On decompositions of λ-dendroids

by

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§ 1. Introduction. All continua considered in this paper are metric. An irreducible continuum I is said to be *of type* λ (see [7], § 43, III, p. 137, the footnote) if it has an upper semi-continuous decomposition into tranches (for the definition of tranches see [7], § 43, IV, p. 139 and also [8], p. 184 and p. 185—the notion of E-subcontinua and C-subcontinua) such that the hyperspace of this decomposition is the unit interval. It is known (see [7], § 43, VII, 3, p. 153), that

(1.1) An irreducible continuum I is of type λ if and only if every indecomposable subcontinuum of I has the empty interior.

A *dendroid* means a continuum X such that

(1.2) For every two points a and b of X there exists exactly one continuum $I(a, b)$ irreducible from a to b,

(1.3) $I(a, b)$ is an arc.

Condition (1.2) is equivalent to the hereditary unicoherence of X (see [8], Theorem 1.1, p. 179). Condition (1.3) means arcwise connectedness of X. Thus a continuum X is a dendroid if and only if it is hereditarily unicoherent and arcwise connected.

Proof. B. Knaster has proposed a generalization of the notion of a dendroid by replacing of condition (1.3) by the following:

(1.4) $I(a, b)$ is of type λ.

He has called such continua λ-dendroids. Hence λ-*dendroid* means a continuum X for which conditions (1.2) and (1.4) hold.

The purpose of this paper is to define and investigate some upper semi-continuous decompositions of λ-dendroids. The decompositions are the finest possible in certain sense. The main result (see Corollary 2) is patterned after the well known theorem concerning upper semi-continuous decompositions of irreducible continua into tranches, described by K. Kuratowski in his papers [5] as well as in [7], p. 139–142.

First of all observe the following properties of λ-dendroids.

(1.5) Every λ-dendroid is hereditarily decomposable.

In fact, suppose a λ-dendroid X contains an indecomposable continuum N. Thus there are points a and b in N such that N is irreducible from a to b. By condition (1.2) there is no other irreducible continuum from a to b in X. But N is not of type λ by (1.1) contrary to (1.4).

(1.6) Every λ-dendroid is a curve, i.e. a continuum of dimension one.

For every continuum of dimension greater than one contains an indecomposable subcontinuum (a theorem of S. Mazurkiewicz; see [7], § 43, V, p. 144).

THEOREM 1. *A continuum is a λ-dendroid if and only if it is hereditarily unicoherent and hereditarily decomposable.*

Indeed, if X is a λ-dendroid, then condition (1.2) implies the hereditary unicoherence of X by above quoted Theorem 1.1 in [8], p. 179, and X is hereditarily decomposable by (1.5). Invertedly, if X is hereditarily unicoherent and hereditarily decomposable, then (1.2) holds by the same Theorem 1.1, and (1.4) follows from the hereditary decomposability of X by virtue of (1.1).

It follows immediately from the definitions that every dendroid is a λ-dendroid. Also every hereditarily unicoherent irreducible continuum of type λ is a λ-dendroid. Now we describe here an example of a λ-dendroid $K(abc)$ which will be used to construct a more complicated λ-dendroid L.

Let $\varrho(x, y)$ denote the distance between points x and y of the Euclidean plane. By an oriented triangle T we mean a triangle (i.e. a 2-cell) in which an ordering $\prec$ of vertices is distinguished. If a, b and c are vertices of T and this ordering is just $a \prec b \prec c$, then we write $T = T(abc)$. Let $T(abc)$ be an arbitrary oriented equilateral triangle and let a_n and c_n be points of the side ac of $T(abc)$ such that $\varrho(a, a_n) = \varrho(a, c)/2n$ and $\varrho(a, c_n) = \varrho(a, c)/(2n-1)$ for $n = 1, 2, \dots$ Join each of points c_n with the vertex b by the segment bc and define

(1.7) $$F(abc) = ab \cup \bigcup_{n=1}^{\infty} bc_n .$$

Thus $F(abc)$ is a dendroid homeomorphic to the harmonic fan (see [1], E1, p. 240). Now put in each of the triangles $T(a_n bc_n)$ a homeomorphic image S_n of the graph of the function $y = \sin(1/x)$ for $0 < x \leqslant 1$ having the segment bc_n as the limit continuum, i.e. such that $\overline{S_n} \cap F(abc) = bc_n = \overline{S_n} \setminus S_n$. Putting

(1.8) $$K(abc) = F(abc) \cup \bigcup_{n=1}^{\infty} S_n$$

we see that $K(abc)$ is a λ-dendroid (see Fig. 1).

Let (u, v) denote a point of the Euclidean plane given by its coordinates u and v in a Cartesian rectangular coordinate system. Admit

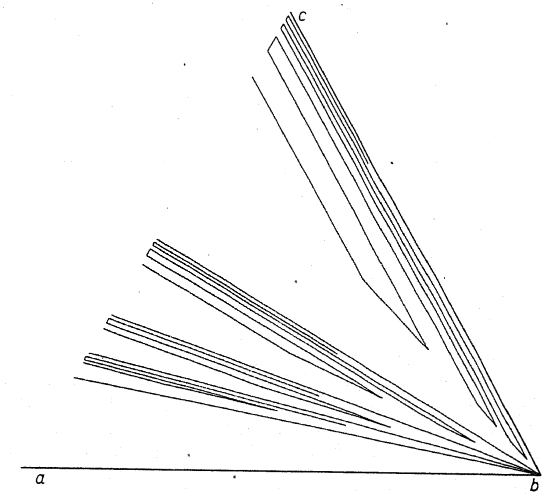


Fig. 1

$p = (0, 0)$, $q = (1, 0)$ and $U = pq$. Consider subsets P_i $(i = 0, 1, 2)$ of U defined as follows. Let k and m be positive integers. The only point of P_0 is p. The abscissae of points belonging to P_1 are the numbers of the form $u = 2^{-k}$. The set P_2 consists of points $(u, 0)$ of U for which $u = 2^{-k} + 2^{-(k+m)}$ and of the point q. Put

$$P = P_0 \cup P_1 \cup P_2 .$$

So P is a closed, countable subset of the unit segment U with the property that for every point $y \in P$ there exists a point $x \in P$ such that

(1.9) $$x \in py$$

and that no point of P is in the interior of the segment xy:

(1.10) $$xy \cap P = (x) \cup (y) .$$

In fact, if $y \in P_0$, i.e. if $y = p$, we have $x = p$. If y is in P_1 and has the form $y = (2^{-k}, 0)$, we have $x = (2^{-(k+1)} + 2^{-(k+2)}, 0)$. Finally if $y \in P_2$ and the abscissa of y is $u = 2^{-k} + 2^{-(k+m)}$, then the abscissa of the point x which is associated with y by (1.9) and (1.10) has the form $u = 2^{-k} + 2^{-(k+m+1)}$. So we see that if $y \neq p$, then $x \neq y$ and that $U = \bigcup \{xy |\ y \in P\}$.

Now let us associate to each point $y \in P$ a point $z = (u, v)$ with $v > 0$ and such that the triangle $T(xyz)$ is equilateral. Further, in each of the triangles $T(xyz)$, where $y \in P$ and x and z are associated with y as above (see Fig. 2), we construct the λ-dendroid $K(xyz)$ defined in the same way as $K(abc)$ was by (1.8). In particular if $y = p$ we have $x = y = z$ and thus $K(xyz) = (p)$. Putting

$$L = \bigcup \{K(xyz) \mid y \in P\} \tag{1.11}$$

it is readily seen that L is a λ-dendroid.

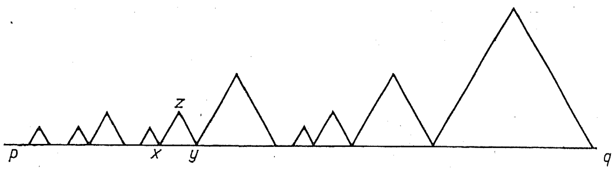


Fig. 2

Remark that if we take into consideration the (similarly defined) sets P_3, P_4 and so on, and if we define P as the union of all these P_i's leaving further constructions without any change, we can get some more complicated examples of λ-dendroids. Here we have stopped on P_2 in the above definition of P to obtain a λ-dendroid which is not too hard to describe and to image it, but also not too trivial. Another example of a λ-dendroid, more complicated than L, is given in [3].

§ 2. The canonical decomposition. Now we shall define an upper semi-continuous decomposition of a λ-dendroid X into continua (called strata of X) such that the hyperspace of this decomposition is a dendroid. We shall prove that the decomposition under consideration has some nice properties, very similar to properties of the decomposition of an irreducible continuum into tranches, described by K. Kuratowski in his papers [5] as well as in [7], pp. 139–142.

Let X be a λ-dendroid and let x be a point of X. To describe the stratum $S(x)$ to which x belongs, we shall define (by the transfinite induction) an increasing sequence of continua $A_\alpha(x)$ each of which contains the point x.

With this in view let us consider in X all irreducible continua I which contain the point x and take in each of them the tranche $T(x)$ to which x belongs. Put

$$A_0(x) = \overline{\bigcup T(x)}, \tag{2.1}$$

where the union in the right side runs over all irreducible continua I such that $x \in I \subset X$. Now suppose sets $A_\beta(x)$ are defined for $\beta < \alpha$, and put

$$A_\alpha(x) = \begin{cases} \bigcup \{\operatorname{Ls}_{n\to\infty} A_\beta(x_n) \mid \lim_{n\to\infty} x_n \in A_\beta(x)\}, & \text{if } \alpha = \beta+1, \\ \overline{\bigcup_{\beta<\alpha} A_\beta(x)}, & \text{if } \alpha = \lim_{\beta<\alpha} \beta, \end{cases} \tag{2.2}$$

where, in the case $\alpha = \beta+1$, the union is taken over all convergent sequences of points $x_n \in X$ with $\lim_{n\to\infty} x_n \in A_\beta(x)$.

So the sets $A_\alpha(x)$ are well-defined for all $\alpha < \Omega$. Observe that it follows from this definition that the sequence $\{A_\alpha(x)\}$ is increasing, i.e. that

$$x \in A_0(x) \subset A_1(x) \subset \ldots \subset A_\alpha(x) \subset \ldots \tag{2.3}$$

Indeed, $x \in A_0(x)$ by (2.1). Assume

$$x \in A_0(x) \subset A_1(x) \subset \ldots \subset A_\beta(x)$$

for all $\beta < \alpha$. If $\alpha = \beta+1$ then putting $x_n = x$ in (2.2) we have $\lim_{n\to\infty} x_n = x \in A_\beta(x)$ and $\operatorname{Ls}_{n\to\infty} A_\beta(x_n) = A_\beta(x)$, whence $A_\beta(x) \subset A_\alpha(x)$. The last inclusion trivially holds by (2.2) also in the case $\alpha = \lim_{\beta<\alpha} \beta$. Thus (2.3) is true.

For example, if the λ-dendroid $K(abc)$ (see (1.8) and (1.7)) is taken as X, we have $A_\alpha(x) = (x)$ for each ordinal α and for each $x \in K(abc) \setminus F(abc)$. For $x \in F(abc)$ we have: $A_0(x) = A_1(x) = F(abc)$ if $x \in (F(abc) \setminus ab) \cup (b)$; $A_0(x) = (x)$ and $A_1(x) = ab$ if $x \in ab \setminus (b)$; and finally $A_\alpha(x) = F(abc)$ for all $\alpha \geqslant 2$ and all $x \in F(abc)$.

As another example of X take the λ-dendroid L defined by (1.11). Fix $x = q$. In order to describe the sets $A_\alpha(q)$ let us admit for $n = 0, 1, 2, \ldots$ $x_n = (2^{-1} + 2^{-(n+1)}, 0)$ and $y_n = (2^{-2^n}, 0)$. Thus we see that, according to the definition, $x_n \in P_2$ and $y_n \in P_1$. Further, denote by z_n a point (u, v) such that $v > 0$ and that the triangle $T(x_{n+1} x_n z_n)$ is equilateral. One can verify that

$$A_n(q) = \bigcup_{i=1}^{2^n} F(x_i x_{i-1} z_{i-1}),$$

$$A_\omega(q) = \overline{\bigcup_{i=1}^{\infty} F(x_i x_{i-1} z_{i-1})} = (y_0) \cup \bigcup \{F(xyz) \mid y \in P \cap y_0 q \setminus (y_0)\},$$

$$A_{\omega+n}(q) = (y_n) \cup \bigcup \{F(xyz) \mid y \in P \cap y_n q \setminus (y_n)\},$$

$$A_{\omega\cdot 2}(q) = \bigcup \{F(xyz) \mid y \in P\},$$

$$A_\eta(q) = A_{\omega\cdot 2}(q) \quad \text{for all} \quad \eta > \omega\cdot 2,$$

where x and z denote points associated with $y \in P$ as it was described in the end of the previous paragraph.

Now we shall prove the following

LEMMA 1. *The sets $A_\alpha(x)$ are continua.*

Proof. Apply transfinite induction. If $\alpha = 0$, then we see that $\bigcup T(x)$ is connected for each $T(x)$ is a connected set and contains the point x, whence $A_0(x)$ is a continuum by (2.1). Assume that $A_\beta(x)$ is a continuum for all $\beta < \alpha$. To prove that $A_\alpha(x)$ is a continuum take firstly $\alpha = \beta+1$. Let $\{x_n\}$ be a convergent sequence of points of x such that

$$\lim_{n\to\infty} x_n \in A_\beta(x) . \tag{2.4}$$

Since $x_n \in A_\beta(x_n)$ by (2.3), hence

$$\lim_{n\to\infty} x_n \in \operatorname*{Li}_{n\to\infty} A_\beta(x_n) \neq \emptyset , \tag{2.5}$$

thus $\operatorname*{Ls}_{n\to\infty} A_\beta(x_n)$ is a continuum (see [7], § 42, II, 6, p. 111). Obviously $\lim_{n\to\infty} x_n \in \operatorname*{Ls}_{n\to\infty} A_\beta(x_n)$ by (2.5), therefore

$$A_\beta(x) \cap \operatorname*{Ls}_{n\to\infty} A_\beta(x_n) \neq \emptyset \tag{2.6}$$

by (2.4). So $A_{\beta+1}(x)$ is the union of continua $\operatorname*{Ls}_{n\to\infty} A_\beta(x_n)$ each of them intersects the continuum $A_\beta(x)$ which is contained in $A_{\beta+1}(x)$ by (2.3). Thus we conclude that $A_{\beta+1}(x)$ is connected (see [7], § 41, II, 2, p. 82).

Now we prove that $A_{\beta+1}(x)$ is compact. Since it is a subset of X, it is sufficient to prove that $A_{\beta+1}(x)$ is closed. Take a convergent sequence of points p_k such that

$$p_k \in A_{\beta+1}(x) \tag{2.7}$$

and put

$$p = \lim_{k\to\infty} p_k . \tag{2.8}$$

We shall prove that

$$p \in A_{\beta+1}(x) . \tag{2.9}$$

It follows from (2.7) and (2.2) that for any fixed natural k there exists a convergent sequence of points $x_{k,n} \in X$ with the limit x_k

$$x_k = \lim_{n\to\infty} x_{k,n} \tag{2.10}$$

such that

$$x_k \in A_\beta(x) \tag{2.11}$$

and

$$p_k \in \operatorname*{Ls}_{n\to\infty} A_\beta(x_{k,n}) . \tag{2.12}$$

Further, it follows from (2.12) that for every fixed natural k there exists a convergent sequence of points $q_{k,n}$ such that

$$q_{k,n} \in A_\beta(x_{k,n}) \tag{2.13}$$

and

$$\lim_{n\to\infty} q_{k,n} = p_k . \tag{2.14}$$

The set $A_\beta(x)$ being a continuum by hypothesis, the sequence of points x_k contains a convergent subsequence $\{x_{k_m}\}$, the limit of which belongs to $A_\beta(x)$. So putting

$$x_0 = \lim_{m\to\infty} x_{k_m} \tag{2.15}$$

we have

$$x_0 \in A_\beta(x) . \tag{2.16}$$

Take now the double sequence of points $x_{k_m,n}$ with $n\to\infty$ and $m\to\infty$, and consider the diagonal sequence $\{x_{k_m,m}\}$. Thus (2.10) and (2.15) lead to

$$x_0 = \lim_{m\to\infty} x_{k_m,m} \tag{2.17}$$

as well as (2.8) and (2.14) give

$$p = \lim_{m\to\infty} q_{k_m,m} . \tag{2.18}$$

It follows from (2.13) that

$$q_{k_m,m} \in A_\beta(x_{k_m,m}) , \tag{2.19}$$

whence

$$p \in \operatorname*{Ls}_{m\to\infty} A_\beta(x_{k_m,m}) \tag{2.20}$$

by (2.18). Therefore we have proved that there exists a convergent sequence of points x'_m, namely $x'_m = x_{k_m,m}$, the limit point of which, namely x_0 by (2.17), belongs to $A_\beta(x)$ by (2.16) and such that $p \in \operatorname*{Ls}_{m\to\infty} A_\beta(x'_m)$ by (2.20). Thus we conclude that

$$p \in \bigcup \{\operatorname*{Ls}_{m\to\infty} A_\beta(x'_m) \mid \lim_{m\to\infty} x'_m \in A_\beta(x)\} ,$$

which proves (2.9) by (2.2).

Take secondly $\alpha = \lim_{\beta<\alpha} \beta$. Each $A_\beta(x)$ being a continuum by hypothesis and x belonging to $A_\beta(x)$ for all $\beta < \alpha$ by (2.3), the union $\bigcup_{\beta<\alpha} A_\beta(x)$ is connected. Thus $A_\alpha(x)$ is a continuum by (2.2), which completes the proof.

We conclude from Lemma 1 and from (2.3) that $\{A_\alpha(x)\}$ is an increasing sequence of continua. Therefore there exists a countable ordinal ξ such that

$$\text{if } \xi < \eta < \Omega , \text{ then } A_\xi(x) = A_\eta(x) . \tag{2.21}$$

We define the *stratum* $S(x)$ of the point x by

$$S(x) = A_\xi(x) . \tag{2.22}$$

Thus we conclude from (2.3) and (2.21) that

$$A_\alpha(x) \subset S(x) \text{ for every ordinal } \alpha < \Omega . \tag{2.23}$$

It follows immediately from the above definition of $S(x)$ that

(2.24) If the λ-dendroid X is an irreducible continuum, then $S(x) = T(x)$ for every $x \in X$, i.e. the notion of a stratum coincides with the notion of a tranche.

(2.25) If the λ-dendroid X is arcwise connected (i.e. if X is a dendroid), then $S(x) = x$ for every $x \in X$.

Now we shall show some properties of strata.

LEMMA 2. *If* $\lim_{n\to\infty} x_n = x$, *then* $\operatorname{Ls}_{n\to\infty} S(x_n) \subset S(x)$.

Proof. Let

$$S(x) = A_\xi(x) \quad \text{and} \quad S(x_n) = A_{\xi_n}(x_n)\,, \tag{2.26}$$

where ξ and ξ_n are as ξ in (2.21). Obviously there exists an ordinal $\eta < \Omega$ such that

$$\xi < \eta \quad \text{and} \quad \xi_n < \eta \quad \text{for } n = 1, 2, \ldots \tag{2.27}$$

Hence we have

$$A_\eta(x) = A_{\eta+1}(x) = S(x) \quad \text{and} \quad A_\eta(x_n) = S(x_n) \tag{2.28}$$

by (2.26) and (2.21). According to (2.2)

$$A_{\eta+1}(x) = \bigcup \{\operatorname{Ls}_{n\to\infty} A_\eta(x_n) \mid \lim_{n\to\infty} x_n \in A_\eta(x)\}\,. \tag{2.29}$$

Thus

$$\text{if } \lim_{n\to\infty} x_n \in A_\eta(x)\,, \text{ then } \operatorname{Ls}_{n\to\infty} A_\eta(x_n) \subset A_{\eta+1}(x)\,. \tag{2.30}$$

We conclude from (2.3) that $x \in A_\eta(x)$, which means $\lim_{n\to\infty} x_n \in A_\eta(x)$ by our hypothesis $x = \lim_{n\to\infty} x_n$. Thus it follows from (2.30) that $\operatorname{Ls}_{n\to\infty} A_\eta(x_n) \subset A_{\eta+1}(x)$, i.e. $\operatorname{Ls}_{n\to\infty} S(x_n) \subset S(x)$ by (2.28).

LEMMA 3. *If* $y \in S(x)$, *then* $x \in S(y) \subset S(x)$.

Proof. Let

$$S(x) = A_{\xi_1}(x) \quad \text{and} \quad S(y) = A_{\xi_2}(y)\,, \tag{2.31}$$

where ξ_1 and ξ_2 are as ξ in (2.21). Obviously there exists an ordinal $\eta < \Omega$ such that

$$\xi_1 < \eta \quad \text{and} \quad \xi_2 < \eta\,.$$

Hence we have

$$A_\eta(x) = A_{\eta+1}(x) = S(x) \quad \text{and} \quad A_\eta(y) = S(y) \tag{2.32}$$

by (2.31) and (2.21). According to (2.2) we see that (2.29) holds, which implies (2.30) as previously. Taking $x_n = y$ we have

$$\lim_{n\to\infty} x_n = y \tag{2.33}$$

and

$$\operatorname{Ls}_{n\to\infty} A_\eta(x_n) = A_\eta(y)\,. \tag{2.34}$$

Thus we can rewrite our hypothesis $y \in S(x)$ in the form $\lim_{n\to\infty} x_n \in A_\eta(x)$ by (2.33) and (2.32). Therefore we conclude from (2.30) and (2.34) that $A_\eta(y) \subset A_{\eta+1}(x)$, i.e. $S(y) \subset S(x)$ by (2.32). So we have shown that

$$\text{if } y \in S(x)\,, \text{ then } S(y) \subset S(x)\,. \tag{2.35}$$

To prove $x \in S(y)$ recall that $S(x) = A_{\xi_1}(x)$ according to (2.31) and apply transfinite induction. If $\xi_1 = 0$, then $y \in A_0(x)$ which is defined by (2.1). Assume firstly that

$$y \in \bigcup T(x)\,.$$

Thus there exists an irreducible continuum I' in X and a tranche $T'(x)$ of the point x in I' such that $y \in T'(x)$. It implies $x \in T'(y)$, where $T'(x) = T'(y)$, whence $x \in A_0(y)$ according to (2.1). Since $A_0(y) \subset S(y)$ by (2.23) with y instead of x, hence $x \in S(y)$.

Assume secondly that

$$y \in \overline{\bigcup T(x)} \setminus \bigcup T(x)\,.$$

Thus there exist a sequence of irreducible continua I_n in X, a sequence of tranches $T_n(x)$ of I_n each of which contains the point x, and a sequence of points y_n such that

$$y_n \in T_n(x) \quad \text{for } n = 1, 2, \ldots \tag{2.36}$$

and

$$\lim_{n\to\infty} y_n = y\,. \tag{2.37}$$

It follows from (2.36) that $x \in T_n(y_n)$ for $n = 1, 2, \ldots$, where $T_n(x) = T_n(y_n)$, whence $x \in A_0(y_n)$ for $n = 1, 2, \ldots$ by the Definition (2.1) of $A_0(y_n)$. Thus we obviously have $x \in \operatorname{Ls}_{n\to\infty} A_0(y_n)$, and, by (2.37) and (2.3), $\lim_{n\to\infty} y_n \in A_0(y)$. So according to the definition of $A_1(y)$ (put $\beta = 0$, $x = y$, $x_n = y_n$ in (2.2)) we conclude that $x \in A_1(y)$, thus $x \in S(y)$ follows from (2.23) with y instead of x.

Assume now the following inductive hypothesis:

(2.38) for every $\beta < \xi_1$ and for any two points x' and y' in X, if $y' \in A_\beta(x')$, then $x' \in S(y')$.

Consider two cases. Firstly, let $\xi_1 = \beta+1$. So our assumption $y \in A_{\beta+1}(x)$ means by (2.2) that there exists a convergent sequence of points x_n such that

$$y \in \operatorname{Ls}_{n\to\infty} A_\beta(x_n) \tag{2.39}$$

and $\lim_{n\to\infty} x_n \in A_\beta(x)$. Hence putting

(2.40) $$\lim_{n\to\infty} x_n = p$$

we have

(2.41) $$p \in A_\beta(x).$$

It follows from (2.3) and (2.23) that $x_n \in S(x_n)$, thus

(2.42) $$p \in \operatorname*{Ls}_{n\to\infty} S(x_n)$$

by (2.40). Further, we conclude from (2.39) that there exists a sequence of points $y_{n_k} \in A_\beta(x_{n_k})$ which converges to y. Without loss of generality we can take y_n as y_{n_k} assuming

(2.43) $$\lim_{n\to\infty} y_n = y$$

and

(2.44) $$y_n \in A_\beta(x_n).$$

Observe that (2.44) and (2.38) imply $x_n \in S(y_n)$, hence $S(x_n) \subset S(y_n)$ by (2.35). It leads immediately to

(2.45) $$\operatorname*{Ls}_{n\to\infty} S(x_n) \subset \operatorname*{Ls}_{n\to\infty} S(y_n).$$

Lemma 2 and (2.43) give

(2.46) $$\operatorname*{Ls}_{n\to\infty} S(y_n) \subset S(y).$$

Thus it follows from (2.42), (2.45) and (2.46) that $p \in S(y)$, therefore

(2.47) $$S(p) \subset S(y)$$

by (2.35). As a corollary from (2.41) and (2.38) we have $x \in S(p)$, whence $x \in S(y)$ by (2.47).

Secondly, let $\xi_1 = \lim_{\beta<\xi_1} \beta$. So our hypothesis $y \in A_{\xi_1}(x)$ means by (2.2) that there exist a sequence of ordinals $\beta_n < \xi_1$ and a convergent sequence of points y_n such that (2.43) holds an $y_n \in A_{\beta_n}(x)$. It implies $x \in S(y_n)$ for every natural n by (2.38), whence

(2.48) $$x \in \operatorname*{Ls}_{n\to\infty} S(y_n).$$

Further, it follows from (2.43) and from Lemma 2 that $\operatorname*{Ls}_{n\to\infty} S(y_n) \subset S(y)$, thus $x \in S(y)$ by (2.48), which completes the proof.

THEOREM 2. *If* $S(x) \cap S(y) \neq \emptyset$, *then* $S(x) = S(y)$.

Proof. Let $z \in S(x) \cap S(y)$. Thus in particular $z \in S(x)$, whence $x \in S(z) \subset S(x)$ by Lemma 3. Using the same lemma once more we see that $x \in S(z)$ implies $S(x) \subset S(z)$. So $S(z) = S(x)$. Replacing x by y we obtain $S(z) = S(y)$ and the theorem follows.

We conclude from this theorem that for various x the strata $S(x)$ are either disjoint, or identical. Since they are continua by definition (2.2) according to Lemma 1, hence we have defined a decomposition of a λ-dendroid X into its strata $S(x)$. Call this decomposition *canonical*.

A decomposition of a continuum X into disjoint continua X_i is said to be upper semi-continuous if for every closed subset A of X the union of all elements X_i of this decomposition which meet A is closed (see [6], § 19, Definition and Theorem 3, p. 185, and [7], § 39, V, p. 42). We prove now

THEOREM 3. *The canonical decomposition of a λ-dendroid X is upper semi-continuous.*

Proof. Take a set $A = \bar{A} \subset X$ and denote by Z the union of all strata $S(x)$ of X for which $S(x) \cap A \neq \emptyset$. Let $\{p_n\}$ be an arbitrary convergent sequence of points of Z and put

(2.49) $$p = \lim_{n\to\infty} p_n.$$

Since $p_n \in Z$ by hypothesis, hence in view of Theorem 2 $S(p_n) \cap A \neq \emptyset$. Let

(2.50) $$a_n \in S(p_n) \cap A.$$

The set A being closed, the sequence $\{a_n\}$ contains a convergent subsequence $\{a_{n_m}\}$. Putting

(2.51) $$a = \lim_{m\to\infty} a_{n_m}$$

we have $a \in A$. It follows from (2.51) and from Lemma 2 that

(2.52) $$\operatorname*{Ls}_{m\to\infty} S(a_{n_m}) \subset S(a).$$

Since $a \in A$ and obviously $a \in S(a)$ hence we see that $S(a) \cap A \neq \emptyset$, thus

(2.53) $$S(a) \subset Z.$$

Further, we have

(2.54) $$S(a_n) = S(p_n)$$

by (2.50) and Theorem 2. As $p = \lim_{m\to\infty} p_{n_m}$ by (2.49) and as $p_{n_m} \in S(p_{n_m})$ we conclude that $p \in \operatorname*{Ls}_{m\to\infty} S(a_{n_m})$ by (2.54), whence $p \in Z$ follows by (2.52) and (2.53).

§ 3. The hyperspace. We shall prove the following

THEOREM 4. *The hyperspace of an arbitrary upper semi-continuous decomposition of a λ-dendroid into continua is a λ-dendroid.*

Proof. Let X be a λ-dendroid and let

(3.1) $$X = \bigcup\{X_d \mid d \in D\}$$

be an upper semi-continuous decomposition of X into continua X_d, where d runs over a set D. Introduce a topology on D as the quotient topology (see [6], § 19, I, p. 183). Since X is a separable metric space, hence also is D (see [9], Theorem (2.2), p. 123).

The projection of X onto D i.e. the mapping $f\colon X \to D$ such that

(3.2) $$f^{-1}(d) = X_d \text{ for each } d \in D$$

is continuous (see [9], (3.1), p. 125) and monotone for the sets $f^{-1}(d)$ are continua. Thus f is a confluent mapping (see [2], pp. 213 and 214, V) of the λ-dendroid X onto the metric space D, whence D is a λ-dendroid (see [2], XIV, p. 217).

THEOREM 5. *The hyperspace D of an upper semi-continuous decomposition* (3.1) *of a λ-dendroid X into continua X_d is a dendroid if and only if*

(3.3) *for every tranche T of an irreducible continuum I in X*

$$T \cap X_d \neq \emptyset \text{ implies } T \subset X_d.$$

Proof. Firstly we prove that if (3.3) holds, then D is a dendroid. D being a λ-dendroid according to Theorem 4, it remains to prove that D is arcwise connected. With this in view consider the projection $f\colon X \to D$ described above in the proof of Theorem 4 by (3.2). Take two different points a and b in D and their inverse images $f^{-1}(a)$ and $f^{-1}(b)$ in X. Further, take a point $x \in f^{-1}(a)$ and a point $y \in f^{-1}(b)$. Let I be the unique irreducible continuum from x to y in X. Obviously $f(I)$ is a continuum which contains a and b. We shall prove that $f(I)$ is the arc ab. It is sufficient to show that if

(3.4) $$c \in f(I)$$

and

(3.5) $$a \neq c \neq b\,,$$

then $f(I)\setminus(c)$ is not connected (see [9], Theorem (6.2), p. 54). In fact, (3.4) implies that $I \cap f^{-1}(c) \neq \emptyset$, thus by the hereditary unicoherence of X the set $I \cap f^{-1}(c)$ is a subcontinuum of I which does not contain points x or y by (3.5). Further, it follows from (3.3) that if $z \in I \cap f^{-1}(c)$, then $T(z)$, the tranche of I which contains z, is contained in $f^{-1}(c)$:

$$T(z) \subset f^{-1}(c)$$

because $f^{-1}(c) = X_c$ is an element of the decomposition. Thus

$$T(z) \subset I \cap f^{-1}(c)\,,$$

$T(z)$ being contained in I. It follows that the continuum $I \cap f^{-1}(c)$ is the union of all such tranches $T(z)$ of I, whence $I\setminus\big(I \cap f^{-1}(c)\big)$ is the union of two disjoint open in I sets U and V such that $x \in U$ and $y \in V$. Thus

$$f(I) = f(U) \cup (c) \cup f(V)\,,$$

$$a \in f(U)\,, \quad b \in f(V)\,, \quad f(U) \cap f(V) = \emptyset\,.$$

Moreover the set $I \cap f^{-1}(c) \cup U$ is of course closed, as well as $I \cap f^{-1}(c) \cup V$, so their images, i.e. sets $(c) \cup f(U)$ and $(c) \cup f(V)$ are also closed. Therefore the sets $f(U)$ and $f(V)$ are open in $f(I)$. Since they are disjoint, hence the set $f(I)\setminus(c)$ is the union of two disjoint open sets $f(U)$ and $f(V)$, therefore it is not connected.

Secondly we prove that if D is a dendroid, then (3.3) holds. Let f be, as previously, the projection of X onto D, and let I be an arbitrary continuum irreducible from x to y in X. The decomposition (3.1) of X into continua X_d, where $d \in D$, being upper semi-continuous, the decomposition of I into sets $I \cap X_d$, where $d \in f(I)$, is also upper semi-continuous, and it follows from the hereditary unicoherence of X that sets $I \cap X_d$ are continua. Since for every $d \in f(I)$ we have $(f|I)^{-1}(d) = I \cap f^{-1}(d) = I \cap X_d$, hence the mapping $f|I$ is monotone. The continuum I being irreducible from x to y, its image $f(I)$ is irreducible from $f(x) = a$ to $f(y) = b$ (see [7], § 43, I, 3, p. 133) and, D being a dendroid, $f(I)$ is the arc ab. Thus we have an upper semi-continuous linear decomposition of I:

$$I = \bigcup\{I \cap X_d \mid d \in ab\}\,.$$

Now, for any fixed d, let T be a tranche of I such that $T \cap X_d \neq \emptyset$. The decomposition of I into tranches is the finest possible decomposition among all upper semi-continuous linear decompositions of I (see [7], § 43, IV, 3, p. 139), therefore $T \cap X_d \neq \emptyset$ leads to $T \subset I \cap X_d$, whence $T \subset X_d$ which finishes the proof.

Let us come back now to the canonical decomposition of a λ-dendroid X, i.e. to the decomposition of X into its strata. Denote the hyperspace of this decomposition by $\Delta(X)$ and the strata of X by S_d, where $d \in \Delta(X)$. So we have

$$X = \bigcup\{S_d \mid d \in \Delta(X)\}\,.$$

COROLLARY 1. *The hyperspace $\Delta(X)$ of the canonical decomposition of a λ-dendroid X is a dendroid.*

Proof. According to Theorem 5 it is sufficient to prove that (3.3) holds for the canonical decomposition. Indeed, if I is an irreducible continuum in X and T is a tranche of I such that $T \cap S_d \neq \emptyset$ for some $d \in \Delta(X)$, then for any point $x \in T \cap S_d$ we have $x \in T = T(x) \subset A_0(x) \subset S(x)$ by (2.1) and (2.23), thus $S(x) = S_d$ by Theorem 2, whence $T \subset S_d$.

§ 4. The main property of the canonical decomposition. Let us go now to the investigation of some relations between the canonical decomposition of a λ-dendroid X and other upper semi-continuous decompositions of X into continua for which the hyperspace are dendroids.

Take namely an arbitrary upper semi-continuous decomposition (3.1) of a λ-dendroid X into continua X_d and assume that the hyperspace D of this decomposition is a dendroid. Thus (3.3) holds according to Theorem 5. Recall that the sets $A_\alpha(x)$ for $\alpha < \Omega$ are defined by (2.1) and (2.2).

We shall prove the following

LEMMA 4. *If $x \in X_d$, then $A_\alpha(x) \subset X_d$ for every $\alpha < \Omega$.*

Proof. Apply transfinite induction. If $\alpha = 0$, then $x \in X_d$ implies $T(x) \cap X_d \neq \emptyset$ where $T(x)$ is a tranche of a point x in some irreducible continuum $I \subset X$. Thus we have $T(x) \subset X_d$ by (3.3). This leads to $\bigcup T(x) \subset X_d$, where the union in the left side runs over all irreducible continua I such that $x \in I \subset X$. The set X_d being closed, we have $\overline{\bigcup T(x)} \subset X_d$, which means $A_0(x) \subset X_d$ by (2.1).

If $\alpha > 0$, assume that

(4.1) $x \in X_d$ implies $A_\beta(x) \subset X_d$ for every $\beta < \alpha$.

Firstly, let $\alpha = \beta+1$. Consider a convergent sequence of points x_n such that $\lim_{n\to\infty} x_n \in A_\beta(x)$. Denote by X_{d_n} a member of the decomposition (3.1) to which x_n belongs:

$$x_n \in X_{d_n} \tag{4.2}$$

and put

$$\lim_{n\to\infty} x_n = p\,. \tag{4.3}$$

Thus we have $p \in A_\beta(x)$, and we conclude from (4.1) that $x \in X_d$ implies

$$p \in X_d\,. \tag{4.4}$$

Using (4.1) once more we see that (4.2) leads to $A_\beta(x_n) \subset X_{d_n}$, whence

$$\operatorname*{Li}_{n\to\infty} A_\beta(x_n) \subset \operatorname*{Li}_{n\to\infty} X_{d_n}\,. \tag{4.5}$$

Further (4.2) and (4.3) give $p \in \operatorname*{Li}_{n\to\infty} X_{d_n}$, thus $X_d \cap \operatorname*{Li}_{n\to\infty} X_{d_n} \neq \emptyset$ by (4.4), so $\operatorname*{Ls}_{n\to\infty} X_{d_n} \subset X_d$ by the upper semi-continuity of the decomposition (3.1) (see [7], § 39, V, 4, p. 45). Since (4.5) holds, we have $\operatorname*{Ls}_{n\to\infty} A_\beta(x_n) \subset X_d$, therefore $A_{\beta+1}(x) \subset X_d$ by (2.2).

Secondly let $\alpha = \lim_{\beta<\alpha} \beta$. The set X_d being compact by hypothesis and the sequence $A_\beta(x)$ for $\beta < \alpha$ being increasing by (2.3), we conclude immediately from (4.1) and (2.2) that $A_\alpha(x) \subset X_d$. Thus the proof is finished.

According to the Definition (2.22) of a stratum Lemma 4 leads to the following

THEOREM 6. *If the hyperspace D of an upper semi-continuous decomposition (3.1) of a λ-dendroid X into continua X_d is a dendroid, and if S_{d_1}, where $d_1 \in \Delta(X)$, means a stratum of the canonical decomposition of X, then*

$$S_{d_1} \cap X_d \neq \emptyset \textit{ implies } S_{d_1} \subset X_d\,.$$

Theorem 6 can be reformulated as

COROLLARY 2. *The canonical decomposition of a λ-dendroid X is the finest possible decomposition among all upper semi-continuous decompositions of X into continua, hyperspaces of which are dendroids.*

It is well known (see e.g. [9], Theorem (3.4), p. 126) that if M is a compact metric space, any upper semi-continuous decomposition of M is equivalent to a continuous mapping defined on M, and conversely — in the sense that, given any upper semi-continuous decomposition $\mathfrak{D}$ of M with hyperspace M', there exists a continuous mapping f of M onto M' whose inverse sets $f^{-1}(x)$, $x \in M'$, are exactly the elements of $\mathfrak{D}$ and, on the other hand, given any continuous mapping f of M onto N, the inverse sets $f^{-1}(x)$ give an upper semi-continuous decomposition of M whose hyperspace is homeomorphic to N. In particular, any monotone mapping on a compact space M is equivalent to an upper semi-continuous decomposition of M into continua. Conversely, any upper semi-continuous decomposition of M into continua with hyperspace N is equivalent to a monotone mapping of M onto N (see [9], (4.1), Theorem, p. 127).

The canonical decomposition (3.1) of the λ-dendroid X into strata S_d being upper semi-continuous, it is equivalent to a monotone mapping φ of X onto $\Delta(X)$ defined by

$$\varphi^{-1}(d) = S_d \quad \text{for every } d \in \Delta(X)\,.$$

Call the mapping φ *canonical.*

Thus we have the following equivalent form of Theorem 6.

COROLLARY 3. *If φ is the canonical mapping of the λ-dendroid X onto the dendroid $\Delta(X)$, then for any monotone mapping f of X onto a dendroid D and for every $x \in X$ we have*

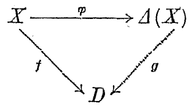
$$\varphi^{-1}\big(\varphi(x)\big) \subset f^{-1}\big(f(x)\big)\,. \tag{4.6}$$

As another form of Corollary 3 (thus of Theorem 6) we have

THEOREM 7. *If φ is the canonical mapping of the λ-dendroid X onto the dendroid $\Delta(X)$, then for every continuous monotone mapping f of X*

onto a dendroid D there exists one and only one continuous mapping g of $\Delta(X)$ onto D such that the diagram

(4.7)



commutes, and g is monotone.

In fact, take an arbitrary point $d \in \Delta(X)$. It follows from (4.6) that $f(\varphi^{-1}(d))$ is a point. Denote this point by $g(d)$. If $d = \varphi(x)$, then $g(d) = f(x)$, thus $g(\varphi(x)) = f(x)$ for every $x \in X$, i.e. diagram (4.7) commutes. The mapping φ being continuous and defined on a metric continuum, it is closed (see [4], Theorem 9, p. 104). Since f is continuous, the continuity of g follows from Theorems 1 and 3 in [6], § 13, XV, p. 119. The uniqueness of g follows from the definition. From the definition of g we conclude also that

$$g^{-1}(y) = \varphi(f^{-1}(y)) \quad \text{for every } y \in D.$$

The mapping f being monotone, $f^{-1}(y)$ is a continuum, hence $\varphi(f^{-1}(y))$ is also a continuum. So $g^{-1}(y)$ is, and g is monotone. Therefore we have proved that Corollary 3 leads to Theorem 7. The opposite way is quite obvious.

COROLLARY 4. *If a dendroid D is the hyperspace of an upper semi-continuous decomposition of a λ-dendroid X into continua, then it is a monotone image of the dendroid $\Delta(X)$.*

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Irreducibly generated algebras

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The study of functions involves, in a most fundamental way, the study of the composition of functions. If Δ is a set of elements, then any mapping of Δ^p (i.e., of the pth product $\Delta \times \ldots \times \Delta$) into Δ is a *p-place function* over Δ. The composite $F_0(F_1, \ldots, F_p)$ of any $p+1$ p-place functions $F_0, F_1, \ldots, F_p$ is again a p-place function defined in the usual manner:

$$F_0(F_1, \ldots, F_p)(x_1, \ldots, x_p) = F_0(F_1(x_1, \ldots, x_p), \ldots, F_p(x_1, \ldots, x_p))$$

for $(x_1, \ldots, x_p)$ in Δ^p. From here it easily follows that composition satisfies the superassociative law (cf. [2]), namely that

$$(1) \qquad (F_0(F_1, \ldots, F_p))(G_1, \ldots, G_p) = F_0(F_1(G_1, \ldots, G_p), \ldots, F_p(G_1, \ldots, G_p))$$

for any p-place functions $F_0, F_1, \ldots, G_p$ over Δ. A set $\mathfrak{S}$ of functions is called an *algebra of functions* if $\mathfrak{S}$ is closed with respect to composition.

Equation (1) serves as a point of departure for the study of a more abstract algebraic structure. Let $\mathfrak{S}$ be a set of elements with a $(p+1)$-ary operation, i.e., an operation which associates with each $(p+1)$-tuple of elements $S_0, S_1, \ldots, S_p$ of $\mathfrak{S}$ an element of $\mathfrak{S}$ denoted by $S_0(S_1, \ldots, S_p)$. If the superassociative law is valid in $\mathfrak{S}$, then $\mathfrak{S}$ will be called a *p-place Menger algebra* and its operation will be called composition. Clearly any algebra of functions is a Menger algebra. That the converse is true was shown by Dicker (cf. [1]) — for any Menger algebra $\mathfrak{S}$ there exists a set Δ such that $\mathfrak{S}$ is isomorphic to an algebra of functions over Δ.

The structure of Menger algebras in general have been studied in [1]–[4]. This paper, however, deals with a particular type of Menger algebra. The Menger algebra $\mathfrak{S}$ is said to be *irreducibly generated* if each subset of $\mathfrak{S}$ is also an algebra, that is, is closed with respect to composition. Therefore, for elements $S_0, S_1, \ldots, S_p$ in $\mathfrak{S}$, the composite $S_0(S_1, \ldots, S_p)$ must be one of the elements $S_0, S_1, \ldots, S_p$ since the set $\{S_0, S_1, \ldots, S_p\}$ forms an algebra. An element S_0 of $\mathfrak{S}$ is *constant* if $S_0(S_1, \ldots, S_p) = S_0$ for each sequence $(S_1, \ldots, S_p)$ of elements from $\mathfrak{S}$; S_0 is called a *k-th place selector* relative to a subset $\mathfrak{J}$ of $\mathfrak{S}$, if $S_0(T_1, \ldots, T_p) = T_k$ for each sequence $(T_1, \ldots, T_p)$ from $\mathfrak{J}$.