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Coanalytic sets that are not Blackwell spaces

by

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A σ-algebra $\boldsymbol{B}$ of subsets of a set X will be said to be *separable* if (i) $\boldsymbol{B}$ is countably generated and (ii) $\{x\} \in \boldsymbol{B}$ for all $x \in X$. Call a measurable space $(X, \boldsymbol{B})$ separable if $\boldsymbol{B}$ is a separable σ-algebra. We shall say that a measurable space $(X, \boldsymbol{B})$ is a *Blackwell space* if (i) $(X, \boldsymbol{B})$ is separable, and (ii) for every separable σ-algebra $\boldsymbol{C} \subset \boldsymbol{B}$, $\boldsymbol{C} = \boldsymbol{B}$. Say that a separable metric space X is a *Blackwell space* if $(X, \boldsymbol{B}_X)$ is a Blackwell space (here, and in the sequel, whenever Z is a metric space, $\boldsymbol{B}_Z$ will denote the σ-algebra of Borel sets of Z). In [1], Blackwell proved that every analytic subset of a Polish space is a Blackwell space. The question then arises if every separable metric space which is a Blackwell space is analytic in its completion (and, consequently, analytic in any Polish space in which it can be imbedded). I do not know the answer to this question. The aim of this note is to exhibit complementary analytic (referred to, hereafter, as coanalytic) subsets of Polish spaces which are not Blackwell spaces. Again, I do not know if every coanalytic non-analytic subset of a Polish space fails to be a Blackwell space.

We first characterise Blackwell spaces. We shall say that g is a *measurable mapping* from a measurable space $(X, \boldsymbol{B})$ to a measurable space $(Y, \boldsymbol{C})$ if g is a function from X into Y and $g^{-1}(\boldsymbol{C}) \subset \boldsymbol{B}$. Call a mapping g an *isomorphism* between $(X, \boldsymbol{B})$ and $(Y, \boldsymbol{C})$ if g is one-to-one on X onto Y and both g and g^{-1} are measurable. In case the range of a mapping is a metric space, the relevant σ-algebra will always be the Borel σ-algebra and it will not be mentioned explicitly.

PROPOSITION. *Let $(X, \boldsymbol{B})$ be a separable measurable space. Then the following conditions are equivalent:*

(a) *$(X, \boldsymbol{B})$ is a Blackwell space.*

(b) *If $(Y, \boldsymbol{C})$ is any separable measurable space and f is a one-to-one measurable mapping from $(X, \boldsymbol{B})$ onto $(Y, \boldsymbol{C})$, then f is an isomorphism between $(X, \boldsymbol{B})$ and $(Y, \boldsymbol{C})$.*

(c) *For every one-to-one $\boldsymbol{B}$-measurable mapping f from $(X, \boldsymbol{B})$ to a Polish space, f is an isomorphism between $(X, \boldsymbol{B})$ and $\big(f(X), \boldsymbol{B}_{f(X)}\big)$.*

(d) *Every countable collection* $\{A_n, n \geq 1\}$ *of sets belonging to* $\boldsymbol{B}$ *which separates points of* X *generates* $\boldsymbol{B}$.

Proof. (a) $\Rightarrow$ (b). Let $(X, \boldsymbol{B})$ be a Blackwell space, $(Y, \boldsymbol{C})$ a separable measurable space and f a one-to-one measurable mapping from $(X, \boldsymbol{B})$ onto $(Y, \boldsymbol{C})$. Let $\boldsymbol{D} = f^{-1}(\boldsymbol{C})$. Then $\boldsymbol{D}$ is a separable σ-algebra and $\boldsymbol{D} \subset \boldsymbol{B}$, so that $\boldsymbol{D} = \boldsymbol{B}$. Consequently, for any $E \in \boldsymbol{B}$, there exists an $F \in \boldsymbol{C}$ such that $f^{-1}(F) = E$. Since f is onto Y, it follows that $F = f(E)$. Hence, f^{-1} is measurable, so that f is an isomorphism between $(X, \boldsymbol{B})$ and $(Y, \boldsymbol{C})$.

(b) $\Rightarrow$ (c). Clear.

(c) $\Rightarrow$ (d). Let $\{A_n, n \geq 1\}$ separate points of X and suppose that $A_n \in \boldsymbol{B}$, $n \geq 1$. Denote by $\boldsymbol{D}$ the σ-algebra generated by $\{A_n, n \geq 1\}$. Note that D is separable. Following Szpilrajn-Marczewski, we let f be the characteristic function of the sequence $\{A_n, n \geq 1\}$, that is, $f = \sum_{n=1}^{\infty} 2I_{A_n}/3^n$, where I_{A_n} denotes the indicator function of A_n. According to a result of Szpilrajn-Marczewski ([6], p. 144), f is an isomorphism between $(X, \boldsymbol{D})$ and $(f(X), \boldsymbol{B}_{f(X)})$. Since $\boldsymbol{D} \subset \boldsymbol{B}$, f is a one-to-one $\boldsymbol{B}$-measurable mapping and hence, by (c), f is an isomorphism between $(X, \boldsymbol{B})$ and $(f(X), \boldsymbol{B}_{f(X)})$. Consequently, $\boldsymbol{D} = \boldsymbol{B}$.

(d) $\Rightarrow$ (a). Let $\boldsymbol{D}$ be a separable σ-algebra and let $\boldsymbol{D} \subset \boldsymbol{B}$. Let $\{E_n, n \geq 1\}$ be a countable set of generators for $\boldsymbol{D}$. Since $\boldsymbol{D}$ is separable, $\{E_n, n \geq 1\}$ separates points of X and hence, by (d), generates $\boldsymbol{B}$. Consequently, $\boldsymbol{D} = \boldsymbol{B}$, so that $(X, \boldsymbol{B})$ is a Blackwell space. This completes the proof.

COROLLARY. *Let A be a coanalytic subset of a Polish space X. In order that A be a Blackwell space, it is necessary that, for every one-to-one $\boldsymbol{B}_A$-measurable function f on A into a Polish space Y, $f(A)$ be coanalytic in Y.*

Proof. If A is a Blackwell space, it follows from the previous proposition that f is an isomorphism between $(A, \boldsymbol{B}_A)$ and $(f(A), \boldsymbol{B}_{f(A)})$. By a result of Kuratowski ([3], p. 343), f admits an extension g such that g is an isomorphism between $(E, \boldsymbol{B}_E)$ and $(F, \boldsymbol{B}_F)$, where E, F are Borel subsets of X, Y, respectively, and $A \subset E$ and $f(A) \subset F$. Note that g is a one-to-one Borel mapping on an absolute Borel set E and so, by a standard result ([3]), p. 398), $f(A) = g(A)$ is coanalytic. This terminates the proof.

We are now ready to prove our main result.

THEOREM. *There exist coanalytic subsets of Polish spaces which are not Blackwell spaces.*

Proof. We shall give two methods of exhibiting such sets.

Method I. Let A be an analytic non-Borel subset of the real line. Denote by J the set of irrationals. Since A is analytic, there is a continuous function f from J onto A. By a theorem of Mazurkiewicz ([3], p. 389), there is a coanalytic subset C of J such that if g is the restriction of f to C, then g is one-to-one and $g(C) = A$. Thus, g is a one-to-one $\boldsymbol{B}_C$-measurable function on C into the line and $g(C)$ is not coanalytic. It now follows from the corollary above that C is not a Blackwell space.

Moreover, if $\boldsymbol{E}$ is the σ-algebra of Borel sets on the line, then $g^{-1}(\boldsymbol{E})$ is a separable sub-σ-algebra of $\boldsymbol{B}_C$ such that $g^{-1}(\boldsymbol{E}) \neq \boldsymbol{B}_C$. Note that $(C, g^{-1}(\boldsymbol{E}))$ is a Blackwell space. This follows from the fact that $(C, g^{-1}(\boldsymbol{E}))$ and $(A, \boldsymbol{B}_A)$ are isomorphic and that every analytic set is a Blackwell space. But, of course, $g^{-1}(\boldsymbol{E})$ is not the Borel σ-algebra of C.

Method II. The second method depends on the existence of a Borel set not containing a graph. The existence of such Borel sets has been shown by many authors, including Novikoff [4], Sierpiński [5], and Blackwell [2].

It follows from the results of Sierpiński [5] and Novikoff [4] that if X, Y are two uncountable Borel subsets of Polish spaces, then there exists a Borel set $D \subset X \times Y$ such that the projection of D to X is X and for every Borel mapping f from X to Y,

$$\operatorname{graph} f = \{(x, y) \in X \times Y\colon\ y = f(x)\} \not\subset D\,.$$

It now follows from the uniformization theorem of Luzin and Sierpiński ([3], p. 398) that there exists a coanalytic set $C \subset D$ which uniformizes D, that is, for every $x \in X$, the vertical section C_x of C at x is a singleton. Let g be the projection to X restricted to C. Then g is a one-to-one continuous function on C and so a one-to-one $\boldsymbol{B}_C$-measurable function from C onto X. Claim that g is not an isomorphism between $(C, \boldsymbol{B}_C)$ and $(X, \boldsymbol{B}_X)$. For if it were, then C would be a Borel set and, consequently, by virtue of a known result ([3], p. 398), C would be the graph of a Borel function from X into Y, contradicting one of the properties of D. Now the proposition proved above implies that C is not a Blackwell space. This completes the proof.

Incidentally, the above proof shows that every uncountable Borel subset of a Polish space is a one-to-one continuous image of some coanalytic non-Borel subset of some Polish space. This augments the result of Mazurkiewicz quoted above.

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A general realcompactification method

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Conventions. The *closure* of a set A in a space X will be denoted by $\mathrm{cl}_X A$. Collections of subsets of a space are indicated by German letters. If $\mathfrak{U}$ is a family of subsets of a space then the symbol $\mathrm{cl}_X\mathfrak{U}$ is used to denote the collection of all $\mathrm{cl}_X U$ for which $U \in \mathfrak{U}$. The union and intersection of a family of sets $\mathfrak{U}$ will be denoted by $\bigcup\mathfrak{U}$ and $\bigcap\mathfrak{U}$, respectively. For further basic conventions in general topology we refer to [6].

Introduction Let X be a T_1-space and let $\mathfrak{S}$ be a subbase for the closed sets of X. If $\mathfrak{S}$ has certain separation properties and is closed for certain set-theoretical operations (for instance, closed for the taking of finite intersections), then there is a standard way [2] to extend X to a compact Hausdorff space. Indeed, we consider all maximal centered systems of members of $\mathfrak{S}$ which have empty intersection in X, and let them serve as the new points for the extended space $\beta(\mathfrak{S})X$. $\beta(\mathfrak{S})X$ endowed with a suitable topology is a Hausdorff compactification of X. In particular, $\beta(\mathfrak{S})X$ is the Čech–Stone compactification of X in case X is completely regular and $\mathfrak{S}$ is the collection of all zero-sets of X [4].

In [5] Aarts and de Groot generalized this construction for the case where $\mathfrak{S}$ is not closed for finite intersections but only has certain separation properties (cf. also [1]). Let M be the collection of all maximal centered systems of members of $\mathfrak{S}$. By adding to each $\mu \in M$ the elements $S \in \mathfrak{S}$ that intersect each member of μ we obtain new collections $\bar{\mu}$. Those $\bar{\mu}$ which have empty intersection in X are in general not centered, but still do have the property that each two elements of it have a non-empty intersection; they are so-called *maximal linked systems* and serve as the new points for the extended space $\beta(\mathfrak{S})X$. By choosing a suitable topology for $\beta(\mathfrak{S})X$ we obtain a Hausdorff compactification of X.

In this paper our purpose is to adapt the above procedure for the realcompact case; thus, starting from a fixed closed subbase $\mathfrak{S}$, to obtain a general realcompactification $v(\mathfrak{S})X$ which depends on $\mathfrak{S}$ (see [4] for the definitions of realcompactness and realcompactification). Of course, we must see to it that $v(\mathfrak{S})X = vX$, the Hewitt realcompactification