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A general realcompactification method

by

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Conventions. The *closure* of a set A in a space X will be denoted by $\mathrm{cl}_X A$. Collections of subsets of a space are indicated by German letters. If $\mathfrak{U}$ is a family of subsets of a space then the symbol $\mathrm{cl}_X\mathfrak{U}$ is used to denote the collection of all $\mathrm{cl}_X U$ for which $U \in \mathfrak{U}$. The union and intersection of a family of sets $\mathfrak{U}$ will be denoted by $\bigcup \mathfrak{U}$ and $\bigcap \mathfrak{U}$, respectively. For further basic conventions in general topology we refer to [6].

Introduction Let X be a T_1-space and let $\mathfrak{S}$ be a subbase for the closed sets of X. If $\mathfrak{S}$ has certain separation properties and is closed for certain set-theoretical operations (for instance, closed for the taking of finite intersections), then there is a standard way [2] to extend X to a compact Hausdorff space. Indeed, we consider all maximal centered systems of members of $\mathfrak{S}$ which have empty intersection in X, and let them serve as the new points for the extended space $\beta(\mathfrak{S})X$. $\beta(\mathfrak{S})X$ endowed with a suitable topology is a Hausdorff compactification of X. In particular, $\beta(\mathfrak{S})X$ is the Čech–Stone compactification of X in case X is completely regular and $\mathfrak{S}$ is the collection of all zero-sets of X [4].

In [5] Aarts and de Groot generalized this construction for the case where $\mathfrak{S}$ is not closed for finite intersections but only has certain separation properties (cf. also [1]). Let M be the collection of all maximal centered systems of members of $\mathfrak{S}$. By adding to each $\mu \in M$ the elements $S \in \mathfrak{S}$ that intersect each member of μ we obtain new collections $\bar{\mu}$. Those $\bar{\mu}$ which have empty intersection in X are in general not centered, but still do have the property that each two elements of it have a non-empty intersection; they are so-called *maximal linked systems* and serve as the new points for the extended space $\beta(\mathfrak{S})X$. By choosing a suitable topology for $\beta(\mathfrak{S})X$ we obtain a Hausdorff compactification of X.

In this paper our purpose is to adapt the above procedure for the realcompact case; thus, starting from a fixed closed subbase $\mathfrak{S}$, to obtain a general realcompactification $v(\mathfrak{S})X$ which depends on $\mathfrak{S}$ (see [4] for the definitions of realcompactness and realcompactification). Of course, we must see to it that $v(\mathfrak{S})X = vX$, the Hewitt realcompactification of X, in case where $\mathfrak{S}$ is the collection of all zerosets of a completely regular space X.

We proceed as follows: Instead of considering all maximal linked systems $\bar{\mu}$ for $\mu \in M$ we rather consider those $\bar{\mu}$ for which μ has the countable intersection property. Let $v(\mathfrak{S})X$ denote this restricted collection of linked systems; then $v(\mathfrak{S})X$ becomes a subspace of $\beta(\mathfrak{S})X$. We shall prove that $v(\mathfrak{S})X$ is a realcompact space and show that it has some properties analogous to the Hewitt realcompactification vX.

Indeed, we have $v(\mathfrak{S})X = vX$ if $\mathfrak{S}$ is the collection of all zero sets of X and X is completely regular. We also prove that $v(\mathfrak{S})X = X$ iff each maximal centered system of members of $\mathfrak{S}$ with the countable intersection property has a non-empty intersection.

This yields an intrinsic characterization of realcompactness which seems to be new.

Furthermore, $v(\mathfrak{S})X$ is maximal in some sense: Let $f\colon X \to Y$ be continuous and $\mathfrak{S}$ and $\mathfrak{T}$ closed subbases of X and Y, respectively, such that $f^{-1}(T) \in \mathfrak{S}$ for each $T \in \mathfrak{T}$. If $v(\mathfrak{S})X$ and $v(\mathfrak{T})Y$ are defined as above, then f has a continuous extension which carries $v(\mathfrak{S})X$ into $v(\mathfrak{T})Y$.

It should be pointed out that the results in this paper intersect with those of my thesis [7]. However, the techniques used to obtain the main theorems are different.

1. Separation conditions for a subbase; centered systems of subbase elements. In this section, we define the separation conditions which are introduced in [5]. Cf. also [1] and [7]. We also prove some auxiliary propositions.

Two subsets A and B of a topological space X are said to be *screened* by a finite family $\mathfrak{E}$ of subsets of X if $\mathfrak{E}$ covers X and each element of $\mathfrak{E}$ meets at most one of A and B.

A subbase $\mathfrak{S}$ for the closed sets of a space X satisfies the *condition of subbase-regularity* [1] provided that each $S \in \mathfrak{S}$ and $x \notin S$ are screened by a finite subcollection of $\mathfrak{S}$. $\mathfrak{S}$ satisfies the *condition of subbase-normality* if each two disjoint elements of $\mathfrak{S}$ are screened by a finite subcollection of $\mathfrak{S}$.

EXAMPLES. 1. The family of all closed sets of a normal space is a closed (sub)base which satisfies the conditions of subbase-regularity and subbase-normality.

2. In a completely regular space the (sub)base of all zerosets [2] satisfies the conditions of subbase-regularity and subbase-normality ([4], p. 17).

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[1] This condition is defined in [7] in a somewhat different way.

[2] A subset Z of X is called a *zeroset* of X if there exists a real-valued continuous function f on X such that $Z = \{x \in X|\ f(x) = 0\}$.

A subbase $\mathfrak{S}$ for the closed sets of a space X satisfies the *countability condition* iff each countable cover of X by members of $\{X \backslash S|\ S \in \mathfrak{S}\}$ has a countable refinement by members of $\mathfrak{S}$.

EXAMPLES. 1. In a countably paracompact normal space, the (sub)base of all closed sets satisfies the countability condition.

2. In a completely regular space the (sub)base of all zerosets satisfies the countability condition.

Recall that if $\mathfrak{S}$ is a family of subsets of a topological space X, then a centered system $\mathfrak{F}$ of members of $\mathfrak{S}$ is *prime* iff each finite cover of X by members of $\mathfrak{S}$ contains a member of $\mathfrak{F}$. As a matter of fact, each maximal centered system is prime.

The following two propositions will be needed in the sequel.

PROPOSITION 1. *If $\mathfrak{S}$ is a subbase for the closed sets of a space X which satisfies the condition of subbase-regularity, then the intersection of every prime centered system $\mathfrak{F}$ of members of $\mathfrak{S}$ consists of at most one point.*

Proof. If $p \in \bigcap \mathfrak{F}$ and if q is a point of X which is different from p, then there exists $S \in \mathfrak{S}$ such that $p \in S$, $q \notin S$ and a finite cover $\{S_1, \ldots, S_n\}$ of X by members of $\mathfrak{S}$ which screens S and q. Since $\mathfrak{F}$ is prime, there exists a natural number i $(1 \leqslant i \leqslant n)$ such that $S_i \in \mathfrak{F}$. Obviously, $p \in S_i$ and $q \notin S_i$. Thus S_i is a member of $\mathfrak{F}$ which does not contain q, i.e., $q \notin \bigcap \mathfrak{F}$.

PROPOSITION 2. *Let $\mathfrak{S}$ be a closed subbase for a space X which satisfies the conditions of subbase-regularity, subbase-normality and the countability condition. Then the following statements are equivalent.*

(i) *Every maximal centered system of members of $\mathfrak{S}$ with c.i.p. (countable intersection property) has a non-empty intersection.*

(ii) *Every prime centered system of members of $\mathfrak{S}$ with c.i.p. has a non-empty intersection.*

Proof. (ii) $\Rightarrow$ (i). Trivial!

(i) $\Rightarrow$ (ii). Let $\mathfrak{F}$ be a prime centered system of members of $\mathfrak{S}$ with the countable intersection property. $\mathfrak{F}$ is contained in a maximal centered system $\mathfrak{G}$ of members of $\mathfrak{S}$; hence, it suffices to show that $\mathfrak{G}$ has the countable intersection property. Suppose, on the contrary, that there exists a countable subcollection $\{G_i|\ i = 1, 2, \ldots\}$ of $\mathfrak{G}$ with empty intersection. Since $\mathfrak{S}$ satisfies the countability condition, the countable cover $\{X \backslash G_i|\ i = 1, 2, \ldots\}$ has a countable refinement $\{S_n|\ n = 1, 2, \ldots\}$ consisting of members of $\mathfrak{S}$. For each $n = 1, 2, \ldots$, select an index i_n such that $S_n \subset X \backslash G_{i_n}$ and a finite cover $\mathfrak{E}_n$ of X by members of $\mathfrak{S}$ which screens S_n and G_{i_n}. Since $\mathfrak{F}$ is prime, for $n = 1, 2, \ldots$ there exists $E_n \in \mathfrak{E}_n$ such that $E_n \in \mathfrak{F}$. Obviously, $E_n \cap G_{i_n} \neq \emptyset$ since $\mathfrak{G}$ is a centered system, and so $E_n \cap S_n = \emptyset$. It follows that $\bigcap \{E_n|\ n = 1, 2, \ldots\} = \emptyset$. This contradicts the fact that $\mathfrak{F}$ has the countable intersection property.

Recall that a completely regular space is *realcompact* iff each maximal centered family of zerosets with the countable intersection property has non-empty intersection. Thus, taking for $\mathfrak{S}$ the subbase of all zerosets, in the previous proposition, we obtain an equivalent condition of realcompactness in terms of prime centered systems. This fact is well known and is used as auxiliary condition to prove that the property of realcompactness is inherited by topological products and closed subspaces. See [4] for further information.

2. The construction of the realcompactification $v(\mathfrak{S})X$. In this section we give an outline of the construction of the realcompactification $v(\mathfrak{S})X$.

Throughout this section, let X be a T_1-space and $\mathfrak{S}$ a closed subbase for X which satisfies the conditions of subbase-regularity, subbase-normality, and the countability condition. For sake of convenience we sometimes use Greek letters to denote centered systems of members of $\mathfrak{S}$.

DEFINITION. A subcollection $\mathfrak{F}$ of $\mathfrak{S}$ is called a *linked system* iff each two members of $\mathfrak{F}$ have a non-empty intersection.

PROPOSITION 3. a. *Each maximal centered system μ of members of $\mathfrak{S}$ is contained in a maximal linked system $\bar{\mu}$ by defining $\bar{\mu} = \{S \in \mathfrak{S} | \ S \cap T \neq \emptyset$ for all $T \in \mu\}$.*

b. *If $\bigcap \mu \neq \emptyset$, then μ consists of all $S \in \mathfrak{S}$ containing a fixed point of X and $\bar{\mu} = \mu$.*

Proof. a. Let us suppose, on the contrary, that there exist $S, T \in \bar{\mu}$ such that $S \cap T = \emptyset$. Because of the condition of subbase-normality, S and T are screened by a finite subcollection $\{S_1, \ldots, S_n\}$ of $\mathfrak{S}$. There exists i $(1 \leqslant i \leqslant n)$ such that $S_i \in \mu$, and so $S_i \cap S \neq \emptyset$, $S_i \cap T \neq \emptyset$ by the definition of $\bar{\mu}$. This contradicts the fact that $\{S_1, \ldots, S_n\}$ screens the pair (S, T).

b. Let $x \in \bigcap \mu$ and suppose that there exists $S \in \mathfrak{S}$ such that $S \in \bar{\mu}$ and $x \notin S$. Because of the regularity condition for $\mathfrak{S}$ there exists a finite subcollection $\{S_1, \ldots, S_m\}$ of $\mathfrak{S}$ which screens the pair (x, S). Obviously, there exists i $(1 \leqslant i \leqslant m)$ such that $S_i \in \mu$, and so $S \cap S_i \neq \emptyset$. Thus $x \notin S_i$ which contradicts $x \in \bigcap \mu$.

Now, let M be the family of all maximal centered systems of members of $\mathfrak{S}$. Let $\beta(\mathfrak{S})X = \{\bar{\mu} | \ \mu \in M\}$ and for $S \in \mathfrak{S}$ $S^{**} = \{\bar{\mu} | \ \mu \in M, S \in \bar{\mu}\}$. Then the collection $\{S^{**} | \ S \in \mathfrak{S}\}$ is a subbase for a topology on $\beta(\mathfrak{S})X$ and $\beta(\mathfrak{S})X$ is a Hausdorff compactification of X. By identifying each $x \in X$ with the linked system $\{S \in \mathfrak{S} | \ x \in S\}$, X becomes a dense subspace of $\beta(\mathfrak{S})X$. For detailed proofs see [5] (3).

(3) N. B. For the construction of $\beta(\mathfrak{S})X$, it is not necessary that $\mathfrak{S}$ satisfies the countability condition.

Next, we consider the subcollection M' of M consisting of those μ with the countable intersection property (c.i.p.).

We define $v(\mathfrak{S})X = \{\bar{\mu} | \ \mu \in M'\}$. Then $v(\mathfrak{S})X$ is a subspace of $\beta(\mathfrak{S})X$ and the family of all $S^* = \{\bar{\mu} | \ \mu \in M', S \in \bar{\mu}\}$ for $S \in \mathfrak{S}$ is a closed subbase. Using the countability condition for $\mathfrak{S}$ one easily verifies that $\bar{\mu} = \mu$ for each $\mu \in M'$; thus the elements of $v(\mathfrak{S})X \setminus X$ are maximal centered systems of members of $\mathfrak{S}$ with c.i.p. that have an empty intersection in X (the corresponding property of $\beta(\mathfrak{S})X$ fails).

The following proposition says that $v(\mathfrak{S})X$ is the intersection of σ-compact subspaces of $\beta(\mathfrak{S})X$. Hence, $v(\mathfrak{S})X$ is a realcompactification of X (see [4], p. 119).

PROPOSITION 4. *Denote by γ the collection of all countable covers of X by members of $\mathfrak{S}$. For $\mathfrak{U} \in \gamma$, let $\mathfrak{U}^{**} = \{S^{**} | \ S \in \mathfrak{U}\}$ and $Y = \bigcap \{\bigcup \mathfrak{U}^{**} | \ \mathfrak{U} \in \gamma\}$. Then $Y = v(\mathfrak{S})X$.*

Proof. Let $\mu = \bar{\mu} \in v(\mathfrak{S})X$. For each $\mathfrak{U} \in \gamma$ there exists $S \in \mathfrak{U}$ such that $S \in \mu$ which implies $\bar{\mu} \in S^{**}$. Thus, $\bar{\mu} \in Y$. On the other hand, if $\bar{\mu} \in Y$, then in order to prove $\bar{\mu} \in v(\mathfrak{S})X$, it is sufficient to show that μ has the countable intersection property. Let us suppose, on the contrary, that there exists $S_i \in \mu$, $i = 1, 2, \ldots$, such that $\bigcap \{S_i | \ i = 1, 2, \ldots\} = \emptyset$. Obviously, $\{X \setminus S_i | \ i = 1, 2, \ldots\}$ is a countable cover of X which, by virtue of the countability condition for $\mathfrak{S}$, has a countable refinement $\{T_j | \ j = 1, 2, \ldots\}$ by members of $\mathfrak{S}$. Since $\bar{\mu} \in Y$, there exists an index m such that $\bar{\mu} \in T_m^{**}$. Thus $T_m \in \bar{\mu}$. There also exists an index n such that $T_m \cap S_n = \emptyset$. Because $S_n \in \mu$ this gives a contradiction.

PROPOSITION 5. a. *$v(\mathfrak{S})X = vX$ if $\mathfrak{S}$ is the collection of all zerosets of X,*

b. *The equality $v(\mathfrak{S})X = X$ holds if and only if the following condition is satisfied:*

Each maximal centered system of members of $\mathfrak{S}$ with the countable intersection property has a non-empty intersection.

Proof. a. See [4].

b. For every maximal centered system μ of members of $\mathfrak{S}$ with the countable intersection property, we have the equivalence (Proposition 3)

$$\bigcap \mu \neq \emptyset \quad \Longleftrightarrow \quad \text{there exists } x \in X \text{ such that } \bar{\mu} = \{S \in \mathfrak{S} | \ x \in S\}.$$

Because we have identified these $\bar{\mu}$ with the points of X the proposition follows.

The foregoing results yield the following intrinsic characterization of realcompactness.

THEOREM 1. *A T_1-space X is a realcompact completely regular space if and only if there exists a closed subbase $\mathfrak{S}$ for its topology that satisfies the conditions of subbase-regularity, subbase-normality and the countability* *condition, and, moreover, satisfies the condition that each maximal centered system of members of $\mathfrak{S}$ with the countable intersection property has a non-empty intersection.*

The following two propositions give us somewhat more information about the structure of $v(\mathfrak{S})X$. If $S \in \mathfrak{S}$ then S^* is defined as above. For subcollections $\mathfrak{S}_1$ of $\mathfrak{S}$ the notation $\mathfrak{S}_1^*$ is used to denote $\{S^*|\ S \in \mathfrak{S}_1\}$.

PROPOSITION 6. a. *$\mathfrak{S}^*$ is a subbase for the closed sets of $v(\mathfrak{S})X$ and satisfies the conditions of subbase-regularity, subbase-normality and the countability condition.*

b. *If $\{S_i|\ i = 1, 2, \ldots\}$ is a countable subcollection of $\mathfrak{S}$ with empty intersection in X, then the collection $\{S_i^*|\ i = 1, 2, \ldots\}$ has an empty intersection in $v(\mathfrak{S})X$.*

c. *Each maximal centered system of members of $\mathfrak{S}^*$ with c.i.p. has an non-empty intersection in $v(\mathfrak{S})X$.*

Proof. a. If $S^* \cap T^* = \emptyset$ for $S, T \in \mathfrak{S}$, then $S \cap T = \emptyset$ and so (S, T) is screened by a finite subcollection $\{S_1, \ldots, S_n\}$ of $\mathfrak{S}$. It follows that $\{S_1^*, \ldots, S_n^*\}$ screens (S^*, T^*) (remark that two disjoint elements of $\mathfrak{S}$ have disjoint stars in $v(\mathfrak{S})X$). Thus we have proved the normality condition for $\mathfrak{S}^*$. The regularity condition for $\mathfrak{S}^*$ is proved similarly. To prove the countability condition for $\mathfrak{S}^*$, let $\{v(\mathfrak{S})X\backslash S^*|\ S \in \mathfrak{S}_1 \subset \mathfrak{S}\}$ be a countable cover of $v(\mathfrak{S})X$. Obviously, $\{X\backslash S|\ S \in \mathfrak{S}_1\}$ is a countable cover of X which has a countable refinement $\mathfrak{T}$ by members of $\mathfrak{S}$. By Proposition 4 it follows that $\mathfrak{T}^*$ covers $v(\mathfrak{S})X$. Since for each $T \in \mathfrak{T}$ there exists $S \in \mathfrak{S}_1$ such that $T \subset X\backslash S$ and also $T^* \subset v(\mathfrak{S})X\backslash S^*$, it follows that $\mathfrak{T}^*$ refines $\{v(\mathfrak{S})X\backslash S^*|\ S \in \mathfrak{S}_1\}$.

b. If $\mu = \bar{\mu} \in \bigcap \{S_i^*|\ i = 1, 2, \ldots\}$, then $S_i \in \mu$ for each $i = 1, 2, \ldots$ which contradicts the countable intersection property of μ.

c. Let $\mathfrak{S}_1$ be a subcollection of $\mathfrak{S}$ such that $\mathfrak{S}_1^*$ is a maximal centered system of members of $\mathfrak{S}^*$ with the countable intersection property. Using b one easily verifies that $\mathfrak{S}_1$ is a maximal centered system of members of $\mathfrak{S}$ with c.i.p. The collection $\mathfrak{S}_1$ is also a maximal linked system and is identified as a point of $v(\mathfrak{S})X$ which is in the intersection of $\mathfrak{S}_1^*$.

PROPOSITION 7. *For each $S \in \mathfrak{S}$ we have $S^* = \mathrm{cl}_{v(\mathfrak{S})X} S$* [4].

Proof. Obviously, $\mathrm{cl}_{v(\mathfrak{S})X} S \subset S^*$. To prove $S^* \subset \mathrm{cl}_{v(\mathfrak{S})X} S$, let $\{S_1, \ldots, S_n\}$ be a finite subcollection of $\mathfrak{S}$ such that $S \subset S_1^* \cup \ldots \cup S_n^*$. Then $S \subset S_1 \cup \ldots \cup S_n$ and we also have $S^* \subset S_1^* \cup \ldots \cup S_n^*$. Indeed, if there would exist $\bar{\mu} \in S^*$ which is not in $S_1^* \cup \ldots \cup S_n^*$, then for each $i = 1, 2, \ldots, n$ there exists $T_i \in \bar{\mu}$ such that $T_i \cap S_i = \emptyset$. Thus $S \cap (T_1 \cap \ldots \cap T_n) = \emptyset$, and consequently $S^* \cap (T_1^* \cap \ldots \cap T_n^*) = \emptyset$ by the previous proposition. This is impossible. Hence, S^* is contained in the unions of all finite covers of S by elements of $\mathfrak{S}^*$. It follows that $S^* \subset \mathrm{cl}_{v(\mathfrak{S})X} S$ because $\mathfrak{S}^*$ is a subbase for the closed sets of $v(\mathfrak{S})X$.

[4] As was pointed out by J. de Groot, in general it is not true that $S^{**} = \mathrm{cl}_{\beta(\mathfrak{S})X} S$.

Remark. If $\mathfrak{S}$ is closed for countable intersections, then a slightly stronger version of b in Proposition 6 is satisfied (this is the case when $\mathfrak{S}$ is the collection of all zerosets of a completely regular space X). One easily verifies that for each countable subcollection $\{S_i|\ i = 1, 2, \ldots\}$ of $\mathfrak{S}$ we have $\bigcap \{S_i^*|\ i = 1, 2, \ldots\} = (\bigcap \{S_i|\ i = 1, 2, \ldots\})^*$.

3. Maximality and uniqueness of $v(\mathfrak{S})X$. In this section we generalize the well-known result [4] which states that a continuous map from a completely regular space X into a completely regular space Y has a continuous extension over the Hewitt realcompactifications of X and Y.

We start with a proposition that gives a general method to form extensions of mappings. Assume X and Y be T_1-spaces and $\mathfrak{T}$ a closed subbase for Y satisfying the conditions of subbase-regularity, subbase-normality and the countability condition.

PROPOSITION 8. *Let f be a mapping from a dense subspace Z of X into Y such that $\bigcap \{\mathrm{cl}_X f^{-1}(T_i)|\ i = 1, 2, \ldots\} = \emptyset$ for each countable subcollection $\{T_i|\ i = 1, 2, \ldots\}$ of $\mathfrak{T}$ with empty intersection in Y. Under the hypothesis that every maximal centered system of members of $\mathfrak{T}$ with c.i.p. has a non-empty intersection in Y (i.e. $v(\mathfrak{T})Y = Y$), f has a continuous extension over X.*

Proof. Let p be an arbitrary point of X. Denote by $\mathfrak{T}_1$ the subcollection of $\mathfrak{T}$ consisting of those sets T for which $p \in \mathrm{cl}_X f^{-1}(T)$. The extra condition on f implies that $\mathfrak{T}_1$ has the countable intersection property. Furthermore, the centered system $\mathfrak{T}_1$ is also prime. Indeed, if $\{T_k|\ k = 1, 2, \ldots, n\}$ is a finite subcollection of $\mathfrak{T}$ which is a cover of Y, then the collection $\{\mathrm{cl}_X f^{-1}(T_k)|\ k = 1, 2, \ldots, n\}$ is a cover of X. Hence, there exists j $(1 \leqslant j \leqslant n)$ such that $p \in \mathrm{cl}_X f^{-1}(T_j)$, and we have $T_j \in \mathfrak{T}_1$. By virtue of Propositions 1 and 2 of Section 1 we can define $f^*(p) = \bigcap \mathfrak{T}_1$. The mapping $f^*\colon X \to Y$ is an extension of f, for if $p \in Z$, then we have

$$f(p) \in \bigcap \{T \in \mathfrak{T}|\ p \in f^{-1}(T)\} = \bigcap \{T \in \mathfrak{T}|\ p \in \mathrm{cl}_X f^{-1}(T)\} = f^*(p)\,.$$

Therefore, it remains to show that f^* is continuous. Let x be an arbitrary point of X and let T be some member of $\mathfrak{T}$ such that $f^*(x) \in Y\backslash T$. In order to prove the continuity of f^*, it suffices to show that there exists a neighborhood of x which is mapped into $Y\backslash T$ by f^*. Since $f^*(x) \notin T$, there exists a screening of the pair $(f^*(x), T)$ by a finite subcollection $\{T_1, T_2, \ldots, T_k\}$ of $\mathfrak{T}$. Let $T_1, \ldots, T_m$ be the elements of this collection that intersect T.

Define

$$U = X\backslash \bigcup \{\mathrm{cl}_X f^{-1}(T_j)|\ j = 1, 2, \ldots, m\}\,.$$

Then U is a neighborhood of x in X which is mapped into $Y\backslash T$ by f^*. This completes the proof.

Proposition 8 together with Propositions 6 and 7 of Section 2 yield the following theorems.

THEOREM 2. *Let $\mathfrak{S}$ and $\mathfrak{T}$ be closed subbases for the T_1-spaces X and Y; and suppose that $\mathfrak{S}$ and $\mathfrak{T}$ satisfy the conditions of subbase-regularity, subbase-normality and the countability condition. If f is a (continuous) map from X into Y such that $f^{-1}(T) \in \mathfrak{S}$ for each $T \in \mathfrak{T}$, then there is a continuous extension f^* of f which carries $v(\mathfrak{S})X$ into $v(\mathfrak{T})Y$.*

THEOREM 3 (UNIQUENESS THEOREM). *The extension $v(\mathfrak{S})X$ of a space X constructed in Section 2 is essentially unique in the sense that if $\mu(\mathfrak{S})X$ is any extension of X satisfying the conditions* a, b *and* c *of Proposition 6 of Section 2 (with the star operator replaced by the closure operator in $\mu(\mathfrak{S})X$), then there is a homeomorphism of $v(\mathfrak{S})X$ onto $\mu(\mathfrak{S})X$ which leaves X pointwise fixed.*

EXAMPLE. If X is a Lindelöf space, then for each closed subbase $\mathfrak{S}$ which satisfies the conditions of subbase-regularity, subbase-normality and the countability condition, we have $v(\mathfrak{S})X = X$. This statement does not generally hold for arbitrary realcompact spaces. Indeed, if X is a discrete space of cardinal $> \aleph_0$, then let $\mathfrak{S}$ be the collection of all singleton points and complements of singleton points in X. It is easy to see that $\mathfrak{S}$ satisfies all required conditions, and $v(\mathfrak{S})X$ is homeomorphic with the one point compactification of X.

THEOREM 4. *Let $\{X_a|\ a \in A\}$ be a collection of topological spaces and $X = \Pi\{X_a|\ a \in A\}$. Suppose that for $a \in A$, $\mathfrak{S}_a$ is a closed subbase for X_a which satisfies the conditions of subbase-regularity, subbase-normality and the countability condition. Then the collection $\mathfrak{S}$ consisting of the sets $\pi_a^{-1}(C)$, where π_a is the natural projection onto the a'th coordinate space and C a member of $\mathfrak{S}_a$, is a closed subbase for X which also satisfies these conditions and $v(\mathfrak{S})X$ is homeomorphic with $\Pi\{v(\mathfrak{S}_a)X_a|\ a \in A\}$.*

Proof. One easily verifies that $\mathfrak{S}$ is a closed subbase for X which satisfies the conditions of subbase-regularity, subbase-normality and the countability condition. By Theorem 2, for each $a \in A$, there exists a continuous extension π_a^* of π_a which carries $v(\mathfrak{S})X$ into $v(\mathfrak{S}_a)X_a$. Define $i^*\colon v(\mathfrak{S})X \to \Pi\{v(\mathfrak{S}_a)X_a|\ a \in A\}$ by the conditions $\big(i^*(x)\big)_a = \pi_a^*(x)$ $(a \in A)$. Proposition 8 gives a method to extend the inclusion map j of X into $v(\mathfrak{S})X$ to a continuous mapping $j^*\colon \Pi\{v(\mathfrak{S}_a)X_a|\ a \in A\} \to v(\mathfrak{S})X$. The composition map $j^* \circ i^*$ has the property that it leaves the dense set X pointwise fixed. Consequently, $j^* \circ i^*$ is the identity map of $v(\mathfrak{S})X$. By applying the same argument to $i^* \circ j^*$ the theorem now follows.

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