A note on the theory of shape of compacta

by

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In this note we shall study some relations between the notion of the shape of a compactum and some other topological notions.

§ 1. Basic definitions. Let X, Y be two compacta lying in the Hilbert cube Q. A sequence of maps $f_k\colon Q \to Q$ is said to be a *fundamental sequence from X to Y* (notation: $\underline{f} = \{f_k, X, Y\}$, or $\underline{f}\colon X \to Y$) (compare [1], p. 225) if for every neighborhood V of Y there is a neighborhood U of X such that

$$f_k/U \simeq f_{k+1}/U \quad \text{in } V \text{ for almost all } k\,.$$

Two fundamental sequences $\underline{f} = \{f_k, X, Y\}$ and $\underline{f}' = \{f'_k, X, Y\}$ are said to be *homotopic* (notation: $\underline{f} \simeq \underline{f}'$) if for every neighborhood V of Y there is a neighborhood U of X such that

$$f_k/U \simeq f'_k/U \quad \text{in } V \text{ for almost all } k\,.$$

If f_k is the identity map $i\colon Q \to Q$ for every $k = 1, 2, \ldots$, then the fundamental sequence $\{f_k, X, X\}$ is said to be the *fundamental identity sequence for X*, and it is denoted by $\underline{i}_X$.

By the *composition* $\underline{g}\underline{f}$ of two fundamental sequences $\underline{f} = \{f_k, X, Y\}$ and $\underline{g} = \{g_k, Y, Z\}$ one understands the fundamental sequence $\{g_k f_k, X, Z\}$. If there exist two fundamental sequences $\underline{f}\colon X \to Y$ and $\underline{g}\colon Y \to X$ such that $\underline{g}\underline{f}\colon X \to X$ is homotopic to the fundamental identity sequence $\underline{i}_X$, then we say that Y *fundamentally dominates* X (notation: $X \underset{F}{\leqslant} Y$). If there exist two fundamental sequences $\underline{f}\colon X \to Y$, $\underline{g}\colon Y \to X$ such that the relations $\underline{g}\underline{f} \simeq \underline{i}_X$ and $\underline{f}\underline{g} \simeq \underline{i}_Y$ both hold, then X and Y are said to be *fundamentally equivalent* (notation: $X \underset{F}{\simeq} Y$).

Replacing in those definitions the compacta X, Y by the pointed compacta $(X, x_0) \subset (Q, x_0)$, $(Y, y_0) \subset (Q, y_0)$ and also the neighborhoods U, V by the pointed neighborhoods (U, x_0), (V, y_0) one gets the notion of the fundamental pointed sequence $\underline{f} = \{f_k, (X, x_0), (Y, y_0)\}$, of the *homotopy of pointed fundamental sequences* and of the *fundamental domination* and of the *fundamental equivalence for pointed compacta.*

By the *shape* of a compactum (see [2]) we understand the collection $\mathrm{Sh}(X)$ of all compacta Y such that there exist two compacta $X', Y' \subset Q$ homeomorphic to X and to Y respectively and satisfying the relation $X' \underset{F}{\simeq} Y'$. The shape of the compactum consisting of only one point is said to be *trivial*. Analogously one gets the notion of the *shape* $\mathrm{Sh}(X, x_0)$ of a pointed compactum (X, x_0).

A closed subset A of a compactum $X \subset Q$ is said to be a *fundamental retract of* X ([3], p. 59) if there exists a fundamental retraction of X to A, i.e. a fundamental sequence $\underline{r} = \{r_k, X, A\}$ satisfying the condition $r_k(x) = x$ for every point $x \in A$ and $k = 1, 2, \ldots$ In particular, every retract of X is a fundamental retract of X, but not conversely. It is also clear that X fundamentally dominates each of its retracts. The fundamental retracts of the AR-sets are said to be *fundamental absolute retracts*, or FAR-*sets* ([3], p. 66), and the fundamental retracts of the ANR-sets are said to be *fundamental absolute neighborhood retracts*, or FANR-*sets* ([3], p. 67).

Let (X, x_0) be a pointed compactum and (Y, y_0) a pointed compactum lying in Q. A sequence of maps $\xi_k: (X, x_0) \to (Q, y_0)$ is said to be an *approximative map of* (X, x_0) *towards* (Y, y_0) ([1], p. 246) if for every neighborhood V of Y the homotopy $\xi_k \simeq \xi_{k+1}$ in (V, y_0) holds for almost all k. We denote this approximative map by $\{\xi_k, (X, x_0) \to (Y, y_0)\}$, or shortly by $\underline{\xi}$. Another approximative map $\underline{\eta} = \{\eta_k, (X, x_0) \to (Y, y_0)\}$ is said to be *homotopic to* $\underline{\xi}$ if for every neighborhood V of Y the homotopy $\xi_k \simeq \eta_k$ in (V, y_0) holds for almost all k. In particular, $\underline{\xi}$ is said to be *null-homotopic* if it is homotopic to the approximative map $\underline{\eta} = \{\eta_k, (X, x_0), (Y, y_0)\}$ given by the formula $\eta_k(x) = y_0$ for every point $x \in X$ and for every $k = 1, 2, \ldots$ The classes of all homotopic approximative maps from (X, x_0) towards (Y, y_0) are said to be *approximative classes from* (X, x_0) *towards* (Y, y_0). In the case where X is the n-dimensional sphere S^n, the approximative maps from (S^n, x_0) towards (Y, y_0) constitute a group $\underline{\pi}_n(Y, y_0)$ called the *n-th fundamental group of* (Y, y_0) ([1], p. 251). If $Y \in \mathrm{ANR}$, then $\underline{\pi}_n(Y, y_0)$ is isomorphic with the nth homotopy group $\pi_n(Y, y_0)$. The group $\underline{\pi}_n(Y, y_0)$ is trivial if and only if every approximative map of (S^n, x_0) towards (Y, y_0) is null-homotopic.

§ 2. Some shape invariants. A pointed compactum $(Y, y_0) \subset (Q, y_0)$ is said to be *approximatively n-connected* (compare [6], 223) if for every neighborhood V of Y there is a neighborhood V_0 of Y such that every map of the pointed n-sphere (S^n, a) into (V_0, y_0) is null homotopic in (V, y_0). A compactum $Y \subset Q$ is said to be *approximatively n-connected* if (Y, y_0) is approximatively n-connected for every point $y_0 \in Y$.

It is clear that the nth fundamental group $\underline{\pi}_n(Y, y_0)$ of any approximatively n-connected pointed compactum (Y, y_0) is trivial. The converse is not true, because if y_0 is a point of the solenoid of Van Dantzig Y_0, then the group $\underline{\pi}_1(Y, y_0)$ is trivial, but (Y, y_0) is not approximatively 1-connected. Let us observe also that approximative 0-connectedness is the same as connectedness.

(2.1) THEOREM. *If* (Y, y_0) *is approximatively n-connected and* (Y, y_0) *fundamentally dominates* (X, x_0), *then* (X, x_0) *is approximatively n-connected.*

Proof. The relation $(Y, y_0) \underset{F}{\geqslant} (X, x_0)$ means that there exist two fundamental sequences $\underline{f} = \{f_k, (X, x_0), (Y, y_0)\}$, $\underline{g} = \{g_k, (Y, y_0), (X, x_0)\}$ such that $\underline{gf} \simeq \underline{i}_{(X, x_0)}$.

Let U be a neighborhood of X. Then there exists a neighborhood $U_1 \subset U$ of X and an index k_1 such that

$$g_k f_k/(U_1, x_0) \simeq i/(U_1, x_0) \quad \text{in} \quad (U, x_0) \text{ for every } k \geqslant k_1. \tag{2.2}$$

Now we can assign to U_1 a neighborhood V_1 of Y and an index $k_2 \geqslant k_1$ such that

$$g_k(V_1) \subset U_1 \quad \text{for every } k \geqslant k_2. \tag{2.3}$$

Since (Y, y_0) is approximatively n-connected, there exists a neighborhood $V_0 \subset V_1$ of Y such that every map $\beta: (S^n, a) \to (V_0, y_0)$ is null-homotopic in (V_1, y_0). We can assign to V_0 a neighborhood $U_0 \subset U_1$ of X and an index $k_0 \geqslant k_2$ such that

$$f_k(U_0) \subset V_0 \quad \text{for every } k \geqslant k_0. \tag{2.4}$$

Consider now a map $a: (S^n, a) \to (U_0, x_0)$. It follows by (2.4) that $f_k a: (S^n, a) \to (V_0, y_0)$. If we recall the definition of the neighborhood V_0, we infer that the map $f_k a$ is null-homotopic in (V_1, y_0). Applying (2.3), we infer that

$$g_k f_k a \simeq 0 \quad \text{in } (U_1, x_0). \tag{2.5}$$

If one recalls that $U_0 \subset U_1$ and that $k_0 \geqslant k_1$, one infers by (2.2) that $g_k f_k/(U_0, x_0) \simeq i/(U_0, x_0)$ in (U, x_0) for every $k \geqslant k_0$. It follows that

$$g_k f_k a \simeq a \quad \text{in } (U, x_0). \tag{2.6}$$

It suffices to recall the inclusion $U_1 \subset U$ in order to obtain from (2.5) and (2.6) the homotopy $g_k f_k a \simeq 0$ in (U, x_0) for every $k \geqslant k_0$. Thus the approximative n-connectedness of (X, x_0) is proved.

A pointed compactum $(Y, y_0) \subset (Q, y_0)$ is said to be *approximatively contractible* if for every neighborhood V of Y in the space Q the map $i/(Y, y_0): (Y, y_0) \to (Y, y_0)$ is null-homotopic in (V, y_0). A compactum $Y \subset Q$

is said to be *approximatively contractible* if (Y, y_0) is approximatively contractible for every point $y_0 \in Y$.

Let us observe that

(2.7) *Approximative contractibility implies approximative n-connectedness for every $n = 0, 1, \ldots$*

Proof. Let (Y, y_0) be an approximatively contractible pointed compactum. Consider an open neighborhood V of Y. Then there is a homotopy

$$\vartheta\colon\ Y \times \langle 0, 1\rangle \to V$$

satisfying the conditions

$$\vartheta(y, 0) = y\,, \quad \vartheta(y, 1) = y_0 \quad \text{for every point } y \in Y$$

and

$$\vartheta(y_0, t) = y_0 \quad \text{for every } 0 \leqslant t \leqslant 1\,.$$

Since V is an absolute neighborhood retract for metric spaces, there exists a neighborhood V_0 of Y such that ϑ can be extended to a homotopy $\vartheta'\colon V_0 \times \langle 0, 1\rangle \to V$ satisfying the conditions:

$$\vartheta'(y, 0) = y\,, \quad \vartheta'(y, 1) = y_0 \quad \text{for every point } y \in V_0\,.$$

Now, if a is a map of (S^n, a) into (V_0, y_0), then setting

$$\chi(x, t) = \vartheta\big(a(x), t\big) \quad \text{for every } (x, t) \in S^n \times \langle 0, 1\rangle\,,$$

we get a homotopy $\chi\colon (S^n \times \langle 0, 1\rangle) \to V$ satisfying the conditions:

$$\chi(x, 0) = a(x)\,, \quad \chi(x, 1) = y_0 \quad \text{for every point } x \in S^n,$$

$$\chi(a, t) = y_0 \quad \text{for every } 0 \leqslant t \leqslant 1\,.$$

Hence a is null-homotopic in (V, y_0). Thus the proof of (2.1) is finished.

Using an analogous argument to that used in the proof of Theorem (2.1), one gets the following

(2.8) THEOREM. *If (Y, y_0) is an approximatively contractible pointed compactum and if $(Y, y_0) \underset{F}{\geqslant} (X, x_0)$, then (X, x_0) is approximatively contractible.*

It follows by Theorems (2.1) and (2.8) that both properties, approximative n-connectedness and approximative contractibility, are monotonous shape-invariants.

§ 3. The case of ANR-sets. Let us prove the following

(3.1) THEOREM. *If y_0 is a point of a compactum $Y \in$ ANR, then (Y, y_0) is approximatively n-connected if and only if the group $\underline{\pi}_n(Y, y_0)$ is trivial. The approximative contractibility of (Y, y_0) is equivalent to the contractibility of Y in itself.*

Proof. We have already observed that the approximative n-connectedness of (Y, y_0) implies that $\underline{\pi}_n(Y, y_0)$ is trivial.

On the other hand, let us assume that $y_0 \in Y \in$ ANR and that the group $\underline{\pi}_n(Y, y_0)$ is trivial. Consider a neighborhood V of Y. There is a neighborhood V_0 of Y and a retraction $r\colon V_0 \to Y$ such that the segment $\overline{yr(y)}$ lies in V for every point $y \in V_0$. Now, if $f\colon (S^n, x_0) \to (V_0, y_0)$ is a map, then setting

$$\varphi(x, t) = (1-t)\cdot f(x) + t\cdot rf(x) \quad \text{for every } (x, t) \in S^n \times \langle 0, 1\rangle\,,$$

one gets a homotopy $\varphi\colon S^n \times \langle 0, 1\rangle \to V$ such that

$$\varphi(x, 0) = f(x)\,, \quad \varphi(x, 1) = rf(x) \quad \text{for every point } x \in S^n$$

and

$$\varphi(x_0, t) = y_0 \quad \text{for every } 0 \leqslant t \leqslant 1\,.$$

It follows that the map f is homotopic in (V, y_0) to the map $f'\colon (S^n, x_0) \to (V_0, y_0)$ given by the formula

$$f'(x) = rf(x) \quad \text{for every point } x \in X\,.$$

Since the values of f' belong to Y and since for $Y \in$ ANR the group $\underline{\pi}_n(Y, y_0)$ is isomorphic to the nth homotopy group $\pi_n(Y, y_0)$, we infer that f' is null-homotopic in (Y, y_0), whence also in (V, y_0). Thus (Y, y_0) is approximatively n-connected.

Passing to the second part of theorem, observe that the contractibility of Y in itself implies that $Y \in$ AR and consequently (Y, y_0) is contractible in itself to y_0, hence also approximatively contractible. On the other hand, if (Y, y_0) is approximatively contractible, then for every neighborhood V of Y there is a homotopy φ contracting Y to y_0 in V. We can select V so that there is a retraction $r\colon V \to Y$. Then the formula $\psi(y, t) = r\varphi(y, t)$ for every $(y, t) \in Y \times \langle 0, 1\rangle$ defines a homotopy ψ contracting Y in itself to y_0.

§ 4. Movable compacta. A compactum $Y \subset Q$ is said to be *movable* ([4], p. 137) if for every neighborhood V of Y there exists a neighborhood V_0 of Y such that for every neighborhood W of Y there is a homotopy $\varphi\colon V_0 \times \langle 0, 1\rangle \to V$ such that $\varphi(y, 0) = y$ and $\varphi(y, 1) \in W$ for every point $y \in V_0$. It is known ([4], p. 137 and 145) that all ANR-sets and all plane compacta are movable, and also that if Y is movable and $X \underset{F}{\leqslant} Y$, then X is movable. The solenoids of Van Dantzig are examples of not movable compacta.

Remark. It is known ([5], p. 226) that if Y is a movable compactum and if y_0, y_0' are two points belonging to one component of Y, then $\operatorname{Sh}(Y, y_0) = \operatorname{Sh}(Y, y_0')$. In particular, for movable continua Y the shape $\operatorname{Sh}(Y, y_0)$ does not depend on the choice of the point y_0. It follows by

Theorems (2.1) and (2.8) that in this case the specification of the point $y_0 \in Y$ is immaterial for the definitions of the approximative n-connectedness of a movable continuum Y and of its approximative contractibility. In this case we can denote the nth fundamental group $\underline{\pi}_n(Y, y_0)$ shortly by $\underline{\pi}_n(Y)$.

The question arises whether in Theorem (3.1) the hypothesis that $Y \in \mathrm{ANR}$ can be replaced by a less restrictive one, namely that Y is movable. Let us prove a theorem giving a partial answer to this question. First let us prove the following

(4.1) LEMMA. *Let $(Y, y_0) \subset (Q, y_0)$ be an approximatively 1-connected pointed continuum and let Y be movable. Then for every neighborhood V of Y there is a neighborhood V_0 of Y such that for every neighborhood W of Y there exists a homotopy $\psi\colon V_0 \times \langle 0, 1\rangle \to V$ satisfying the following conditions:*

$$\psi(y, 0) = y, \quad \psi(y, 1) \in W \quad \text{for every point } y \in V_0,$$
$$\psi(y_0, t) = y_0 \quad \text{for every } 0 \leqslant t \leqslant 1.$$

Proof. Since (Y, y_0) is approximatively 1-connected, there exists for a given open neighborhood V of Y a neighborhood $V_1 \subset V$ of Y such that every loop in V_1 with basic point y_0 is null-homotopic in (V, y_0). Since Y is movable, there exists a neighborhood $V_0 \subset V_1$ of Y such that for every open neighborhood $W \subset V_1$ of Y there is a homotopy

$$\varphi\colon V_0 \times \langle 0, 1\rangle \to V_1$$

such that $\varphi(y, 0) = y$ and $\varphi(y, 1) \in W$ for every point $y \in V_0$. Since Y is a continuum, we can assume that W is an open connected subset of Q. One can easily see that there exists a homotopy

$$\vartheta\colon W \times \langle 0, 1\rangle \to W$$

such that

$$\vartheta(y, 0) = y \quad \text{for every point } y \in W,$$
$$\vartheta(\varphi(y_0, 1), 1) = y_0.$$

Setting

$$\varphi'(y, t) = \varphi(y, 2t) \qquad \text{for} \quad y \in V_0 \text{ and } 0 \leqslant t \leqslant \tfrac{1}{2},$$
$$\varphi'(y, t) = \vartheta[\varphi(y, 1), 2-2t] \quad \text{for} \quad y \in V_0 \text{ and } \tfrac{1}{2} \leqslant t \leqslant 1,$$

we get a homotopy $\varphi'\colon V_0 \times \langle 0, 1\rangle \to V_1$ such that $\varphi'(y, 0) = y$, $\varphi'(y, 1) \in W$ for every point $y \in V_0$ and $\varphi'(y_0, 1) = y_0$.

Consider now the closed subset Z of $V_0 \times \langle 0, 1\rangle$ given by the formula

$$Z = (V_0 \times (0)) \cup ((y_0) \times \langle 0, 1\rangle) \cup (V_0 \times (1)).$$

Setting

$$\lambda(t) = \varphi'(y_0, t) \quad \text{for} \quad 0 \leqslant t \leqslant 1,$$

we get a map of $\langle 0, 1\rangle$ into V_1 satisfying the condition $\lambda(0) = \lambda(1) = y_0$. Hence λ is a loop in V_1 with y_0 as its basic point. We infer that there is a homotopy $\alpha\colon \langle 0, 1\rangle \times \langle 0, 1\rangle \to V$ satisfying the conditions:

$$\alpha(0, t) = \lambda(t), \quad \alpha(1, t) = y_0 \quad \text{for every } 0 \leqslant t \leqslant 1,$$
$$\alpha(s, 0) = \alpha(s, 1) = y_0 \quad \text{for every } 0 \leqslant s \leqslant 1.$$

It follows that the map $\varphi'/Z\colon Z \to V_1$ is homotopic in V to the map $\hat{\psi}\colon Z \to V_1$ given by the formulas:

$$\text{(4.2)} \quad \begin{aligned} &\hat{\psi}(y, 0) = \varphi'(y, 0), \quad \hat{\psi}(y, 1) = \varphi'(y, 1) \quad \text{for every point } y \in V_0, \\ &\hat{\psi}(y_0, t) = y_0 \quad \text{for every } 0 \leqslant t \leqslant 1. \end{aligned}$$

Since V, as an open subset of Q, is an absolute neighborhood retract (for metric spaces), we infer by the homotopy extension theorem that $\hat{\psi}$ can be extended to a map $\psi\colon V_0 \times \langle 0, 1\rangle \to V$. Formulas (4.2) and the inclusion $\varphi'(V_0, 1) \subset W$ imply that ψ satisfies all the required conditions.

(4.3) THEOREM. *If $Y \subset Q$ is a movable and approximatively 1-connected continuum, then for every $n > 1$ the group $\underline{\pi}_n(Y)$ is trivial if and only if Y is approximatively n-connected.*

Proof. Let y_0 be a point of Y. We have already observed that the approximative n-connectedness of (Y, y_0) implies that $\underline{\pi}_n(Y, y_0)$ is trivial. It remains to prove that if (Y, y_0) is not approximatively n-connected, then the group $\underline{\pi}_n(Y, y_0)$ is not trivial. In this case there is a neighborhood V of Y such that for every neighborhood W of Y there exists a map $\xi\colon (S^n, a) \to (Q, y_0)$ with values in W which is not null-homotopic in (V, y_0). Since Y is movable and (Y, y_0) is approximatively 1-connected, we infer by Lemma (4.1) that there exists a sequence $V = V_0 \supset V_1 \supset V_2 \supset \ldots$ of neighborhoods of Y such that every neighborhood of Y contains V_n for almost all n and, moreover, there exists for every $n = 1, 2, \ldots$ a homotopy $\psi_n\colon V_n \times \langle 0, 1\rangle \to V_{n-1}$ such that $\psi_n(y, 0) = y$, $\psi_n(y, 1) \in V_{n+1}$ for every point $y \in V_n$ and $\psi_n(y_0, t) = y_0$ for every number $0 \leqslant t \leqslant 1$. Let $\xi_1\colon (S^n, a) \to (Q, y_0)$ be a map with values in V_1 which is not null-homotopic in (V, y_0). We define the sequence of maps $\xi_1, \xi_2, \ldots$ by induction, setting

$$\xi_{n+1}(x) = \psi_n(\xi_n(x), 1) \quad \text{for every point } x \in S^n.$$

Then $\xi_n\colon (S^n, a) \to (Q, y_0)$ and $\xi_n \simeq \xi_{n+1}$ in V_{n-1} for every $n = 1, 2, \ldots$, but ξ_n is not null-homotopic in V. It follows that

$$\underline{\xi} = \{\xi_n, (S^n, a) \to (Y, y_0)\}$$

is an approximative map which is not null-homotopic, whence the group $\underline{\pi}_n(Y, y_0)$ is not trivial.

Thus the proof of Theorem (4.3) is finished.

§ 5. A lemma on extension of maps. Now let us prove the following

(5.1) LEMMA. *Let $(Y, y_0) \subset (Q, y_0)$ be a pointed compactum which is approximatively n-connected for every $n = 0, 1, 2, \ldots$ Then, for every $m = 0, 1, 2, \ldots$ and for every neighborhood V of Y there exists a neighborhood V_m of Y such that if T is a triangulation of a polyhedron P and R is a polyhedron which is the union of some simplices of T satisfying the condition $\dim(P \setminus R) \leqslant m$, then every map $f\colon R \to V$ with values in V_m can be extended to a map of P into V.*

Proof. Since (Y, y_0) is approximatively 0-connected, Y is a continuum. It follows that we can assume that V is open in Q and connected. By the homotopy-extension theorem, we can replace f by any map homotopic to f. Consequently we can assume that f maps all vertices of the triangulation T belonging to R into y_0.

If $m = 0$, then $P \setminus R$ is a finite set of vertices of T and we can extend f onto P assigning to each of those vertices the value y_0. Now let us assume that our proposition is true for an m. Let P' denote the union of R and of all simplices of T with dimensions $\leqslant m$. Since (Y, y_0) is approximatively $(m+1)$-connected, there exists a neighborhood $\hat{V}$ of Y such that every map $\xi\colon (S^n, a) \to (\hat{V}, y_0)$ is null-homotopic in (V, y_0). By the hypothesis of induction, there exists a neighborhood V'_m of Y such that for every polyhedron P with a triangulation T and for every polyhedron R which is the union of some simplices of T, every map $f\colon R \to V$ with values in V'_m can be extended to a map f' of the polyhedron P' which is the union of R and of all simplices of T with dimensions $\leqslant m$ into V. We can assume also that $f'(x) = y_0$ for every vertex x of T. Now let us assume that $\dim(P \setminus R) \leqslant m+1$. Then $P \setminus P'$ is the union of a finite number of $(m+1)$-dimensional simplices Δ with boundaries $\Delta^{\bullet}$ lying in P'. For every such simplex Δ the restriction $f'/\Delta^{\bullet}$ may be considered as a map of (S^m, a), where a denotes one of the vertices of T, into (Y, y_0). Since the values of f' belong to V, we infer that $f'/\Delta^{\bullet}$ can be extended to a map of Δ with values in V. If we apply this procedure to each $(m+1)$-dimensional simplex Δ of T which is not contained in R, we get the required extension of the map f with values in V.

Thus the proof of Lemma (5.1) is finished.

§ 6. FAR-sets. A relation between the notions of approximative contractibility and approximative n-connectedness on the one hand and the notion of the fundamental absolute retract on the other hand is given by the following

(6.1) THEOREM. *Every* FAR-*set* $Y \subset Q$ *is approximatively contractible (hence also approximatively n-connected). Every finite-dimensional, non-empty compactum $Y \subset Q$ which is approximatively n-connected for every $n = 0, 1, \ldots$ is an* FAR-*set.*

Proof. If $Y \in \mathrm{FAR}$, then Y is a fundamental retract of Q, whence $Y \underset{F}{\leqslant} Q$. Since Q is contractible in itself, we infer by (2.8) that Y is approximatively contractible, whence also approximatively n-connected for every $n = 0, 1, 2, \ldots$

In order to prove the second part of Theorem (6.1), we may assume that there is a natural m such that Y is a subset of the m-dimensional cube $Q^m = Q \cap E^m$. Consider a null-triangulation T of the set $Q^m \setminus Y$ (that is, a countable triangulation of the set $Q^m \setminus Y$ with the diameters of simplices converging to zero). Let $V_0 = Q^m \supset V_1 \supset V_2 \supset \ldots$ be a decreasing sequence of closed neighborhoods of Y in Q^m such that $Y = \bigcap_{k=1}^{\infty} V_k$ and that V_k is the union of Y and of almost all simplices of T for every $k = 1, 2, \ldots$; we say that V_k is a *polyhedrical neighborhood* of Y. By Lemma (5.1), there exists a neighborhood W_k of Y in Q^m such that every map defined on a subpolyhedron R of a polyhedron P with $\dim P \leqslant m+1$, with values in W_k, can be extended to a map of P with values in V_k.

Let g_1 denote the identity map $i\colon Q^m \to Q^m$ and let us assume that for an index k a map

$$g_k\colon Q^m \to Q^m$$

is already defined, satisfying the conditions:

$$g_k(y) = y \quad \text{for every point } y \in Y \quad \text{and} \quad g_k(Q^m) \subset W_k .$$

Then there exists a neighborhood W'_k of Y (in Q^m) such that for every polyhedra P, R as above, each map of R with values in W'_k can be extended to a map of P with values in W_k.

Consider now the inclusion map $j\colon Y \to Q^m$. Then j can be extended to a map g' of a neighborhood U of Y (a neighborhood in the space Q^m) with values in W'_k. Let W be a polyhedrical neighborhood of Y contained in U. Setting

$$R = [Q^m \times (0)] \cup [W \times (1)] \subset P = Q^m \times \langle 0, 1\rangle$$

and

$$\varphi(x, 0) = g_k(x) \quad \text{for every point } x \in Q^m,$$
$$\varphi(x, 1) = g'(x) \quad \text{for every point } x \in W,$$

we can extend the map $\varphi\colon R \to Q^m$ to a map $\hat{\varphi}\colon Q^m \times \langle 0, 1\rangle \to Q^m$ with values in W_k. It follows that the map g_k is homotopic in V_k to the map $g_{k+1}\colon Q^m \to Q^m$ given by the formula

$$g_{k+1}(x) = \hat{\varphi}(x, 1) \quad \text{for every point } x \in Q^m,$$

that is,

(6.2) $$g_k \simeq g_{k+1} \quad \text{in } V_k \text{ for every } k = 1, 2, \ldots$$

Moreover, the condition $g_{k+1}(y) = y$ is for every point $y \in Y$ satisfied. Now let us consider the map $p\colon Q \to Q^m$ given by the formula

$$p(x) = (x_1, x_2, \ldots, x_m, 0, \ldots) \quad \text{for every point } x = (x_1, x_2, \ldots) \in Q .$$

Setting

(6.3) $$f_k(x) = g_k p(x) \quad \text{for every point } x \in Q ,$$

we get a sequence of maps $f_k\colon Q \to Q$. In order to finish our proof, it suffices to show that $\{f_k, Q, Y\}$ is a fundamental retraction of Q to Y.

Let U be a neighborhood of Y in Q. Then there is an index k_0 such that

$$V_{k_0} \subset U .$$

Since $V_k \subset V_{k_0}$ for every $k \geqslant k_0$, we infer by (6.2) and (6.3) that

$$f_k \simeq f_{k+1} \quad \text{in } U \text{ for every } k \geqslant k_0 .$$

Hence $\{f_k, Q, Y\}$ is a fundamental sequence. Moreover, since $g_k(y) = y$ for every point $y \in Y$, we infer by (6.3) that also $f_k(y) = y$ for every point $y \in Y$. Hence $\underline{f} = \{f_k, Q, Y\}$ is an extension of the fundamental identity sequence $\{i, Y, Y\}$, that is, $\underline{f}$ is a fundamental retraction. Thus the proof of Theorem (6.1) is finished.

(6.4) COROLLARY. *In order that a finite-dimensional continuum $Y \subset Q$ be an FAR-set it is necessary and sufficient that Y be movable and approximatively 1-connected and that all groups $\underline{\pi}_n(Y, y_0)$, where $y_0 \in Y$, be trivial.*

This follows from Theorems (4.3) and (6.1).

(6.5) PROBLEM. *Does corollary (6.4) remain true if one cancels the hypothesis of finite dimension?*

§ 7. The shape of FAR-sets. Let us prove the following

(7.1) THEOREM. *A compactum Y is an FAR-set if and only if the shape $\mathrm{Sh}(Y)$ of Y is trivial.*

Proof. First let us show that

(7.2) *If $\mathrm{Sh}(X) \geqslant \mathrm{Sh}(Y)$ and $\mathrm{Sh}(X)$ is trivial, then $\mathrm{Sh}(Y)$ is trivial.*

Proof. We can assume that X contains only one point a. If $\mathrm{Sh}(X) \geqslant \mathrm{Sh}(Y)$, then there exist two fundamental sequences

$$\underline{f} = \{f_k, X, Y\} \quad \text{and} \quad \underline{g} = \{g_k, Y, X\}$$

such that $\underline{fg}$ is homotopic to the fundamental identity sequence $\underline{i}_Y$. Moreover, we can replace $\underline{g}$ by any fundamental sequence homotopic to it, in particular (since $X = (a)$), we may assume that $g_k(y) = a$ for every point $y \in Q$. In order to prove (7.2), it remains to show that $\underline{gf} \simeq \underline{i}_X$. But this is obvious, because $g_k f_k(x) = a$ for every point $x \in X$, whence the fundamental sequences $\underline{gf}$ and $\underline{i}_X$ are both generated by the same map $i\colon (a) \to (a)$, whence they are homotopic ([1], p. 226).

Since every set $Y \subset \mathrm{FAR}$ is a fundamental retract of Q and since $\mathrm{Sh}(Q)$ is trivial, we infer by (7.2) that $\mathrm{Sh}(Y)$ is trivial.

On the other hand, if the shape of Y is trivial, then for an arbitrarily selected point $a \in Y$, there exist two fundamental sequences

$$\underline{f} = \{f_k, (a), Y\} \quad \text{and} \quad \underline{g} = \{g_k, Y, (a)\}$$

such that $\underline{fg} \simeq \underline{i}_Y$ and $\underline{gf} \simeq \underline{i}_{(a)}$. We can assume that $g_k(x) = a$ for every point $y \in Y$. Then the homotopy $\underline{fg} \simeq \underline{i}_Y$ implies that $\underline{i}_Y$ is homotopic to the fundamental sequence $\underline{g}' = \{g'_k, Y, Y\}$ given by the formula

$$g'_k(x) = f_k(a) \quad \text{for every point } x \in Q .$$

It follows that $\{g'_k, Q, Y\}$ is an extension of $\underline{g}'$. By the homotopy extension theorem for fundamental sequences ([7], p. 87) we infer that $\underline{i}_Y$ can be extended to a fundamental sequence $\underline{r}\colon Q \to Y$. Since $\underline{r}$ is a fundamental retraction, we infer that Y is an FAR-set and the proof of Theorem (7.1) is finished.

(7.3) COROLLARY. *If $X \in \mathrm{FAR}$ and $\mathrm{Sh}(X) \geqslant \mathrm{Sh}(X)$ then $Y \in \mathrm{FAR}$.*

(7.4) PROBLEM. *Is it true that $X \in \mathrm{FANR}$ and $\mathrm{Sh}(X) \geqslant \mathrm{Sh}(Y)$ implies that $Y \in \mathrm{FANR}$?*

§ 8. Strong movability. A compactum $X \subset Q$ is said to be *strongly movable* if for every neighborhood U of X there exists a neighborhood U_0 of X such that for every neighborhood W of X there is a homotopy

$$\varphi\colon U_0 \times \langle 0, 1\rangle \to U$$

satisfying the following conditions:

(8.1) $$\varphi(x, 0) = x , \quad \varphi(x, 1) \in W \quad \text{for every point } x \in U_0 ,$$

(8.2) $$\varphi(x, 1) = x \quad \text{for every point } x \in X .$$

It is clear that every strongly movable compactum is movable.

EXAMPLES 1. Every ANR-set X is strongly movable. In fact, $X \in \mathrm{ANR}$ implies that there is a neighborhood $U_0 \subset U$ of X and a retraction $r\colon U_0 \to X$ such that all segments $\overline{xr(x)}$, where $x \in U_0$, lie in U. Now it suffices to set

$$\varphi(x, t) = t \cdot r(x) + (1-t) \cdot x \quad \text{for} \quad (x, t) \in U_0 \times \langle 0, 1\rangle ,$$

in order to obtain a homotopy φ satisfying (8.1) and (8.2).

2. If X is strongly movable, then all Betti numbers $p_n(X)$ of X are finite. In fact, if $p_n(X) = \infty$ and if U is a neighborhood of X, then for every neighborhood U_0 of X there is in X a true n-dimensional cycle (with rational coefficients) $\underline{\gamma}$ homologous to zero in U_0, but not homologous to zero in X. Then there is a neighborhood W of X such that

(8.3) $$\underline{\gamma} \not\sim 0 \quad \text{in } W.$$

One can easily see that if a homotopy $\varphi\colon U_0 \times \langle 0, 1\rangle \to U$ satisfies both conditions (8.1) and (8.2), then $\underline{\gamma} = \varphi(\underline{\gamma}, 1) \sim 0$ in the set $\varphi(U_0, 1)$ lying in W, contrary to (8.3).

Examples 1 and 2 are both special cases of the following

(8.4) THEOREM. *A compactum $X \subset Q$ is an* FANR*-set if and only if X is strongly movable.*

Proof. First let us assume that $X \in$ FANR. Then there exist a closed neighborhood V of X (in Q) and a fundamental retraction $\underline{r} = \{r_k, V, X\}$. Consider a neighborhood U of X. Then there exist a neighborhood U_0 of X and an index k_0 such that

$$r_k/U_0 \simeq r_{k_0}/U_0 \quad \text{in } U \text{ for every } k \geqslant k_0.$$

Since $r_{k_0}(x) = x$ for every point $x \in X$, the neighborhood U_0 may be selected so that $U_0 \subset V$ and that all segments $\overline{xr_{k_0}(x)}$, with $x \in U_0$, lie in U. It follows that r_{k_0}/U_0 is homotopic in U with the inclusion map $j\colon U_0 \to U$. Consequently

(8.5) $$r_k/U_0 \simeq j \quad \text{in } U \text{ for every } k \geqslant k_0.$$

Now let W be a neighborhood of X. Then $r_k(V) \subset W$ for almost all k, and since $U_0 \subset V$, we infer that there is an index $k_1 \geqslant k_0$ such that

(8.6) $$r_{k_1}(U_0) \subset W.$$

It follows by (8.5) and (8.6) that there exists a homotopy

$$\varphi\colon U_0 \times \langle 0, 1\rangle \to U$$

such that $\varphi(x, 0) = x$ and $\varphi(x, 1) = r_{k_1}(x)$ for every point $x \in X$. Since $r_{k_1}(U_0) \subset W$ and $r_{k_1}(x) = x$ for every point $x \in X$, we infer that conditions (8.1) and (8.2) are both satisfied, whence X is strongly movable.

Now let us assume that X is strongly movable. Then there exists a decreasing sequence of closed neighborhoods

$$V_0 = Q \supset V_1 \supset V_2 \supset \ldots$$

of X such that each neighborhood X contains V_n for almost all n and, moreover, there exists for every $n = 0, 1, 2, \ldots$ a homotopy

$$\varphi_n\colon V_{n+1} \times \langle 0, 1\rangle \to V_n$$

such that

(8.7) $$\varphi_n(x, 0) = x, \quad \varphi_n(x, 1) \in V_{n+2} \quad \text{for every point } x \in V_{n+1}$$

and

(8.8) $$\varphi_n(x, 1) = x \quad \text{for every point } x \in X.$$

Now let us define (by induction) a sequence of maps $f_k\colon V_1 \to Q$ as follows:

$$f_1(x) = \varphi_0(x, 1) \quad \text{for every point } x \in V_1.$$

Then

$$f_1(x) = x \quad \text{for every point } x \in X,$$
$$f_1(V_1) \subset V_2.$$

Let us assume that for an index $k \geqslant 1$ a map $f_k\colon V \to Q$ is already defined such that

(8.9) $$f_k(x) = x \quad \text{for every point } x \in X,$$

(8.10) $$f_k(V_1) \subset V_{k+1}.$$

Setting

$$f_{k+1}(x) = \varphi_k[f_k(x), 1] \quad \text{for every point } x \in V_1,$$

we get a map $f_{k+1}\colon V_1 \to Q$.

It follows by (8.8) and (8.9) that $f_{k+1}(x) = x$ for every point $x \in X$ and, by (8.7) and (8.10), that $f_{k+1}(V_1) \subset V_{k+2}$. Moreover, (8.7) implies that $f_k(x) = \varphi_k[f_k(x), 0]$ for every point $x \in V_1$. Since the values of φ_k belong to V_k, we infer that

(8.11) $$f_{k+1} \simeq f_k \quad \text{in } V_k \text{ for every } k = 1, 2, \ldots$$

If we recall that V_1 is a closed subset of Q, we can extend the map $f_k\colon V_1 \to Q$ to a map $r_k\colon Q \to Q$. Hence $r_k/V_1 = f_k$ and we infer by (8.11) and (8.9) that

$$r_k/V_1 \simeq r_{k+1}/V_1 \quad \text{in } V_k \text{ for every } k = 1, 2, \ldots$$

and

$$r_k(x) = x \quad \text{for every point } x \in X.$$

Since each neighborhood of X (in Q) contains V_k for almost all k, we infer that $\{r_k, V_1, X\}$ is a fundamental retraction of V_1 to X. Thus we have shown that X is a fundamental retract of its neighborhood V_1, whence X is an FANR-set, and the proof of Theorem (8.4) is finished.

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Regulated bases and completions of regular spaces

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0. Introduction. One can construct the completion of a metric space from any topological base of open balls knowing only the binary relation $S \Subset T$ on the base B defined to mean S is uniformly interior to T with the diameter of S at most half that of T. Motivated by such constructions involving a "regulator" $S \Subset T$ we introduce here the "abstract regulated base" and show that it has a representation as a base of regularly open subsets of a regular Hausdorff space with the regulator on the base somewhat like a semi-topogenous order [2]. $S \Subset T$ always implies $\bar{S} \subseteq T$, the weakest regulator.

The representation theorem yields a technique for "completing" a regular Hausdorff space relative to a base of regularly open subsets and a regulator on the base. Such completions include all Hausdorff compactifications and local compactifications as well as all metric completions, but not all completions of uniform spaces.

The concept of abstract regulated base comes under the program set forth by K. Menger [8]. The compingent algebra of H. de Vries [13] is a special kind of abstract regulated base. Our representation theorem subsumes that of de Vries and thereby that of M. H. Stone [11].

Our "end" is a generalization of the concept introduced to construct compactifications by H. Freudenthal [4] and P. S. Alexandroff [1]. (See [6] and Chapter 21 of [12].)

1. Abstract regulated bases. An *abstract regulated base* $(\boldsymbol{B}, \ll)$ consists of a non-empty set $\boldsymbol{B}$ and a binary relation $\ll$ on $\boldsymbol{B}$ subject under Definition 1 to the four axioms A_1-A_4 listed below.

Definition 1. Given a, b in $\boldsymbol{B}$ we say that a *meets* b if there exists c in $\boldsymbol{B}$ with both $c \ll a$ and $c \ll b$.

A_1. If $a \ll b$ and $b \ll c$ then $a \ll c$.

A_2. If $c \ll a$ is equivalent to $c \ll b$ for all c in $\boldsymbol{B}$ then $a = b$.

A_3. If $a \ll b$ and c meets every $z \ll b$ then $a \ll c$.

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