On a certain property of the derivative

by

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1. A very interesting property of the derivative f' of a real function f of single variable is that the sets

$$\{x\colon f'(x) < a\} \quad \text{and} \quad \{x\colon f'(x) > \beta\}$$

are either void or of positive measure for arbitrary values of a and β. Also, a derivative function is of Baire Class 1 and possesses the Darboux property. These three properties possessed by f' ensure another property possessed by f' [1], known as the Denjoy–Clarkson property, viz. the set

$$\{x\colon a < f'(x) < \beta\}$$

is either void or of positive measure for arbitrary values of a and β, $a < \beta$. Clarkson [1] has pointed out that this property cannot be deduced from the Darboux property alone. However, he has used the Darboux property to prove his theorem. Zahorski [8] has obtained a more general result by using the Darboux property and Lagrange's mean value theorem for f'. In the present paper it has been possible to obtain a necessary and sufficient condition under which a function will satisfy the Denjoy–Clarkson property. In the proof of the theorem the Darboux property is not used and so the class of functions considered in this paper is larger than the class of derivative functions.

Throughout the paper R will denote the set of real numbers and μ will denote the usual Lebesgue measure, linear or 2-dimensional as the case may be.

2. We shall introduce the following definitions and notations.

Definition. A measurable function $f\colon R \to R$ will be said to satisfy the Denjoy–Clarkson property iff given any two reals a, β, $a < \beta$ the set

$$\{x\colon a < f(x) < \beta\}$$

is either void or of positive measure.

A measurable set E is said to satisfy the property $\mathcal{C}$ iff $A \cap E$ is either void or of positive measure whenever A is an interval.

A measurable function f is said to satisfy the property $\mathcal{C}$ iff for arbitrary α, β, $\alpha < \beta$, the set

$$\{x:\ \alpha < f(x) < \beta\}$$

satisfies the property $\mathcal{C}$.

We shall write

$$E(f;\ *, \alpha) = \{x:\ f(x) < \alpha\},$$
$$E(f;\ \beta, *) = \{x:\ f(x) > \beta\},$$
$$E(f;\ \alpha, \beta) = \{x:\ \alpha < f(x) < \beta\}.$$

THEOREM 1. *A necessary and sufficient condition that a function f of Baire Class 1 satisfy the property $\mathcal{C}$ is that the sets $E(f;\ *, \alpha)$ and $E(f;\ \beta, *)$ for arbitrary α, β satisfy the property $\mathcal{C}$.*

Proof. Let f satisfy the property $\mathcal{C}$ and let A be any interval. If $A \cap E(f;\ *, \alpha)$ is non-void, let $\xi \in A \cap E(f;\ *, \alpha)$. Then $f(\xi) < \alpha$. Choose γ such that $\gamma < f(\xi) < \alpha$. Since f satisfies the property $\mathcal{C}$, $A \cap E(f;\ \gamma, \alpha)$ is of positive measure and hence $A \cap E(f;\ *, \alpha)$ is of positive measure. Similar arguments hold for the set $E(f;\ \beta, *)$.

Conversely, let the sets $E(f;\ *, \alpha)$ and $E(f;\ \beta, *)$ for arbitrary α, β satisfy the property $\mathcal{C}$. Let $\alpha < \beta$. If possible, let us assume that the set $E(f;\ \alpha, \beta)$ does not satisfy the property $\mathcal{C}$. So, there is an interval A such that

$$A \cap E(f;\ \alpha, \beta) \neq 0, \qquad \mu[A \cap E(f;\ \alpha, \beta)] = 0.$$

We may consider the set A to be the domain of f. Let us define

$$F_\alpha = \{x:\ f(x) \leqslant \alpha\}, \qquad F_\beta = \{x:\ f(x) \geqslant \beta\}.$$

Let Q be any non-degenerate component of F_α. Then Q is an interval. Let a and b be the end-points of Q where $a < b$. Now the set $[a, b] \cap E(f;\ \alpha, *)$ contains at most two points, namely a and b, and hence is of measure zero. So $[a, b] \cap E(f;\ \alpha, *) = 0$. Hence $f(a) \leqslant \alpha$, $f(b) \leqslant \alpha$ and, since Q is a component, we conclude that $a \in Q$, $b \in Q$. Thus if Q is a non-degenerate component of F_α, then Q is a closed interval. A similar argument holds for F_β.

Let $\{Q\}$ be the collection of all non-degenerate components of F_α and F_β. Let $P = \sim \cup Q^0$, where Q^0 is the interior of Q. Then P is closed.

We shall now show that $P \subset \overline{F}_\alpha \cap \overline{F}_\beta$, where $\overline{F}$ is the closure of F. If possible, let $x_0 \in P$ but $x_0 \notin \overline{F}_\alpha$. Then there is an open interval J such that $x_0 \in J$ and $J \cap F_\alpha$ is void. Since $\mu[E(f;\ \alpha, \beta)] = 0$, $f(x) \geqslant \beta$ almost everywhere in J and hence $f(x) \geqslant \beta$ for all $x \in J$. So $J \subset F_\beta$ and consequently $J \subset Q^0$ for some $Q \in \{Q\}$. So $x_0 \in Q^0$, showing that $x_0 \notin P$, which is a contradiction. Hence $P \subset \overline{F}_\alpha$. Similarly $P \subset \overline{F}_\beta$.

Let $x_0 \in P$ and let I be an interval containing x_0 in its interior. Then we shall show that $I \cap P \cap F_\alpha$ and $I \cap P \cap F_\beta$ are non-void. Since $x_0 \in P$, $x_0 \in \overline{F}_\alpha$. So, there is a point ξ of F_α in I. If $\xi \in P$, then $\xi \in I \cap P \cap F_\alpha$. Otherwise $\xi \in Q^0$ for some $Q \in \{Q\}$ and $Q \subset F_\alpha$. Since $x_0 \notin Q^0$, there is a point η which is one of the end-points of Q such that either $x_0 \leqslant \eta < \xi$ or $\xi < \eta \leqslant x_0$. Then $\eta \in I \cap P \cap F_\alpha$. Similarly $I \cap P \cap F_\beta$ is also non-void.

Hence

$$\inf_{x \in I \cap P} f(x) \leqslant \alpha, \qquad \sup_{x \in I \cap P} f(x) \geqslant \beta.$$

This shows that f has no point of continuity on P relative to P. This contradicts the fact that f is of Baire Class 1.

The above theorem can be stated in the following way:

THEOREM 1'. *A necessary and sufficient condition that a function f of Baire Class 1 satisfy the property $\mathcal{C}$ is that each of the sets $E(f;\ *, \alpha)$ and $E(f;\ \beta, *)$ for arbitrary α, β be metrically dense in itself.*

COROLLARY. *The finite approximate derivative f'_{ap} of a function f of single variable satisfies the property $\mathcal{C}$.*

Proof. If is known [6] that if f has an approximate derivative f'_{ap} which is non-negative almost everywhere, then f is non-decreasing. From this fact we deduce that for arbitrary α and β the sets $I \cap E(f'_{ap};\ *, \alpha)$ and $I \cap E(f'_{ap};\ \beta, *)$ are either void or of positive measure, where I is an interval. Also it is known [3] that f'_{ap} is of Baire Class 1. Hence, by Theorem 1, f'_{ap} satisfies the property $\mathcal{C}$.

The above result has been proved in [4] and [7]. Zahorski [8] refined the Denjoy Clarkson property for a derived function f' in the sense that if $x \in E(f';\ \alpha, \beta)$ and if $\{I_n\}$ is a sequence of intervals not containing x such that $\{I_n\}$ converge to x and $\mu[I_n \cap E(f';\ \alpha, \beta)] = 0$ for all n, then $\mu[I_n]/d(x, I_n) \to 0$ as $n \to \infty$. Weil [7] has shown that this refinement regarding the set $E(f;\ \alpha, \beta)$ for an arbitrary function f of Baire Class 1 possessing the Denjoy–Clarkson property is not possible.

3. We now prove the Denjoy–Clarkson property for approximate partial derivatives in R^2.

DEFINITION. A measurable function f: $R^2 \to R$ will be said to satisfy the Denjoy-Clarkson property iff given any two reals α, β, $\alpha < \beta$, the set

$$\{(x, y):\ \alpha < f(x, y) < \beta\}$$

is either void or of positive measure.

LEMMA. *Let $G \subset R^2$ be a domain. If f: $G \to R$ is continuous relative to the second variable y and the approximate partial derivative with respect to x, $\left(\frac{\partial f}{\partial x}\right)_{ap}$ exists and is finite throughout G, then the set*

$$\left\{(x, y):\ \left(\frac{\partial f}{\partial x}\right)_{ap}(x, y) < \lambda\right\}$$

is either void or of positive measure for arbitrary constant λ.

Proof. We shall prove the lemma for $\lambda = 0$; for $\lambda \neq 0$, the proof follows by considering $f(x,y) - \lambda x$ instead of $f(x,y)$. Let $(\xi, \eta) \in \left\{(x,y)\colon \left(\frac{\partial f}{\partial x}\right)_{ap}(x,y) < 0\right\}$. Let $g(x) = f(x,\eta)$. Then g'_{ap} exists for all x for which $(x,\eta) \in G$. So, by the corollary to Theorem 1, g'_{ap} satisfies the property $\mathfrak{C}$ on $\{x\colon (x,\eta) \in G\}$. Hence since $g'_{ap}(\xi) < 0$, the set $\{x\colon g'_{ap}(x) < 0;\ (x,\eta) \in G\}$ is of positive linear measure. Also since $g'_{ap}(\xi) < 0$, there is a ξ_1 such that $(\xi_1, \eta) \in G$, $\xi < \xi_1$ and $f(\xi,\eta) > f(\xi_1,\eta)$. Let $\varepsilon = \{f(\xi,\eta) - f(\xi_1,\eta)\}/2$. Then $\varepsilon > 0$. Since $f(x,y)$ is continuous relative to y, corresponding to ε there is a $\delta > 0$ such that

$$f(\xi, y) > f(\xi,\eta) - \varepsilon \quad \text{and} \quad f(\xi_1, y) < f(\xi_1,\eta) + \varepsilon$$

whenever $|y - \eta| < \delta$ and the rectangle with vertices $(\xi, \eta-\delta)$, $(\xi, \eta+\delta)$, $(\xi_1, \eta-\delta)$ and $(\xi_1, \eta+\delta)$ lie in G.

Now consider the rectangle $\left\{(x,y)\colon \xi \leqslant x \leqslant \xi_1;\ \eta - \frac{\delta}{2} \leqslant y \leqslant \eta + \frac{\delta}{2}\right\}$ and let $\left\{(x,y)\colon \xi \leqslant x \leqslant \xi_1;\ y = k;\ \eta - \frac{\delta}{2} \leqslant k \leqslant \eta + \frac{\delta}{2}\right\}$ be any section of the above rectangle.

Clearly

$$f(\xi, k) > \frac{f(\xi,\eta) + f(\xi_1,\eta)}{2} > f(\xi_1, k).$$

So, there is at least one point ξ_2, $\xi \leqslant \xi_2 \leqslant \xi_1$, such that $\varphi'_{ap}(\xi_2) = 0$, where $\varphi(x) = f(x,k)$. For, if $\varphi'_{ap}(x) \geqslant 0$ for all x, $\xi \leqslant x \leqslant \xi_1$, then φ would be non-decreasing [3]. Hence the set of points x, $\xi \leqslant x \leqslant \xi_1$, for which $\varphi'_{ap}(x) < 0$ has positive linear measure. Since this is true for all k, $\eta - \frac{\delta}{2} \leqslant k \leqslant \eta + \frac{\delta}{2}$, the set

$$\left\{(x,y)\colon \xi \leqslant x \leqslant \xi_1;\ \eta - \frac{\delta}{2} \leqslant y \leqslant \eta + \frac{\delta}{2}\right\} \cap \left\{(x,y)\colon \left(\frac{\partial f}{\partial x}\right)_{ap} < 0\right\}$$

is of positive measure. This completes the proof.

THEOREM 2. *Let $G \subset R^2$ be a domain. Let $f\colon G \to R$ be continuous relative to y and let the approximate partial derivative with respect to x, $\left(\frac{\partial f}{\partial x}\right)_{ap}$ exist and be finite throughout G. If $\left(\frac{\partial f}{\partial x}\right)_{ap}\colon G \to R$ sends connected sets into connected sets, then $\left(\frac{\partial f}{\partial x}\right)_{ap}$ satisfies the Denjoy–Clarkson property.*

Proof. Let α, β be any two reals such that $\alpha < \beta$. Let us write $\varphi = \left(\frac{\partial f}{\partial x}\right)_{ap}$ and $z = (x,y)$. Let

$$E = \{z\colon z \in G;\ \alpha < \varphi(z) < \beta\},$$
$$E_\alpha = \{z\colon z \in G;\ \varphi(z) \leqslant \alpha\},$$
$$E_\beta = \{z\colon z \in G;\ \varphi(z) \geqslant \beta\}.$$

Then the sets E, E_α and E_β are mutually disjoint and $G = E_\alpha \cup E \cup E_\beta$.

If possible, let us suppose that E is non-void but is of plane measure zero. We shall show that under the condition $E \subset E'_\alpha \cap E'_\beta$ where E'_α and E'_β are the sets of limiting points of E_α and E_β respectively. If possible, suppose $z_0 = (x_0, y_0) \in E$ but $z_0 \notin E'_\alpha$. Then there is an open circle S such that $z_0 \in S$, $S \subset G$ and $S \cap E_\alpha = 0$. Since $\mu E = 0$, $\varphi(z) \geqslant \beta$ almost everywhere on S and hence the set $\{z\colon \varphi(z) < \beta\} \cap S$ is of measure zero. So, by the Lemma $\{z\colon \varphi(z) < \beta\} \cap S = 0$. But this is a contradiction, since $z_0 \in S$ and $\varphi(z_0) < \beta$. Thus $E \subset E'_\alpha$. Similarly $E \subset E'_\beta$. So, $E \subset E'_\alpha \cap E'_\beta$. From this we conclude that $\bar{E} \subset E'_\alpha \cap E'_\beta$, where $\bar{E}$ is the closure of E.

Let $\zeta = (\xi, \eta) \in \bar{E}$. Then since $\bar{E} \subset E'_\alpha \cap E'_\beta$, if S is any open circle containing ζ, we have

$$\inf_{z \in S} \varphi(z) \leqslant \alpha, \qquad \sup_{z \in S} \varphi(z) \geqslant \beta.$$

We conclude that $\inf_{z \in S \cap E} \varphi(z) = \alpha$. For, if $\inf_{z \in S \cap E} \varphi(z) = k > \alpha$, then $(\alpha, k) \cap \varphi(S) = 0$, while $(\alpha, k) \subset \left(\inf_{z \in S} \varphi(z),\ \sup_{z \in S} \varphi(z)\right)$, which contradicts the fact that φ sends connected sets to connected sets. Similarly $\sup_{z \in S \cap E} \varphi(z) = \beta$. So we conclude that

$$\inf_{z \in S \cap \bar{E}} \varphi(z) \leqslant \alpha, \qquad \sup_{z \in S \cap \bar{E}} \varphi(z) \geqslant \beta.$$

So, the saltus of φ at each point of $\bar{E}$ relative to $\bar{E}$ is at least $\beta - \alpha$. Hence the function $\varphi/\bar{E}$ is discontinuous at each point of $\bar{E}$. Now φ belongs to Baire Class 1 [5] and hence the points of discontinuity of φ considered over a closed subset form a set of the first category relative to the subset. Since $\bar{E}$ is a set of the second category on itself, this gives a contradiction. This completes the proof.

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One-to-one continuous images of a line

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Even if the metric space X is a one-to-one continuous image of a line, it still may be rather complicated topologically. However, Lelek and McAuley [3] have pointed out that when X is also both locally connected and locally compact, X must be one of five quite simple plane curves: an open interval, a figure eight, a dumbbell, a theta curve, or a noose. In fact, I have indicated in [1] that if X is embeddable in the plane, local compactness is not needed. Consequently one suspects that in the presence of local compactness, local connectivity is too strong and that something weaker should suffice. This turns out to be the case. For if X is aposyndetic and locally compact then X must be one of the five curves. On the other hand, there are plane continuous one-to-one images of the line which are aposyndetic but not locally connected and hence *not* one of the five curves.

While one of the objects of this paper is to extend the Lelek–McAuley result to the larger class of aposyndetic spaces, it is also one of my purposes to give a more complete argument for the result announced in [1] since the "indication of proof" given there seems not to have been sufficiently suggestive.

DEFINITION. A connected topological space X is aposyndetic at the point x of X provided that if y is a point of $X-x$ there exists a closed and connected set H which contains x in its interior but does not contain y (i.e., $x \in H^0 \subset H \subset X-y$). If X is aposyndetic at each of its points, then X is said to be *aposyndetic* [2].

THEOREM 1. *If the locally compact, aposyndetic, metric space X is a one-to-one continuous image of a line, then X is homeomorphic with an open interval, a figure eight, a dumbbell curve, a theta curve, or a noose (a figure nine).*

Proof. Let f denote a one-to-one continuous function from the real numbers onto X. Suppose that U is an open subset of X such that (1) $\overline{U}$ is compact and (2) there exist a divergent sequence $t_1 < t_2 < t_3 < \ldots$ of positive numbers and a divergent sequence $t_{-1} > t_{-2} > t_{-3} > \ldots$ of negative numbers such that for each positive integer n, both $f(t_n)$