may ask is *it* strong enough. The answer is "no". The figure above indicates how to construct a one-to-one continuous plane image of a line which is semi-locally-connected but not one of the desired curves.

Question: Can 2-aposyndesis be substituted for local connectivity in Theorem 2? That is, if the one-to-one continuous s-l-c plane image X of a line has the property that for x in X and y and z in $X-x$, there exists a closed (rel. X) and connected subset of $X-(y+z)$ which contains x in its interior (rel. X), then is X both locally connected and locally compact?

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UNIVERSITY OF CALIFORNIA RIVERSIDE
and
THE AUSTRALIAN NATIONAL UNIVERSITY

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Maximal chains in atomic Boolean algebras

by

George W. Day (Laramie, Wyoming) *

J. Jakubik [3] has given an example of a Boolean algebra with atoms, which also has a maximal dense-in-itself chain of elements. The principal result of this note is a necessary and sufficient condition that a chain be isomorphic to a maximal chain of an $\aleph^-$-complete atomic Boolean algebra. (A Boolean algebra is said to be *$\aleph^-$-complete* if it is $\aleph'$-complete whenever $\aleph' < \aleph$.) In addition, several examples pertinent to related questions on Boolean algebras are given. Our notation will follow that of Dwinger [2].

THEOREM. *In order that the chain C be isomorphic to a maximal chain in an $\aleph^-$-complete atomic Boolean algebra, it is necessary and sufficient that C is $\aleph^-$-complete, has a maximal and a minimal element, and has no complete dense-in-itself interval.*

Proof. We need only establish that these conditions are sufficient. Let B be the Boolean algebra of finite unions of half-open intervals, $[a, b)$, of C. Then $\{[0, c): c \in C\}$ is a maximal chain in B, which we hereafter identify with C. Let $S(B)$ be the Stone space of B, that is, the set of prime ideals of B with the usual topology. Let P be any set of prime ideals of B such that

(i) *if $[a, b]$ is a dense-in-itself interval of C, then there is an element I of P such that $a+\bar{b} \in I$,* and

(ii) *if $I \in P$, then $C \cap I$ has no maximal element and $C-I$ has no minimal element.*

Note that since C generates B, no two elements of P have the same intersection with C. Next, let F be the field of sets generated by finite subsets of P and open-and-closed sets in $S(B)$. For each $x \in B$, let $O(x)$ be the corresponding open-and-closed set in $S(B)$; that is, let $O(x) = \{I: I \in S(B)$ and $x \notin I\}$. Denoting symmetric difference by $\oplus$, it is readily shown that $F = \{S \oplus O(x): S$ is a finite subset of P and $x \in B\}$.

LEMMA. *F is atomic.*

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Proof. The singletons of P and the isolated points of $S(B)$ are atoms of F. If S is a finite subset of P and $x \in B$, and $S - O(x)$ is not empty, each singleton of that set is an atom in $S \oplus O(x)$. If $S - O(x)$ is empty and $S \oplus O(x)$ is not, then S is a finite proper subset of $O(x)$ and for some non-zero element y of B, we have that $O(y) \subseteq O(x) - S = S \oplus O(x)$. There exist elements a and b of C such that $a < b$ and $\bar{a}b \leqslant y$. Either $[a, b]$ contains a jump, so that y contains an atom of B and $O(y)$ contains an isolated point of $S(B)$, or $[a, b]$ is dense-in-itself, so that by property (i) of P there is an $I \in P$ such that $\bar{a}b \notin I$ and $I \in O(y)$.

LEMMA. *If $x \in B$ and $O(x) \subseteq P$, then $x = 0$.*

Proof. For each $c \in C$ other than the maximal element of C, let $I(c)$ be the unique prime ideal of B whose intersection with C is $\{y\colon y \in C$ and $y \leqslant c\}$. The set $\{I(c)\colon c \in C$ and $c \neq 1\}$ is dense in $S(B)$ and, by property (ii) of P, is disjoint from P.

LEMMA. *$C' = \{O(c)\colon c \in C\}$ is a maximal chain in F.*

Proof. Suppose that for some finite subset S of P and some $z \in B$, $S \oplus O(z)$ is ordered with every element of C'. Then, if $c \in C$, either $S \oplus O(z) \subseteq O(c)$, so that $O(\bar{c}z) \subseteq S$ and $z \leqslant c$ by the preceding lemma; or $O(c) \subseteq S \oplus O(z)$, so that $O(c\bar{z}) \subseteq S$ and $c \leqslant z$. Since C is a maximal chain in B, we thus see that $z \in C$, and that $S \oplus O(z)$ is also ordered with $O(z)$.

If $S \oplus O(z) \subseteq O(z)$, then $S \subseteq O(z)$. In this case, if $c \in C$ and $c < z$, then $O(\bar{c}z) \nsubseteq S$, by the preceding lemma; thus, $S \oplus O(z) = O(z) - S \nsubseteq O(c)$. Since $S \oplus O(z)$ is ordered with each element of C', it follows that $O(c) \subseteq O(z) - S$. Hence, if $I \in S$, then $I \cap C = \{c\colon c \in C$ and $c < z\}$. Hence, $C - I$ has a minimal element, and I is not an element of P. Thus, $S = \emptyset$ and $S \oplus O(z) \in C'$. The dual argument similarly disposes of the case in which $O(z) \subseteq S \oplus O(z)$.

F, then, is an atomic Boolean algebra with a maximal chain C', that is isomorphic to C. It may be noted in passing that if C is infinite, then the cardinality of F is the larger of the cardinalities of C and P. Moreover, since P can be chosen so that its cardinality is no larger than that of C, C can be embedded as a maximal chain in an atomic Boolean algebra of the same cardinality as C.

LEMMA. *C' is a maximal chain in F', the $\aleph^-$-completion of F.*

Proof. Suppose that for some $b \in F'$ there is a partition $C_1 \cup C_2$ of C' such that if $d \in C_1$, then $d < b$, and if $d \in C_2$, then $b < d$. If C_1 has a maximal element, $O(c)$ for some $c \in C$, then since F is dense in F', there is an element a of F such that $0 < a \leqslant O(\bar{c})b$. Since a is a subset of every element of C_2 and is disjoint from every element of C_1, which contains a maximal element, a can contain no singleton from P; thus we may assume that there are elements x and y in C such that $x < y$ and $O(\bar{x}y) \leqslant O(\bar{c})b$. Since no elements of C' lie in the open interval between $O(c)$ and b, we can conclude that $x = c$ and $O(y) = b$, contrary to the definition of b. Hence, C_1 has no maximal element, and by similar argument, C_2 has no minimal element. It follows that both C_1 and C_2 are $\aleph^-$-complete.

Consider now the ideal J of F' defined by $J = \{x\colon x \leqslant c + \bar{d}$ for some $c \in C_1$ and $d \in C_2\}$. This ideal is readily seen to be $\aleph^-$-complete; thus $J \cup \bar{J}$ is an $\aleph^-$-regular subalgebra of F' that contains C' but not b. $J \cup \bar{J}$ will also contain every P-singleton unless $I \in P$, where I is the prime ideal of B for which $I \cap C$ corresponds to C_1 through the natural correspondence between C and C'.

Suppose that $I \in P$; letting $a = \{I\}$, we see that $a \notin J \cup \bar{J}$. Let $F'' = \{x\colon x \in F'$ and for some $y \in J \cup \bar{J}$, $\bar{a}y \leqslant x \leqslant a + y\}$. Then F'' is an $\aleph^-$-complete, $\aleph^-$-regular subalgebra of F', and since $a \in F''$ and $J \cup \bar{J} \subseteq F''$, it follows that $F'' = F'$. Consequently, for some element b' of $J \cup \bar{J}$, either $b = \bar{a}b'$ or $b = a + b'$. Thus, $b' = \bar{a}b$ or $b' = a + b$; in either case, we find that if $d \in C_1$ then $d \leqslant b'$ and if $d \in C_2$ then $b' \leqslant d$. From the definition of J, b' must be a maximal element of C_1 or a minimal element of C_2, contrary to the hypothesis that $I \in P$.

We now see that F and all P-singletons are contained in $J \cup \bar{J}$; thus, $J \cup \bar{J} = F'$. Hence, $b \in J \cup \bar{J}$, which contradicts our original assumption on b. We conclude that C' is a maximal chain in F'.

Since F' is an $\aleph^-$-complete atomic Boolean algebra, this result completes the proof of our theorem.

We close with a few examples relevant to some questions about maximal chains in Boolean algebras.

EXAMPLE. It is well known that every countable atomless Boolean algebra is isomorphic to the free Boolean algebra on countably many generators, and that this algebra is generated by a chain isomorphic to the $[0, 1]$ rational number chain. A special case of the above procedure yields a countable atomic Boolean algebra with a maximal chain that is isomorphic to that chain.

EXAMPLE. To generalize the theorem, one might seek to answer the question, "Under what conditions on a distributive lattice D can D be embedded in an atomic Boolean algebra B so that each maximal chain of D is also maximal in B?" Of course, every finite distributive lattice can be so embedded. Consider, on the other hand, $D = \{(r, s)\colon r$ and s are rational numbers, $r \in [0, \frac{1}{2}]$, $s \in [\frac{1}{2}, 1]$ and $r \neq s\}$, with the ordering "$(r, s) \leqslant (t, u)$ iff $r \leqslant t$ and $s \leqslant u$". It is easily verified that D is a distributive lattice with maximal chain $C = \{(r, \frac{1}{2})\colon r \in [0, \frac{1}{2})\} \cup \{(\frac{1}{2}, s)\colon s \in (\frac{1}{2}, 1]\}$; moreover, in any lattice-embedding of D in a Boolean algebra B, the B-complement of the element $(0, 1)$ of D is ordered with every element of C.

EXAMPLE. One may also ask to what extent the maximal chains of a Boolean algebra characterize the structure of the algebra. Several obvious remarks may be made: a Boolean algebra is atomless if and only if every maximal chain of it is dense-in-itself; a Boolean algebra is $\aleph^-$-complete if and only if every maximal chain in it is $\aleph^-$-complete; a Boolean algebra is superatomic (see [1]) if and only if no maximal chain in it has a dense-in-itself subchain. Nevertheless, the order types of the maximal chains of a Boolean algebra together with the cardinal number of maximal chains of each order type does not completely determine the structure of the algebra: letting $\aleph$ be any cardinal that is the limit of a strictly increasing ω-sequence of infinite cardinals, so that $\aleph < \aleph^{\aleph_0}$, it is easily shown that if B and B' are the Boolean algebras of finite and cofinite subsets of sets of cardinalities $\aleph$ and $\aleph^{\aleph_0}$, respectively, then both B and B' will have maximal chains only of type $\omega+\omega^*$, and both will have exactly $\aleph^{\aleph_0}$ of these.

EXAMPLE. J. Jakubik, in [3], has constructed Boolean algebras having maximal chains of varying length. A method of construction alternate to that presented there is indicated by the following observation: if $\{B_a: a \in A\}$ is a set of atomless Boolean algebras, and B is their Boolean product, and C is a maximal chain in B_a for some $a \in A$, then C is a maximal chain in B.

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On smooth dendroids

by

J. J. Charatonik and Carl Eberhart (Lexington, Ky.)

§ 1. Introduction. Investigating fans, especially very simple ones, called smooth (see [7], p. 7 and [9]) we have observed that the notion of smoothness of fans can be easily extended to smoothness of dendroids. Smooth dendroids are very close to some partially ordered spaces, called generalized trees, which were studied by Ward (see [16], p. 801). He assumed that the considered space is Hausdorff but not necessarily metrizable, and defined a generalized tree as a hereditarily unicoherent continuum which admits a closed, order-dense partial order with unique minimal element. Koch and Krule in [11], p. 679 have replaced the condition "order-dense" by the weaker one, "monotone", and have proved (op. cit. p. 680) the following

THEOREM 1. (Koch and Krule). *Let X be a hereditarily unicoherent continuum, and let $p \in X$. The following are equivalent:*

(1) *$\leqslant_p$ is a monotone, closed partial order on X;*

(2) *there exists a monotone, closed partial order $\leqslant$ on X with unique minimal element p;*

(3) *X is arcwise connected, and for every net $\{x_\gamma\}$ in X it is true that $px_\gamma \to px$ if $x_\gamma \to x$.*

If further (1), (2) *or* (3) *holds, then X is locally connected at p.*

The weak cut point order on X with respect to p, $\leqslant_p$, is defined by $x \leqslant_p y$ if and only if $x \in py$, where py denotes the intersection of all subcontinua of X containing p and y (see e.g. [11], p. 680). If X is a dendroid, then $\leqslant_p$ is a partial order.

This paper contains investigations of smooth dendroids, i.e. metric generalized trees in sense (3) of above Theorem. Some of our theorems are generalizations of known theorems concerning fans, contained in [7].

§ 2. Definitions and preliminary properties. All continua considered in this paper are metric, provided the opposite is not said. The distance from x to y will be denoted by $d(x, y)$. A *dendroid* is a hereditarily unicoherent and arcwise connected continuum. It follows that it