausdorff (not necessarily metric) hereditarily unicoherent continuum which admit a monotone, closed partial order with unique minimal element.

The following corollaries can be drawn from the above definitions.

COROLLARY 1. *If the space is metric, then the notion of smooth dendroid and of generalized tree coincide.*

In fact, it follows from definitions of smooth dendroid and of generalized tree by Theorem 1.

COROLLARY 2. *Every smooth fan X is a smooth dendroid.*

Namely the top of X can be taken as an initial point of X.

COROLLARY 3. *No non-smooth fan is a smooth dendroid.*

In other words, if a dendroid is smooth and if it is a fan, then it is a smooth fan. Indeed, let X be a smooth dendroid with an initial point p, and suppose X to be a fan with the top t. If $p = t$, then X is a smooth fan by definition. If $p \neq t$, take a convergent sequence of points a_n of X for which (2.1) holds. If all points a_n, for sufficiently great n, lie in the same arc starting from t, then $\operatorname*{Lim}_{n\to\infty} ta_n = ta$ trivially. If not, we have $pa_n = pt \cup ta_n$, and since (2.3) holds, hence we conclude that $\operatorname*{Lim}_{n\to\infty} ta_n = ta$.

COROLLARY 4. *Every dendrite X is a smooth dendroid. The initial set of X is equal to the whole X.*

For every dendrite is a dendroid (see [4], p. 239). Conditions (2.2) and (2.3) follow from (2.1) for an arbitrary point $p \in X$ by the local connectedness of X.

COROLLARY 5. *If a dendroid X is smooth and if the initial set of X is equal to the whole X, then X is a dendrite.*

For every initial point of X is a point of the local connectedness of X by Theorem 1.

COROLLARY 6. *If a dendroid X is smooth, then every subdendroid of X is also smooth (the heredity of smoothness for dendroids).*

In fact, let a dendroid X with an initial point p be smooth, and let Y be a subcontinuum of X. Thus Y is a dendroid (see [4], (49), p. 240). If $p \in Y$, then p is an initial point of Y too. If $p \in X \setminus Y$, then take an arbitrary point $y \in Y$, since X is hereditarily unicoherent, $py \cap Y$ is a continuum, thus it is an arc (or a point) as a subcontinuum of the arc py. Let p' be an end point of this arc which is different from y (if $py \cap Y$ is a point, we put $p' = y$). Let $\{a_n\}$ be a sequence of points of Y for which (2.1) holds. Thus we have $pa_n = pp' \cup p'a_n$ for $n = 1, 2, \ldots$, and $pa = pp' \cup p'a$, whence we conclude that the sequence of arcs $p'a_n$ is convergent and $\operatorname*{Lim}_{n\to\infty} p'a_n = p'a$ by (2.2) and (2.3).

§ 3. The initial set. Let $N(X)$ be the set of points of X at which X is not locally connected.

THEOREM 2. *If X is a smooth dendroid with an initial point p, then the constituent of the set $X \setminus N(X)$ containing p is the initial set of X.*

Proof. Let P be the initial set of X. Thus $P \subset X \setminus N(X)$ by Theorem 1. Denote by C the constituent of the set $X \setminus N(X)$ which contains p. To prove $P \subset C$ take a point $q \in P$ and a point $a \in pq$. For an arbitrary convergent sequence of points a_n satisfying (2.1) we have

$$\operatorname*{Lim}_{n\to\infty} pa_n = pa \tag{3.1}$$

and

$$\operatorname*{Lim}_{n\to\infty} qa_n = qa \tag{3.2}$$

since p and q are initial points. Further

$$aa_n \subset pa \cup pa_n$$

whence

$$\operatorname*{Ls}_{n\to\infty} aa_n \subset pa \cup \operatorname*{Ls}_{n\to\infty} pa_n$$

i.e.

$$\operatorname*{Ls}_{n\to\infty} aa_n \subset pa \tag{3.3}$$

by (3.1). Similarly

$$aa_n \subset qa \cup qa_n$$

whence

$$\operatorname*{Ls}_{n\to\infty} aa_n \subset qa \cup \operatorname*{Ls}_{n\to\infty} qa_n$$

i.e.

(3.4) $$\operatorname*{Ls}_{n\to\infty} aa_n \subset qa$$

by (3.2). Thus we conclude from (3.3) and (3.4) that

$$\operatorname*{Ls}_{n\to\infty} aa_n \subset pa \cap qa$$

i.e.

$$\operatorname*{Ls}_{n\to\infty} aa_n = a$$

since $a \in pq$. It implies that X is locally arcwise connected at a, which shows $P \subset C$.

To prove $C \subset P$ take a point $q \in C$ and a convergent sequence of points a_n with the limit a. Thus (2.2) and (2.3) holds. Define points b_n and b by

(3.5) $$pb_n = pq \cap pa_n \quad \text{and} \quad pb = pq \cap pa\,.$$

So, we have

(3.6) $$qa_n = qb_n \cup b_n a_n \quad \text{and} \quad qa = qb \cup ba\,.$$

Since $b \in pq \subset C$, hence b is a point of local arcwise connectedness of X. It follows that

(3.7) $$\lim_{n\to\infty} b_n = b\,.$$

Indeed, let ε be a positive number and let U be an ε-neighborhood of b such that every point of U can be joined with b by an arc lying entirely in U. Since $b \in pa = \operatorname{Lim} pa_n$ by (3.5) and (2.3), all arcs pa_n, by a finite number, intersect U. Points of the sets $pa_n \cap U$ can be joined with b by arcs contained in U, we see that $b_n \in U$ by the definition (3.5) of b_n, which proves (3.7). It implies that

(3.8) $$\operatorname*{Lim}_{n\to\infty} qb_n = qb\,.$$

To prove

(3.9) $$\operatorname*{Lim}_{n\to\infty} a_n b_n = ab\,,$$

observe that $ab \subset \operatorname*{Li}_{n\to\infty} a_n b_n \subset \operatorname*{Ls}_{n\to\infty} a_n b_n$ by Corollary 1 in [7], p. 7; so we should prove

(3.10) $$\operatorname*{Ls}_{n\to\infty} a_n b_n \subset ab\,.$$

Obviously $\operatorname*{Ls}_{n\to\infty} a_n b_n \subset \operatorname*{Ls}_{n\to\infty} pa_n = pa = pb \cup ba$. But $pb \subset pq$, whence we conclude that every point $x \in pb \setminus b$ is a point of local arcwise connectedness of X, so a small neighborhood of x can intersects only a finite number of arcs $a_n b_n$. Thereby (3.10) follows, which proves (3.9). Equalities (3.8) and (3.9) give immediately $\operatorname*{Lim}_{n\to\infty} qa_n = qa$, thus $q \in P$.

COROLLARY 7. *The initial set of a smooth dendroid is arcwise connected.*

COROLLARY 8. *If the initial set P of a smooth dendroid X is closed, then P is a dendrite.*

Indeed, if P is closed, then it is a continuum by Corollary 7, thus it is a dendroid as a subcontinuum of X (see [4], (49), p. 240). But if $p \in P$, then X is locally connected at p by Theorem 1. Hence P must be locally connected at p, which implies that P is a dendrite.

COROLLARY 9. *If a fan X is smooth, then the initial set of X is equal to the constituent of the set $X \setminus N(X)$ containing the top of X.*

§ 4. The T-relation on dendroids. We recall here the relation of nonaposyndeticity of F. B. Jones (see [10], p. 404) and use it to characterize the smooth dendroids among all dendroids.

Let X be a continuum. For points x and y of X define xTy if and only if every subcontinuum of X which contains y in its interior also contains x. Put

$$T_x = \{y \mid xTy\}\,.$$

The following theorem is a well-known result (see [8], p. 115).

THEOREM 3. *The relation T is closed, and the set T_x is a continuum for every $x \in X$.*

Recall that a closed quasi order on X is a transitive, reflexive relation $\leqslant$ whose graph is closed in $X \times X$. If $\leqslant$ is a closed quasi order on X and A is a subset of X, then the set $L(A) = \{x \mid x \leqslant a$ for some $a \in A\}$ is said to be the lower set of A. It is easily seen that the lower set of a closed set is closed, and that X contains elements which are minimal relative to $\leqslant$; that is, there exists an element p in X such that if $x \leqslant p$, then $p \leqslant x$.

The next theorem gives a useful relation between the T-relation on X and certain closed quasi-orders on X.

THEOREM 4. *If $\leqslant$ is a closed quasi-order on X such that*

(4.1) *$L(x)$ is connected for each $x \in X$,*

and

(4.2) *p and q are minimal elements of X implies that $p \leqslant q$ and $q \leqslant p$,*

then $x \leqslant y$ for each $x \in X$ and each $y \in T_x$.

Proof. Suppose $x \not\leqslant y$. Then there exists an open set U about y such that for each point $a \in \overline{U}$ we have $x \not\leqslant a$. Now $L(\overline{U})$ is a closed set containing y in its interior. Further, $x \in X \setminus L(\overline{U})$. To see that $L(\overline{U})$ is connected, take points a and b in $L(\overline{U})$. The sets $L(a)$ and $L(b)$ are connected by (4.1), closed, and they contain minimal elements p and q respectively. Since $p \leqslant q$ by (4.2) and $L(q)$ is connected by (4.1), we have $L(a) \cup L(q) \cup L(b)$ is a connected set containing a and b and lying in $L(\overline{U})$. Thus $L(\overline{U})$ is connected. So $y \in X \setminus T_x$ and the proof is finished.

The next theorem contains a partial converse to the previous theorem.

THEOREM 5. *Let X be a dendroid and $p \in X$. Then p is an initial point of X if and only if*

(4.3) $$px \cap T_x = x \quad \textit{for all } x \in X .$$

Proof. If p is an initial point of X, then $\leqslant_p$ is a closed partial order of X with connected lower sets and unique minimal element. Hence by the previous theorem $x \leqslant_p y$ for each $y \in T_x$. Thus $x \in py$ for each $y \in T_x$, therefore condition (4.3) is satisfied.

Conversely, suppose (4.3) holds and $x \leqslant_p y$. Then $y \in X \setminus T_x$ and hence there is a continuum K containing y in its interior which fails to contain x. Let $K' = K \cup yp$. Note that K' is a continuum containing y in its interior which fails to contain x. Hence $V = (X \setminus K') \times \operatorname{Int} K$ is an open set in $X \times X$ which contains (x, y). Let $(z, w) \in V$. If $z \leqslant_p w$, then $z \in pw$. But $pw \subset K'$; hence $z \in K'$. This is a contradiction and we conclude that $z \not\leqslant_p w$ for all $(z, w) \in V$. Hence $\leqslant_p$ is closed, that is, p is an initial point of X.

We remark that there are easily constructed examples of non-closed partial orders on the circle with connected lower sets and unique minimal elements. For locally connected continua the T-relation is trivial ($T_x = x$ for all x) and hence the condition that $x \leqslant y$ for all $y \in T_x$ is not sufficient to guarantee that $\leqslant$ is closed in general.

Observe that if the equality $T_x = x$ holds for all points x of a continuum X, then X is locally connected. Thus

(4.4) *A dendroid X is a dendrite if and only if $T_x = x$ for all $x \in X$.*

The next theorem also characterizes smooth dendroids in terms of T_x sets.

THEOREM 6. *A dendroid X has an initial point* (i.e. *X is smooth*) *if and only if*

(4.5) *for each $x, y \in X$ either $xy \cap T_x = x$ or $xy \cap T_y = y$.*

Proof. Firstly suppose the dendroid X has an initial point p. Let x and y be points of X and admit $xy \cap T_x \neq x$. Then since T_x is a continuum and X is hereditarily unicoherent, the intersection $xy \cap T_x$ is an arc xz where $z \neq x$. By Theorem 5 we have $px \cap T_x = x$. Hence $px \cap xy = x$; in other words $py = px \cup xy$. By Theorem 5 again, $py \cap T_y = y$ and therefore $xy \cap T_y = y$, thus (4.5) is satisfied.

Secondly suppose (4.5) is true. Let

$$N = \{x \in X \mid T_x \neq x\}.$$

Assume $N \neq \emptyset$, for otherwise X is a dendrite by (4.4) and each point of X is an initial point. Define a relation $\leqslant$ on N by $x \leqslant y$ if and only if $xy \cap T_x \neq x$ or $x = y$. We claim that $\leqslant$ is a partial order on N.

Suppose $x \leqslant y$ and $y \leqslant x$. If $x \neq y$, then $xy \cap T_x \neq x$ and $xy \cap T_y \neq y$ which contradicts (4.5). Hence $\leqslant$ is antisymmetric. Suppose $x \leqslant y$ and $y \leqslant z$. Assume $x \neq y \neq z$. Then $xy \cap T_x \neq x$ and $yz \cap T_y \neq y$. Since T_y is a continuum and X is hereditarily unicoherent, $yz \cap T_y$ is an arc yt where $t \neq y$. Since $xy \cap T_x \neq x$ and (4.5) holds, $xy \cap T_y = y$. Hence $xy \cap yz = y$; in other words, $xz = xy \cup yz$. Now $xz \cap T_x \supset xy \cap T_x \neq x$. Thus $x \leqslant z$ and $\leqslant$ is transitive.

Choose a maximal chain C in N. Consider two cases:

Case 1. C has a minimum element p. We claim that p is an initial point of X. Suppose $x \in X$ and $px \cap T_x \neq x$. Then $x \leqslant p$ and $x \neq p$, which contradicts the maximality of C. Hence $px \cap T_x = x$ for all $x \in X$ and so p is an initial point of X by Theorem 5.

Case 2. C has no minimum element. Then C is a set directed by $\leqslant$ (i.e. for each x and y in C there is z in C which is less than both x and y). Let p be a cluster point of the net C. We claim that p is an initial point of X. Choose a sequence of points a_i in C so that $a_{i+1} < a_i$ and $p = \lim_{i\to\infty} a_i$. It follows from the proof of transitivity of $\leqslant$ that $a_i a_{i+1} \cap a_{i+1}a_{i+2} = a_{i+1}$ for all $i = 1, 2, \ldots$ Thus

$$a_1 a_n = \bigcup_{i=1}^{n-1} a_i a_{i+1}$$

for each n. By a result of Borsuk (see [1], Lemma, p. 18) $\overline{\bigcup_{n=1}^{\infty} a_1 a_n}$ is an arc, and since p is the limit point of a_n, we have that $\overline{a_{i+1}} \in a_i p$ for each i. Suppose there exists a point $x \in X$ such that $px \cap T_x \neq x$. If $a_1 p \cap px \neq p$, then $a_n \in px$ for some n. But this is impossible since we would then have $a_n x \cap T_x \neq x$ and $a_n x \cap T_{a_n} \neq a_n$ contrary to (4.5). Hence $a_1 x = a_1 p \cup px$. But then $x < a_\alpha$ for each $a_\alpha \in C$, a contradiction. Thus (4.3) holds and so p is an initial point of X by Theorem 5.

§ 5. Countably generated dendroids and semi-smoothness. A dendroid X is *countably generated* provided X is irreducible about a countable closed subset A of X. If the set A has n cluster points, where n is either finite or countably infinite, then X is called *n-countably generated*. For example, a harmonic fan is 1-countably generated, a dendroid composed of two harmonic fans joined at their tops is 2-countably generated, while a Cantor fan is not countably generated at all.

Let Y be a subcontinuum of a continuum X. Denote by $T_x(Y)$ the set of all points $y \in Y$ such that if K is a subcontinuum of Y containing y in its Y-interior, then $x \in K$.

LEMMA 1. *Let x and y be points of a dendroid X with $y \in T_x \setminus (x)$. There is a 1-countably generated subdendroid Y of X containing x and y so that $y \in T_x(Y)$.*

Proof. Let U_n be the $(1/n)$-neighbourhood of y and let K_n be the subcontinuum of X irreducible about U_n. Note that $K_n = \overline{\bigcup \{ya \mid a \in U_n\}}$ and that $x \in K_n$. Hence there is a point $y_n \in U_n$ such that $d(yy_n, x) < 1/n$. Let Y be the subcontinuum of X irreducible about the set of points $y, y_1, y_2, \ldots$ Then Y is 1-countably generated since $\lim_{n\to\infty} y_n = y$. Further $x \in Y$ since $Y = \overline{\bigcup_{n=1}^{\infty} yy_n}$ and $d(yy_n, x) < 1/n$. Now suppose that C is a continuum in Y containing y in its Y-interior. Then C contains all but a finite number of the points y_n, hence all but a finite number of arcs yy_n. We conclude that $x \in C$. Thus $y \in T_x(Y)$.

THEOREM 7. *A dendroid X is smooth if and only if every 2-countably generated subdendroid of X is smooth.*

Proof. Since smoothness is hereditary one way is clear. Suppose X is not smooth. Then by Theorem 6 there exist two points x and y in X with $xy \cap T_x \neq x$ and $xy \cap T_y \neq y$. Choose $x' \in (xy \cap T_x)\setminus(x)$ and $y' \in (xy \cap T_y)\setminus(y)$. By Lemma 1 there are 1-countably generated subdendroids Y_1 and Y_2 of X containing x, x' and y, y' respectively so that $x' \in T_x(Y_1)$ and $y' \in T_y(Y_2)$. Put $Y = Y_1 \cup xy \cup Y_2$. So Y is a subdendroid of X. Denote by A_i, where $i = 1$ or 2, a countable set which generates Y_i (i.e. such that Y_i is irreducible about A_i). Then $A_1 \cup \{x, y\} \cup \cup A_2$ generates Y, hence Y is 2-countably generated. If C is a continuum in Y containing x' in its Y-interior, then $C \cap Y_1$ is a continuum in Y_1 containing x' in its Y_1-interior; hence $x' \in T_x(Y)$. Similarly $y' \in T_y(Y)$. Since points x' and y' lie in the arc xy, we conclude that neither $xy \cap T_x(Y)$ is x nor $xy \cap T_y(Y)$ is y. Thus Y is not smooth by Theorem 6, which finishes the proof.

On the basis of the last theorem one might be tempted to think that the property of not being smooth is finite in the class of dendroids in the sense that there is a finite set of 2-countably generated non-smooth dendroids such that a dendroid X is not smooth if and only if it contains a copy of a member of this set. To state this notion more precisely, let $\mathcal{A}$ be a class of spaces and let $\mathcal{P}$ be a property. Call $\mathcal{P}$ *finite in the class* $\mathcal{A}$ provided there is a finite set $\mathcal{F}$ of members of $\mathcal{A}$ such that a member X

of $\mathcal{A}$ has property $\mathcal{P}$ if and only if X contains a homeomorphic copy of some member of $\mathcal{F}$. For example, a result of Kuratowski (see [13], Theorem A, p. 278) can be restated as: the property of not being embeddable in the plane is finite in the class of local dendrites.

THEOREM 8. *The property of not being smooth is not finite in the class of dendroids.*

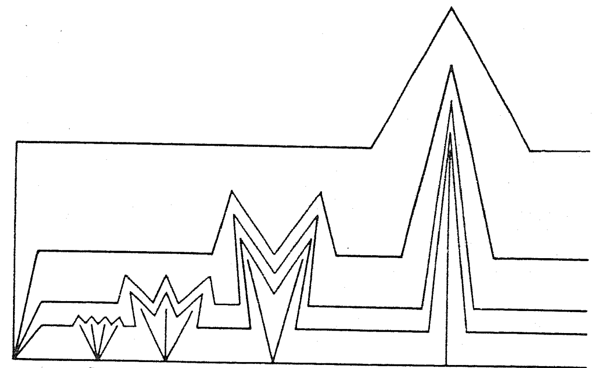

Fig. 1

Proof. Let D_0 be the 1-countably generated dendroid illustrated in the figure 1. In general let D_n denote the 1-countably generated dendroid which looks like D_0 except that p_i has $n+i$ arcs erected above it instead of i arcs. We observe that D_n is non-smooth and that each non-smooth subdendroid of D_n must contain some D_m and

(5.1) $$\text{if } D_m \subset D_n, \text{ then } m \geqslant n .$$

Further, we observe that D_n is homeomorphic with D_m if and only if $n = m$.

Now suppose $\mathcal{F} = \{F_1, F_2, \ldots, F_j\}$ is a set of dendroids such that a dendroid X is non-smooth if and only if X contains a copy of some member of $\mathcal{F}$.

Take an arbitrary D_n. Since it is non-smooth, D_n contains a copy of some member of $\mathcal{F}$, say F_i:

(5.2) $$F_i \subset D_n .$$

Let $\mathcal{F}'$ be a subset of $\mathcal{F}$ such that $F_i \in \mathcal{F}'$ if and only if there exists a natural n such that $F_i \subset D_n$. As a subset of $\mathcal{F}$, $\mathcal{F}'$ is a finite set. Since every member F_i of $\mathcal{F}'$ is a non-smooth subdendroid of some D_n, it must contain some D_m. Let us assign to every F_i a natural m_i defined by

$$m_i = \inf\{m \mid D_m \subset F_i\}. \tag{5.3}$$

So we have a correspondence from the set $\mathcal{F}'$ to the set of all such naturals m_i. Thus the last mentioned set must be finite.

But it follows from (5.3) in particular, that $D_{m_i} \subset F_i$, which leads to $m_i \geqslant n$ by (5.2) and (5.1). So we conclude that for every natural n there exists some m_i such that $m_i \geqslant n$. Hence the set of all such m_i is infinite, a contradiction.

Define a dendroid X to be *semismooth* provided there exists in X a point p such that whenever a_n converges to a, then $\operatorname{Ls}_{n\to\infty} pa_n$ is an arc. It is readily seen that smooth dendroids are semismooth; it can also be seen that none of the dendroids D_n is semismooth. As an example of non-semismooth fan we can take the non-planar fan constructed by Borsuk in [2].

It is known that every smooth fan can be embedded into the Cantor fan, thus into the plane (it follows from [7], Theorem 9, p. 27 and [9], Corollary 4). So the question can be posed whether every semismooth fan is imbeddable in the plane. The answer is negative. Namely take in the plane a Cartesian rectangular coordinate system with a point $p(0,0)$ as the origin and consider points $a = (-1, 0)$, $c = (1, 0)$, $a_n = \big(-(n+1)/n, -1/n\big)$, $b_n = \big(-(n+1)/n, 1/n\big)$ and $c_n = (1, 1/n)$. Thus we have $a = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$ as well as $c = \lim_{n\to\infty} c_n$. Joining a and c, p and a_n, a_n and b_n, b_n and c_n by straight segments we obtain a sequence of polygonal lines pc_n with the limit segment ac. So

$$X = ac \cup \bigcup_{n=1}^{\infty} pc_n$$

is a semismooth planar fan. Observe that the point a is not accessible by an arc, i.e. there exist no arc in the plane which has only the point a in common with X, even if we replace X by an arbitrary its homeomorphic image in the plane. Thus adding an arc ab to X with property $ab \cap X = (a)$ we shall have a non-planar dendroid $ab \cup X$ being a semismooth fan.

The next natural question concerning the possibility of the embedding of some dendroids into the plane is whether every smooth dendroid is imbeddable in the plane. The answer to this question is also negative. To show this, take in the plane a system of polar coordinates r, φ with the pole at a point p, and consider points $a = (2, 0)$, $a_n = (2, -\operatorname{arc\,tan} 1/n)$, $b_n = (1/n, 0)$, $c_n = (1/n, 3\pi/2)$. Join points a and p as well as a_n and c_n by straight segments, and points b_n and c_n by arcs consisting of all points (r, φ), where $r = 1/n$ and $0 \leqslant \varphi \leqslant 3\pi/2$. The dendroid Y obtained in such a way, namely

$$Y = pa \cup \bigcup_{n=1}^{\infty} a_n b_n,$$

is smooth and has p as an initial point. As previously, p is not accessible. Thus $pq \cup Y$, where pq is an arc such that $pq \cap Y = (p)$, is a smooth, non-planar dendroid.

Another example of a smooth, non-planar dendroid can be obtained in the same form $pq \cup Y$ by taking the Cantorian swastika (see [12], p. 4) with the only initial point p, as Y.

The following figures (see Fig. 2) are some of the simplest possible non-smooth semismooth dendroids.

The question which remains unanswered is whether the property of non-smoothness is finite in the class of semismooth dendroids; in particular does the set $\mathcal{F}$ of the nine dendroids in Fig. 2 demonstrate the finiteness?

We remark that one can also ask whether nonimbeddability in the plane is finite in the class of dendroids (smooth dendroids, semismooth fans or semismooth dendroids).

§ 6. Images of smooth dendroids. In this section we investigate various kinds of mappings on dendroids with the aim of finding out the extent to which smoothness is an invariant of these mappings.

First we prove two known results about monotone mappings.

PROPOSITION 1. *If f is a monotone mapping from a hereditarily unicoherent continuum X into a space Y, then the restriction of f to any subcontinuum of X is also monotone.*

Indeed, if K is a subcontinuum of X, then $(f|K)^{-1}(y) = f^{-1}(y) \cap K$ is a continuum by hereditary unicoherence of X.

PROPOSITION 2. *If f is a monotone mapping from a dendroid X onto a Hausdorff space Y, then Y is a dendroid, and if $x, y \in X$, then $f(xy) = f(x)f(y)$.*

Proof. Y is a metric continuum since it is the Hausdorff image of a metric continuum. Thus Y is a dendroid (see [6], Corollary 2, p. 219). The final assertion is a special case (using Proposition 1) of a theorem (see [4], § 43, II, 3, p. 133) that the monotone image of a continuum irreducible between x and y is a continuum irreducible between $f(x)$ and $f(y)$.

Through out the remainder of the section X and Y are dendroids and f is a continuous mapping from X into Y.

PROPOSITION 3. *f is monotone if and only if $f(xy) = f(x)f(y)$.*

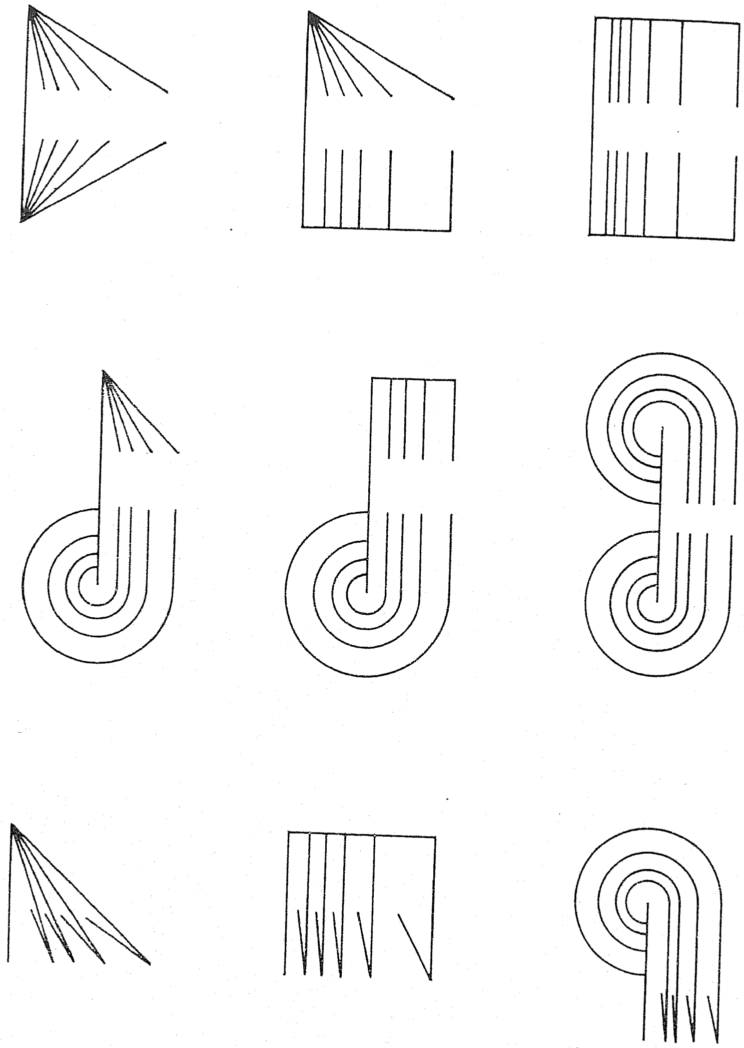

Fig. 2

Proof. One way follows immediately from Proposition 2. Suppose $f(xy) = f(x)f(y)$ for all $x, y \in X$. Let $z \in Y$ and $x, y \in f^{-1}(z)$. Then $f(x)f(y) = (z)$. But $xy \subset f^{-1}(f(xy)) = f^{-1}(f(x)f(y)) = f^{-1}(z)$. Hence $f^{-1}(z)$ is connected and so f is monotone.

Definition. Let $p \in X$. A mapping f will be called order-preserving with respect to p (or simply $\leqslant_p$-preserving) if and only if $x \leqslant_p y$ implies $f(x) \leqslant_{f(p)} f(y)$.

Proposition 4. *The following are equivalent:*

(i) *$f(px) = f(p)f(x)$ for each $x \in X$,*

(ii) *f is $\leqslant_p$ preserving,*

(iii) *$f|px$ is monotone for each $x \in X$.*

Proof. (i) implies (ii). Suppose $x \leqslant_p y$. Then $x \in py$. Hence $f(x) \in f(py) = f(p)f(y)$, so $f(x) \leqslant_{f(p)} f(y)$.

(ii) implies (iii). Suppose $z, w \in px$ with $z \in pw$ and $f(z) = f(w)$. If $t \in zw$, then $z \leqslant_p t \leqslant_p w$. Hence $f(z) \leqslant_{f(p)} f(t) \leqslant_{f(p)} f(w)$. So $f(z) = f(t) = f(w)$ and $f|px$ is monotone.

(iii) implies (i). It follows from a theorem of Kuratowski, cited above in the proof of Proposition 2.

Note that f is monotone if and only if f is $\leqslant_p$-preserving for all $p \in X$.

Theorem 9. *If f is $\leqslant_p$-preserving mapping of X onto Y and p is an initial point of X, then $f(p)$ is an initial point of Y.*

Proof. Since p is an initial point of X, we have that the graph of $\leqslant_p$ is closed in $X \times X$. To show that $f(p)$ is an initial point of Y, it suffices to prove that for each $q \in Y$ we have $T_q \cap qf(p) = q$ (see Theorem 5). So suppose that $z \in qf(p)$ with $q \neq z$. Then $q \in Y \backslash zf(p)$. Let $Q = f^{-1}(q)$ and $Z = f^{-1}(z)$. Note that Q and Z are compact subsets of X with the property that $(Q \times Z) \cap (\text{graph} \leqslant_p) = \emptyset$ (if $(x, y) \in Q \times Z$ and $x \leqslant_p y$, then $q = f(x) \leqslant_{f(p)} f(y) = z$). Hence there are open sets U and V in X such that $Q \times Z \subset U \times V$ and $(\overline{U} \times \overline{V}) \cap (\text{graph} \leqslant_p) = \emptyset$. Let $K = \{x \in X \mid x \leqslant_p v \text{ for some } v \in \overline{V}\}$. K is easily seen to be a continuum containing Z in its interior and containing no point of Q. Therefore $f(K)$ is a continuum containing z in its interior and such that $q \in Y \backslash f(K)$. Hence $z \in Y \backslash T_q$. Thus $T_q \cap qf(p) = q$, so $f(p)$ is an initial point of Y.

As a consequence of Proposition 3 and Theorem 9 we have

Corollary 10. *If f is a monotone mapping of a smooth dendroid X onto Y, then Y is a smooth dendroid and $f(P) \subset P'$, where P and P' denote the initial sets of X and Y respectively.*

Remark. Examples show that "confluent" may not be substituted for "monotone" in Corollary 10. Namely let (x, y) denotes a point with rectangular coordinates in the plane. Join points $(-1, 0)$, $(1, 0)$ and $(1, 1/n)$ with the origin $(0, 0)$ by straight segments. The continuum obtained is a smooth fan having $(-1, 0)$ as an initial point. The mapping f defined by $f(x, y) = (|x|, y)$ takes $(-1, 0)$ to $(1, 0)$, a point of non-local connectedness of the image.

However we have no example to show that a confluent image of a smooth dendroid need not be smooth.

Now we show that every smooth dendroid can be obtained as the image of the Cantor fan F_C under an $\leqslant_t$-preserving map, where t is the top of F_C.

Recall that a metric d on a dendroid X is radially convex with respect to a point $p \in X$ provided that $x \in py$ and $x \neq y$ implies $d(p, x) < d(p, y)$.

THEOREM 10. *A dendroid X is smooth with an initial point p if and only if X has a metric which is radially convex with respect to p.*

Proof. By Theorem 1, $\leqslant_p$ has a closed graph. Now the result follows one way from the Carruth's Theorem 1 (see [3], p. 229) which says that if $\leqslant$ is a closed partial order on the compact metric space X, then there exists an equivalent metric on X which is radially convex with respect to $\leqslant$.

Conversely suppose X has a metric, radially convex with respect to p. To show that p is an initial point of X it suffices to show, according to Theorem 5, that $px \cap T_x = x$ for all $x \in X$. Let $q \in px$ with $q \neq x$. Then $d(p, q) < d(p, x)$. Hence $K = \{z \in X|\ d(p, z) \leqslant d(p, q)+\varepsilon\}$, where $\varepsilon = \big(d(p, x)-d(p, q)\big)/2$ is a continuum containing q in its interior and not containing x. Hence $q \in X\setminus T_x$.

THEOREM 11. *If a dendroid X has a metric d which is radially convex with respect to some point $p \in X$, then there is an $\leqslant_t$-preserving map f of F_C onto X such that $f(t) = p$.*

Proof. Take in the plane a system of polar coordinates r, φ with the pole at a point t. Consider the Cantor discontinuum C in the arc $0 \leqslant \varphi \leqslant 1$ of the circumference $r = 1$, i.e. the set of points $c = (1, \varphi)$, where $\varphi = \sum_{i=0}^{\infty} 2n_i/3^i$ and $n_i = 0$ or 1. Joining all points $c \in C$ with t by straight segments tc we have the Cantor fan F_C. We see that $r \leqslant 1$ for all points of F_C.

Assume $d(p, x) \leqslant 1$ for all $x \in X$. Denote by g an arbitrary continuous mapping from C onto X. Extend g to $f\colon F_C \to X$ as follows. If $(r, \varphi) \in tc \subset F_C$, then $f(r, \varphi)$ is the point x on $pg(c)$ such that

$$d(p, x) = rd\big(p, g(c)\big)\,. \tag{6.1}$$

To see that f is continuous, suppose a sequence of points $(r_n, \varphi_n) \in tc_n$ is convergent to a point $(r, \varphi) \in tc$ in F_C. If $r \neq 0$, then $\varphi = \lim\limits_{n\to\infty} \varphi_n$, whence $c = \lim\limits_{n\to\infty} c_n$ and, g being continuous, $g(c) = \lim\limits_{n\to\infty} g(c_n)$. Since X is smooth with the initial point p, we have $pg(c) = \operatorname{Lim}\limits_{n\to\infty} pg(c_n)$. Hence every cluster point of the sequence of points $f(r_n, \varphi_n) = x_n$ lies in $pg(c)$. Let y be a cluster point of x_n. Then $d(p, y) = \lim\limits_{n\to\infty} d(p, x_n) = \lim\limits_{n\to\infty} r_n d\big(p, g(c_n)\big) = \lim\limits_{n\to\infty} r_n \lim\limits_{n\to\infty} d\big(p, g(c_n)\big) = rd\big(p, g(c)\big)$. Therefore the sequence $\{x_n\}$ has only one cluster point, namely the point x in $pg(c)$ such that (6.1) holds. If $r = 0$ then $\lim\limits_{n\to\infty} r_n = 0$, thus $\lim\limits_{n\to\infty} (r_n, \varphi_n) = t$. By (6.1) $d(p, x_n) = r_n d\big(p, g(c_n)\big)$. But $d\big(p, g(c_n)\big) \leqslant 1$, so $\lim\limits_{n\to\infty} d(p, x_n) = 0$. Thus $\lim\limits_{n\to\infty} x_n = p$, whence $f(t) = p$.

COROLLARY 11. *Let X be a dendroid. The following are equivalent:*

(i) *X is smooth,*

(ii) *X admits a radially convex metric,*

(iii) *X is image of F_C under an $\leqslant_t$-preserving map.*

COROLLARY 12. *Every smooth dendroid is contractible.*

Indeed, X has a radially convex metric d with respect to an initial point p, such that $d(p, x) \leqslant 1$ for all $x \in X$.

So, for $x \in X$ and $0 \leqslant s \leqslant 1$ put $h(x, s) = y$, where y is a point of the arc px such that

$$d(p, y) = d(p, x)\cdot(1-s)\,.$$

It is easy to see that $h\colon X\times[0, 1] \to X$ is continuous and $h(x, 0) = x$ as well as $h(x, 1) = p$ for all $x \in X$.

The inverse is not true, as it shows an example of non-smooth contractible fan in [7], p. 31.

A class $\mathcal{A}$ of continua has a *common model* M under continuous mappings if there exists a continuum M belonging to $\mathcal{A}$ with property that every member of $\mathcal{A}$ is a continuous image of M. Corollary 11 says that if $\mathcal{A}$ denotes the class of smooth dendroids, then there exists a common model for $\mathcal{A}$, namely the Cantor fan F_C. Since it is a fan, it can be taken as a common model for the class of smooth fans. The latter result is known (see [7], Theorem 10, p. 28 and [9], Corollary 4). The problems whether common models exist in the classes of semismooth fans, of uniformly arcwise connected fans (for the definition see § 8 here), of all fans, of semismooth dendroids, of uniformly arcwise connected dendroids or of all dendroids are open.

A class $\mathcal{A}$ of continua has a *universal element* U if there exists a continuum U belonging to $\mathcal{A}$ with property that every member of $\mathcal{A}$ can be homeomorphically embedded into U. It is known that a universal element, namely the Cantor fan, does exist in the class of smooth fans (see [7], Theorem 9, p. 27 and [9], Corollary 4). The problem whether a universal element exists in the class of smooth dendroids is open. Also the questions concerning the existence of universal elements in all classes of the particular kinds of dendroids mentioned above are unanswered.

§ 7. Further characterizations. A necessary and sufficient condition for fans to be smooth was given in [7], Theorem 1, p. 7. A very similar condition characterizes smooth dendroids, and the proof of a corresponding theorem can be made in the same way as for fans. However this proof is rather long, so we shall give here some other one using the result for fans and Theorem 10. The condition in matter is given by

THEOREM 12. *A dendroid X is smooth if and only if there exists a point $p \in X$ such that for any two convergent sequences $\{a_n\}$ and $\{b_n\}$ conditions*

(7.1) $$\lim_{n\to\infty} a_n = a\,, \quad \lim_{n\to\infty} b_n = b$$

and

(7.2) $$a_n \in pb_n \quad \text{for } n = 1, 2, \ldots$$

imply

(7.3) $$\operatorname*{Lim}_{n\to\infty} a_n b_n = ab\,.$$

Proof. If there is a point p in X such that the above implication holds, then putting $a_n = p$ and denoting b_n by a_n as well as b by a we see that (7.3) gives (2.3), so X is smooth by definition.

Conversely let X be smooth and let p be an initial point of X. By Theorem 10, there is a metric d on X radially convex with respect to p.

Let a_n, b_n, a and b be points in X which satisfy (7.1) and (7.2). We wish to show that $\operatorname*{Ls}_{n\to\infty} a_n b_n = \operatorname*{Li}_{n\to\infty} a_n b_n = ab$.

By Corollary 1 in [7], p. 7, we have that

(7.4) $$ab \subset \operatorname*{Li}_{n\to\infty} a_n b_n\,.$$

From $a_n b_n \subset pb_n$, we obtain

(7.5) $$\operatorname*{Ls}_{n\to\infty} a_n b_n \subset \operatorname*{Ls}_{n\to\infty} p b_n\,.$$

Since p is an initial point of X,

(7.6) $$\operatorname*{Ls}_{n\to\infty} p b_n = pb\,.$$

Let $x \in \operatorname*{Ls}_{n\to\infty} a_n b_n$. Then there exist a sequence $c_{n_m} \in a_{n_m} b_{n_m}$ converging to x. Hence

(7.7) $$\lim_{m\to\infty} d(c_{n_m}, p) = d(x, p)\,.$$

But for each m,

(7.8) $$d(a_{n_m}, p) \leqslant d(c_{n_m}, p)\,,$$

by the radial convexity of d. Now (7.8) implies that

(7.9) $$d(a, p) \leqslant d(x, p)\,.$$

So we conclude from (7.5), (7.6) and (7.9) that $x \in ab$. Consequently,

(7.10) $$\operatorname*{Ls}_{n\to\infty} a_n b_n \subset ab \subset \operatorname*{Li}_{n\to\infty} a_n b_n\,.$$

Thus (7.3) follows from (7.10).

It is easy to see that Theorem 12 is very similar to Theorem 1 in [7], p. 7. Remark however, that we cannot replace (7.2) by

(7.11) $$a_n b_n \setminus (p) \quad \text{is connected}$$

as it was done for fans (see loco cit., (3.1)) because (7.11) does not imply (7.2) for smooth dendroids (for fans we can establish (7.2) without loss of generality, having (7.11), see op. cit. (3.3), p. 8). It can be seen by the following example, in which (x, y) denotes a point with rectangular coordinates.

Join point $p = (0, 1)$ with $(0, 0)$ and with points $(1/n, 0)$ by straight segments. The continuum F_1 obtained in this way is a harmonic fan with the top p. Next join the point $q = (0, 1/2)$ with $(0, 0)$ and with points $(-1/n, 0)$ by straight segments. We get another harmonic fan F_2 with the top q. Put $X = F_1 \cup F_2$ and let

$$a_n = (-1/(2n-1), 0) \quad \text{and} \quad b_n = (-1/2n, 0)\,.$$

Thus (7.11) holds, we have neither (7.2) nor $b_n \in pa_n$, (7.1) holds with $a = b = (0, 0)$, hence the arc ab reduces to the point a and $\operatorname*{Lim}_{n\to\infty} a_n b_n$ is the straight segment qa. Whence (7.3) is false.

As a corollary similar to Corollary 3 in [7], p. 9 we have

COROLLARY 13. *If X is a smooth dendroid with an initial point p, and if for any two sequences conditions (7.1) and (7.2) hold, then*

(7.12) $$a \in pb\,,$$

thus

(7.13) $$ab \setminus (p) \ \text{is connected}\,.$$

Indeed, (7.1) and (7.2) imply (7.12) because the partial order $\leqslant_p$ is closed. But it is not true that (7.1) and (7.11) imply (7.13) for smooth dendroids as they do for smooth fans. As an example consider points

$a = (0,1)$, $b = (0,-1)$, $c = (1,0)$, $p = (0,0)$, $a_n = (1/n, 1)$, $b_n = (1/n, -1)$ and put

$$X = ab \cup cp \cup \bigcup_{n=1}^{\infty} a_n b_n ,$$

where the arcs ab, cp and $a_n b_n$ are straight segments. Thus X is a smooth dendroid with p as an initial point, (7.1) and (7.11) hold true, but $p \in ab \setminus \big((a) \cup (b)\big)$, whence (7.13) is false.

Theorem 2 in [7], p. 9 gives another condition which characterizes smooth fans. As a generalization of this theorem to smooth dendroids we shall prove the following

THEOREM 13. *A dendroid X has a radially convex metric d with respect to a point $p \in X$ if and only if it has a metric d' such that for any number $\varepsilon > 0$ there exists a number $\eta > 0$ such that for any two points a and b of X conditions*

$$(7.14) \qquad ap \subset bp \quad \text{or} \quad bp \subset ap$$

$$(7.15) \qquad d'(a, b) < \eta$$

imply

$$(7.16) \qquad \delta(ab) < \varepsilon .$$

Proof. Firstly, let d be a radially convex metric on X with respect to p. We shall show that d satisfies the condition, i.e. that d can be taken as d'. Suppose the condition is not true. So there is an $\varepsilon > 0$ that for every natural n we can find two points a_n and b_n such that

$$(7.17) \qquad a_n p \subset b_n p \quad \text{or} \quad b_n p \subset a_n p ,$$

$$(7.18) \qquad d(a_n, b_n) < 1/n ,$$

$$(7.19) \qquad \delta(a_n b_n) \geqslant \varepsilon .$$

It follows from (7.17) that there exists a subsequence $\{n_k\}$ of naturals for which either

$$(7.20) \qquad a_{n_k} p \subset b_{n_k} p \quad \text{for } k = 1, 2, \ldots$$

or

$$(7.21) \qquad b_{n_k} p \subset a_{n_k} p \quad \text{for } k = 1, 2, \ldots$$

Assuming (7.20) we conclude from (7.19) that there are points $c_k \in a_{n_k} b_{n_k}$ with property

$$(7.22) \qquad d(a_{n_k}, c_k) \geqslant \varepsilon/2 .$$

The metric d being radially convex with respect to p, we have

$$(7.23) \qquad d(p, a_{n_k}) < d(p, c_k) < d(p, b_{n_k}) .$$

It follows from the triangle axiom that

$$d(p, b_{n_k}) \leqslant d(p, a_{n_k}) + d(a_{n_k}, b_{n_k}) ,$$

thus

$$d(p, b_{n_k}) \leqslant d(p, a_{n_k}) + 1/n_k$$

by (7.18). Hence

$$d(p, a_{n_k}) < d(p, c_k) < d(p, a_{n_k}) + 1/n_k$$

by (7.23), and we see that $d(a_{n_k}, c_k)$ must be arbitrarily small for sufficiently large n_k, contrary to (7.22).

Secondly, let d' be a metric for which the condition in matter holds and suppose there is no radially convex metric with respect to p on X. Thus p is not an initial point of X according to Theorem 10. By Theorem 5 there exists a point x in X with

$$px \cap T_x \neq x .$$

Thus we can find a point $q \in px$ with $q \neq x$ and $q \in T_x$. Take

$$\varepsilon = d'(q, x)/3 .$$

So there is an $\eta > 0$ such that for any two points a and b of X conditions (7.14) and (7.15) imply (7.16). Take n such that

$$(7.24) \qquad 2/n < \min(\eta, \varepsilon)$$

and let U_n be the $1/n$-neighbourhood of q. So

$$(7.25) \qquad d'(a, q) < \eta \quad \text{for every } a \in \overline{U}_n .$$

Denote by K_n the unique continuum irreducible about $\overline{U}_n$. Thus

$$(7.26) \qquad K_n = \overline{\bigcup \{aq|\ a \in \overline{U}_n\}} ,$$

and $q \in \operatorname{Int} K_n$. Since $q \in T_x$ we have $x \in K_n$ according to the definition of T_x.

Further, let V_n be the $1/n$-neighbourhood of x. The union $\bigcup \{aq|\ a \in \overline{U}_n\}$ being a set dense in K_n by (7.26), for any n satisfying (7.24) there exists a point $a_n \in \overline{U}_n$ such that

$$(7.27) \qquad a_n q \cap V_n \neq \emptyset .$$

It follows that

$$(7.28) \qquad \delta(a_n q) \geqslant \varepsilon .$$

The metric d' satisfying the condition, we conclude from (7.25) and (7.28) that the negation of (7.13) must hold for points q and a_n, i.e.

$$(7.29) \qquad \text{neither } a_n p \subset pq , \quad \text{nor } pq \subset a_n p .$$

Thus the common part of arcs pq and $a_n p$ is an arc being a proper subarc of each of them.

Let x_n be a point of $a_n q \cap V_n$ which exists by (7.27). So the sequence of points x_n is convergent and

$$\lim_{n\to\infty} x_n = x .$$

It is easy to see that points x_n lie in arcs $a_n p$ because otherwise we would have $x_n \in pq$, thus $x \in pq$ contrary to the definition of q.

Consider now $\operatorname{Li}_{n\to\infty} px_n$ which is obviously a non-empty set. By Corollary 1 in [7], p. 7 it is a continuum and we have

$$px \subset \operatorname{Li}_{n\to\infty} px_n .$$

Since $q \in px$, thus for any n there is a natural i such that if $k > i$, then

$$px_{n+k} \cap U_n \neq \emptyset.$$

Let $b_{n+k} \in px_{n+k} \cap U_n$. Thus we have

$$pb_{n+k} \subset px_{n+k} \subset pa_{n+k} , \tag{7.30}$$

so (7.14) is satisfied for a_{n+k} and b_{n+k}. Since points a_{n+k} and b_{n+k} both belong to U_n, we see that

$$d'(a_{n+k}, b_{n+k}) < \eta$$

by (7.24), thus (7.15) is satisfied. But $x_{n+k} \in a_{n+k} b_{n+k}$ by (7.30) and $x_{n+k} \in V_{n+k} \subset V_n$ by definition, whence

$$\delta(a_{n+k} b_{n+k}) \geqslant \varepsilon$$

contrary to (7.16).

COROLLARY 14. *A dendroid X is smooth if and only if there exist a point $p \in X$ and a metric d such that for every number $\varepsilon > 0$ there is a number $\eta > 0$ such that for any two points a and b of X conditions $a \in pb$ and $d(a, b) < \eta$ imply $\delta(ab) < \varepsilon$.*

§ 8. Uniform arcwise connectedness. Recall that a set X is said to be *uniformly arcwise connected* (see [5], p. 193 and [7], p. 12) if it is arcwise connected and if for every number $\varepsilon > 0$ there is a natural k such that every arc A in X contains points $a_0, a_1, \ldots, a_k$ such that

$$A = \bigcup_{i=0}^{k-1} a_i a_{i+1} ,$$

and

$$\delta(a_i a_{i+1}) < \varepsilon \quad \text{for every } i = 0, 1, \ldots, k-1 .$$

It is known that every dendroid X is irreducible about the set $E(X)$ of all end points of X (see [15], Theorem 3.5, p. 193). Hence if we take an arbitrary point $x \in X$, then we have

$$X = \bigcup \{xe|\ e \in E(X)\} .$$

The following theorems give sufficient and necessary conditions under which a dendroid is uniformly arcwise connected.

THEOREM 14. *A dendroid $X = \bigcup \{xe|\ e \in E(X)\}$ is uniformly arcwise connected if and only if for every number $\varepsilon > 0$ there is a natural k such that every arc xe contains points $a_0, a_1, \ldots, a_k$ such that*

$$xe = \bigcup_{i=0}^{k-1} a_i a_{i+1} ,$$

and

$$\delta(a_i a_{i+1}) \leqslant \varepsilon \quad \textit{for every } i = 0, 1, \ldots, k-1 . \tag{8.1}$$

Proof. If X is uniformly arcwise connected, then the condition in question is satisfied by the definition of uniform arcwise connectedness. Inversely, take an arbitrary arc $A = ab \subset X$ and let e_1 and e_2 be end points of X such that $a \in xe_1$ and $b \in xe_2$. Hence $A \subset xe_1 \cup xe_2$. Thus putting $\varepsilon' = 2\varepsilon$ and $k' = 2k$ we have a decomposition of the arc A into at most k' arcs with diameters less than or equal to ε, i.e. less than ε', which proves the uniform arcwise connectedness of X.

COROLLARY 15. *A dendroid $X = \bigcup \{xe|\ e \in E(X)\}$ is uniformly arcwise connected if and only if for every number $\varepsilon > 0$ there is a natural k such that every arc xe contains points $a_0, a_1, \ldots, a_j$, where $j \leqslant k$, such that*

$$xe = \bigcup_{i=0}^{j-1} a_i a_{i+1}$$

and $\delta(a_i a_{i+1}) \leqslant \varepsilon$ for every $i = 0, 1, \ldots, j-1$.

In the same manner as Theorem 14 we can prove

THEOREM 15. *A dendroid $X = \{xe|\ e \in E(X)\}$ is uniformly arcwise connected if and only if the condition from Theorem 14 is satisfied with $\delta(a_i a_{i+1}) < \varepsilon$ for every $i = 0, 1, \ldots, k-1$ instead of (8.1).*

LEMMA 2. *Let a dendroid X be smooth with p as an initial point of X and let ε be a positive number. If we take in every arc ab such that $ab \setminus (p)$ is connected, points*

$$a = x_0, x_1, \ldots, x_j, x_{j+1} = b$$

such that

$$x_i \in x_{i-1} x_{i+1} \quad \textit{for } i = 1, 2, \ldots, j ,$$

$$d(x_i, x_{i+1}) = \varepsilon \quad \textit{for } i = 0, 1, \ldots, j-1 \quad \textit{and } d(x_j, x_{j+1}) \leqslant \varepsilon ,$$

then there exists a natural k with property $j \leqslant k$ for all arcs ab.

Proof of this lemma is exactly the same as the proof of Lemma 2 in [7], p. 13 for smooth fans.

Lemma 2 leads to the following corollary, similar to Corollary 5 in [7], p. 14.

COROLLARY 16. *Every smooth dendroid is uniformly arcwise connected.*

Obviously the inverse is not true.

§ 9. Brush continua. The notion of the brush continuum was introduced in [4], p. 234. Namely given any continuum Q, a continuum $B(Q)$ is called a *brush continuum* with the base Q if $Q \subset B(Q)$ and if $B(Q)$ is the union of straight segments xy called its generators, such that

(9.1) there is a number $\delta > 0$ such that for every x the inequality $d(x, Q) \geqslant \delta$ holds,

(9.2) there is a number $\eta > 0$ such that for every generator xy of $B(Q)$ and for every point z of xy the inequality $d(z, Q) \geqslant \eta \cdot d(z, y)$ holds,

(9.3) for every point x the set $xy \setminus (y)$ is a component of the set $B(Q) \setminus Q$.

It follows from the above definition that

(9.4) $xy \cap Q = (y)$ for every generator xy of $B(Q)$.

It is known that for every continuum Q there exists a brush continuum $B(Q)$ with the base Q (see [4], Theorems 1 and 2, p. 234 and 235), and if Q is a dendroid, then $B(Q)$ is also a dendroid (op. cit., Theorem 5, p. 241).

Now we shall prove the following

THEOREM 16. *If a dendroid X is smooth, then every brush continuum $B(X)$ is also smooth and the initial set of X is contained in the initial set of $B(X)$.*

Proof. Let X be a smooth dendroid with an initial point p, and let $B(X)$ be an arbitrary brush continuum with the base X. Thus, by the definition, $B(X)$ is the union of straight segments xy where $y \in X$, which satisfy (9.1)–(9.4) for $Q = X$. Take a convergent sequence of points a_n of $B(X)$ with a limit point a, and let

$$a_n \in x_n y_n \quad \text{as well as} \quad a \in xy .$$

Thus (2.1) holds by hypothesis. We shall prove that (2.1) implies

(9.5) $$\lim_{n\to\infty} y_n = y .$$

Indeed, consider two subsequences $\{y_{n_k}\}$ and $\{y_{m_k}\}$ of the sequence $\{y_n\}$ such that

(9.6) $$\lim_{k\to\infty} y_{n_k} = y' \quad \text{and} \quad \lim_{k\to\infty} y_{m_k} = y'' .$$

Hence (2.1) and (9.6) imply

$$\operatorname*{Lim}_{k\to\infty} a_{n_k} y_{n_k} = ay' \quad \text{and} \quad \operatorname*{Lim}_{k\to\infty} a_{m_k} y_{m_k} = ay''$$

the arcs $x_n y_n$ being straight segments. If $a \neq y$, then the point a belongs to the only one generator, namely to xy. Thus segments ay' and ay'' lie in the segment xy and we have $y' = y'' = y$ by (9.4), which proves (9.5). If $a = y$, suppose one of points y' and y'' satisfying (9.6) is different from y. Say $y' \neq y$. Hence for sufficiently great k we have

(9.7) $$d(a_{n_k}, y_{n_k}) > \tfrac{1}{2} \cdot d(a, y')$$

by (2.1) and (9.6). Substituting a_{n_k} and y_{n_k} in place of z and y respectively in (9.2) we have

$$d(a_{n_k}, X) \geqslant \eta \cdot d(a_{n_k}, y_{n_k}) ,$$

which leads to the inequality

(9.8) $$d(a_{n_k}, X) > \eta \cdot \tfrac{1}{2} d(a, y')$$

for sufficiently great k. Since $a = y$, hence $a \in X$ and we have

$$d(a_{n_k}, a) \geqslant d(a_{n_k}, X) ,$$

thus by (9.8)

$$d(a_{n_k}, a) > \eta \cdot \tfrac{1}{2} d(a, y') ,$$

which contradicts (2.1). Therefore (9.5) is established.

To prove that $B(X)$ is smooth we shall show that (2.1) implies (2.2) and (2.3). Observe that for every n we have

(9.9) $$pa_n = py_n \cup y_n a .$$

The dendroid X being smooth by hypothesis, and (2.1) implying (9.5) we see that the sequence of arcs py_n is convergent and

(9.10) $$\operatorname*{Lim}_{n\to\infty} py_n = py .$$

Further, the arcs $y_n a_n$ being straight segments, we have from (2.1) and (9.5)

(9.11) $$\operatorname*{Lim}_{n\to\infty} y_n a_n = ya .$$

Thus (9.10) and (9.11) imply (2.2) by (9.9) and we have

$$\operatorname*{Lim}_{n\to\infty} pa_n = py \cup ya .$$

Since $py \subset X$ and $ya \cap X = (y)$ by (9.4), we conclude $py \cup ya = pa$, so (2.3) holds and the proof of the smoothness of $B(X)$ is finished.

Moreover, we see that just the point $p \in X$ was taken as an initial point of $B(X)$, which implies that the initial set of X is contained in the initial set of $B(X)$.

Since every initial point of $B(X)$ is a point of the local connectedness of $B(X)$ by Theorem 1, hence Theorem 16 and Corollary 7 imply the following.

COROLLARY 17. *If a dendroid X is smooth, then the initial set of an arbitrary brush continuum $B(X)$ is composed of the initial set of X and of all generators xy of $B(X)$ every point of which is a point of the local connectedness of $B(X)$.*

§ 10. Countable smooth combs. Let X be a dendroid and let $R(X)$ denote the set of all ramification points of X. If there exists an arc ab in X such that $R(X) \subset ab$, then X is said to be a *comb*. A comb X is called *countable* if the set $R(X)$ is countable and if $\mathrm{Ord}_x X \leqslant \aleph_0$ for every $x \in R(X)$. In other words, a comb X is countable if and only if $E(X)$, the set of all end-points of X, is countable. For example if we put in rectangular coordinates $a = (0, 0)$, $b = (1, 0)$, $c = (0, 1)$, $x_n = (1/n, 0)$, $y_n = (1/n, 1)$, then the union of straight segments $ab \cup bc \cup \bigcup_{n=1} x_n y_n$ is a countable comb. Obviously it is also smooth.

It follows from the above definition of a comb that if a dendroid has only one ramification point, i.e. if it is a fan, then it is obviously a comb. It is known that every fan F can be written as the union of arcs in the form

$$F = \bigcup \{te \mid e \in E(F)\},$$

where t is the top of the fan F, and e is an end-point of it. Similarly every comb X may be written as the union of some arcs. Let ab be the arc irreducible about $R(X)$ and put $R'(X) = a \cup b \cup R(X)$. For every point $t \in R'(X)$ denote by F_t the union of all arcs te, where $e \in E(X)$, such that $te \cap ab = t$. Thus if $a \neq t \neq b$ and $\mathrm{Ord}_t X = 3$, then F_t reduces to an arc. Also if $a \in X \setminus R(X)$ (or $b \in X \setminus R(X)$) then F_a (or F_b) is an arc (or a point, if a or b are in $E(X)$). So we have

$$X = ab \cup \bigcup \{F_t \mid t \in R'(X)\}. \tag{10.1}$$

Let $E_t = F_t \cap E(X)$. Thus

$$F_t = \bigcup \{te \mid e \in E_t\} \tag{10.2}$$

for every $t \in R'(X)$. Substituting (10.2) into (10.1) we have an expression of a comb X as the union of arcs:

$$X = ab \cup \bigcup \{\bigcup [te \mid e \in E_t] \mid t \in R'(X)\}.$$

In the case when a comb X is countable, we see that $R'(X)$ is countable and E_t is countable for every $t \in R'(X)$. Thus a countable comb X can be written in the form

$$X = ab \cup \bigcup_{n=1}^{\infty} \Big(\bigcup_{m=1}^{\infty} t_n e_{nm}\Big).$$

Let $\tau(X)$ be the degree of the non-local connectedness of X defined in [5], p. 190. If a comb X is countable, then $\tau(X)$ is a countable ordinal and inversely. Thus from Theorem 18 in [5], p. 192 we infer the following

COROLLARY 18. *If a comb Y is a continuous image of a comb X and if $E(X)$ is countable, then $E(Y)$ is at most countable.*

The following theorem is a generalization of Theorem 5 in [7], p. 15:

THEOREM 17. *If a dendroid X is smooth with countable $\overline{E(X)}$ and if a smooth dendroid Y with countable $E(Y)$ is a continuous image of X, then the set $\overline{E(Y)}$ is at most countable.*

Proof of this theorem goes letter by letter exactly like the proof of Theorem 5 in [7], p. 15 if we take t and t' there as initial points of X and Y respectively.

Observe that the countability of $E(Y)$ is an essential hypothesis in the above Theorem: a dendrite with the set of end points being the Cantor discontinuum is a continuous image of an arc.

Theorem 17 and Corollary 18 imply:

COROLLARY 19. *If a smooth comb Y is a continuous image of a smooth comb X and if $\overline{E(X)}$ is countable then $\overline{E(Y)}$ is not at most countable.*

Remark that in the same manner as for fans (see [7], p. 21) one can construct a family of n (for each $n = 2, 3, \ldots$) countable combs incomparable in the sense that none of them is a continuous image of another.

Further, remark that it can be proved also exactly as for fans in [7], p. 30 that there exists neither a common model nor a universal element (for the definitions see the final part of § 6 here) in the class of countable smooth combs. However, in the class of all smooth combs, the question about the existence of a common model has the positive answer (namely the Cantor fan is a common model in this class by Corollary 11, thus also the Cantor comb is), but the question concerning the existence of a universal element in this class is open.

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INSTITUTE OF MATHEMATICS OF THE WROCŁAW UNIVERSITY
UNIVERSITY OF KENTUCKY

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Über die Mächtigkeiten und Unabhängigkeitsgrade der Basen freier Algebren, II *

von

Peter Burmeister (Bonn)

Einleitung. Im Zusammenhang mit den im ersten Teil behandelten Problemen tauchte auch die Frage auf, ob verschieden mächtige Basen einer Algebra verschiedene Unabhängigkeitsgrade besitzen können und was man über die Unabhängigkeitsklassen der $\mathfrak{A}$-Basen der $\mathfrak{A}$-frei erzeugten Algebren einer nichttrivialen primitiven Klasse $\mathfrak{A}$ aussagen kann. Dabei sei die Unabhängigkeitsklasse — die auch Unabhängigkeitsgrad genannt wird — einer Teilmenge M einer Algebra (A, f) (¹) definiert als

$$\mathrm{ind}_{(A,f)} M := \{(B, g) \mid M \text{ ist } (B, g)\text{-freie Teilmenge } (^2) \text{ von } (A, f)\}.$$

Es ist nun leicht einzusehen, daß tatsächlich die Unabhängigkeitsklassen verschieden mächtiger Basen einer Algebra verschieden sein können. Dazu betrachte man etwa das in der Einleitung von Teil I erwähnte Beispiel von C. J. Everett: Sei nämlich (E, f) — kurz: E — ein Links—C—Modul über einem Ring C, wie ihn Everett in [5] angegeben

* Dissertation Bonn (D 5) 1966, Teil II. Gegenüber dem entsprechenden Teil (§ 6) der Dissertation ist die vorliegende Arbeit jedoch etwas erweitert worden, was an einigen Stellen auch Änderungen der Beweise zur Folge hatte. Bezüglich der verwendeten Begriffe aus der Allgemeinen Algebra wird auf Teil I verwiesen bzw. auf die im Literaturverzeichnis angegebenen Arbeiten von J. Schmidt und J. Słomiński, insbesondere [8], [9] und [11]. Wegen der Begriffe aus der Theorie der Kardinalzahlen vergleiche man H. Bachmann [1].

(¹) Man beachte, daß unter einer Algebra (A, f) stets eine nicht notwendig vollständige universelle Algebra mit der Trägermenge A und der Struktur $f = (f_i)_{i \in I}$ verstanden werden soll. Wenn eine Algebra vollständig sein soll, so wird extra darauf hingewiesen. Werden mehrere Algebren gleichzeitig betrachtet, so sollen sie — sofern nichts anderes vermerkt ist — vom gleichen Typus (d. h. ähnlich) sein. Häufig wird, wenn keine Verwechselungen möglich sind, anstelle von (A, f) auch kurz A geschrieben:

(²) Zur Erinnerung: M heißt (B, g)-freie (auch: (B, g)-unabhängige) Teilmenge der Algebra (A, f), wenn jede Abbildung β: $M \to B$ zu einem Homomorphismus $\bar{\beta}$ von der von M in (A, f) erzeugten Unteralgebra $(C_f M, f|_{C_f M})$ in (B, g) fortgesetzt werden kann. M heißt (B, g)-Basis von (A, f), wenn M darüber hinaus (A, f) erzeugt: $C_f M = A$; ist dabei $(B, g) = (A, f)$, so heißt M Basis von (A, f).