If $\mathrm{Sh}(Z, z_0) = \mathrm{Sh}(X, x_0) + \mathrm{Sh}(Y, y_0)$, then $\mathrm{Sh}(X, x_0)$ and $\mathrm{Sh}(Y, y_0)$ will be said to be *constituents* of $\mathrm{Sh}(Z, z_0)$. And if $\mathrm{Sh}(Z, z_0) = \mathrm{Sh}(X, x_0) \times \mathrm{Sh}(Y, y_0)$, then $\mathrm{Sh}(X, x_0)$ and $\mathrm{Sh}(Y, y_0)$ will be said to be *factores* of $\mathrm{Sh}(Z, z_0)$. Thus (8.1) implies that every constituent and also every factor of a pointed shape is less than or equal to that pointed shape.

Let us say that a pointed shape $\mathrm{Sh}(X, x_0)$ is *movable* if (X, x_0) is movable. It follows by (2.3) and (8.1) that all constituents and all factors of a movable pointed shape are movable.

A pointed shape $\mathrm{Sh}(X, x_0)$ is said to be *simple* if each of its constituents either is trivial or coincides with $\mathrm{Sh}(X, x_0)$. A pointed shape $\mathrm{Sh}(X, x_0)$ is said to be *prime* if it is non-trivial and each of its factors either is trivial or coincides with $\mathrm{Sh}(X, x_0)$.

Let us formulate some problems concerning those notions:

1. Is it true that every pointed non-trivial shape has at least one non-trivial simple constituent and at least one non-trivial prime factor?

2. Is it true that there is at most one decomposition of a pointed shape into a finite sum of simple pointed shapes?

3. Is it true that for every compact manifold X the shape $\mathrm{Sh}(X, x_0)$ is simple?

4. Is it true that the shape of every acyclic curve is trivial?

5. Is true that $\mathrm{Sh}(X, x_0) = \mathrm{Sh}(Y, y_0) + \mathrm{Sh}(Z, z_0)$ implies that the fundamental dimension $\mathrm{Fd}(X)$ of X is equal to $\mathrm{Max}(\mathrm{Fd}(Y), \mathrm{Fd}(Z))$?

By the *fundamental dimension* of X we understand here the number $\mathrm{Fd}(X)$ given by the formula (compare [3])

$$\mathrm{Fd}(X) = \min_{\mathrm{Sh}(X) \leqslant \mathrm{Sh}(Y)} \dim Y .$$

6. Is it true that if $Z \in \mathrm{ANR}$ and $\mathrm{Sh}(Z, z_0) = \mathrm{Sh}(X, x_0) + \mathrm{Sh}(Y, y_0)$, then $\mathrm{Sh}(X, x_0)$ is determined by $\mathrm{Sh}(Y, y_0)$ and $\mathrm{Sh}(Z, z_0)$?

7. Is it true that for every ANR-set X the shape $\mathrm{Sh}(X, x_0)$ has only a finite number of simple constituents and prime factors?

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The global dimension of the group rings of abelian groups III

by

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This paper is a continuation of papers [1], [2] and is concerned with computation of the global dimension of the group ring of arbitrary abelian group with commutative Noetherian coefficient ring. Also the dimension of those rings as algebras is computed.

In this paper all rings and groups are assumed to be commutative.

For any R-algebra Λ, we denote by $\dim \Lambda$ or R-$\dim \Lambda$ the *projective dimension of* Λ as Λ^e-module. If $\Lambda = R(\Pi)$ is a group ring, then it is known (see [4]) that $\dim R(\Pi) = \dim_{R(\Pi)} R$ where Π operates trivially on R.

1. In this section we prove some preliminary lemmas.

LEMMA 1. *Let Π be a group and $\mathfrak{a} \subset R$ be an ideal of a ring R. If $\bar{R} = R/\mathfrak{a}$, then R-$\dim R(\Pi) \geqslant \bar{R}$-$\dim \bar{R}(\Pi)$.*

Proof. If P is a projective resolution of $R(\Pi)$-module R, then $P \otimes_R \bar{R}$ is a $\bar{R}(\Pi)$-projective complex. Since $H_n(P \otimes_R \bar{R}) = \mathrm{Tor}_n^R(R, \bar{R})$, then $P \otimes_R \bar{R}$ is a projective resolution of $\bar{R}$ and the lemma follows.

LEMMA 2. *If Π_0 is a subgroup of a group Π, then*

$$\mathrm{gl.}\dim R(\Pi) \geqslant \mathrm{gl.}\dim R(\Pi_0) ,$$

$$\dim R(\Pi) \geqslant \dim R(\Pi_0) .$$

Proof. It is easy to prove the formula

$$\dim_{R(\Pi_0)} A = \dim_{R(\Pi)} A \otimes_{R(\Pi_0)} R(\Pi)$$

for any $R(\Pi_0)$-module A and this implies the first inequality. The second one follows by the fact that any $R(\Pi)$-projective module is $R(\Pi_0)$-projective.

LEMMA 3. *If R is a field and $mR = R$ if m is an order of an element in a group Π, then in the group ring $R(\Pi)$ any set of orthogonal idempotents is at most countable.*

Proof. It is easy to see that all idempotents of $R(\Pi)$ belong to the subring $R(T)$ where T is the maximal torsion subgroup of Π. The group T

is a union of a countable increasing sequence of groups, which are direct sums of finite cyclic groups. Consequently, it is sufficient to prove the lemma in the case of the group T which is a direct sum $\bigoplus_{j\in J}\{t_j\}$ of finite cyclic groups.

We define support of an element $r\in R(T)$ as the (finite) set of all such $j_0\in J$ that $r\notin R(\bigoplus_{j\neq j_0}\{t_j\})$. If supports of elements r, r' are disjoint then $rr'\neq 0$. Using this and semi-simplicity of rings $R(\{t_j\})$ it is easy to prove that the ring $R(T)$ contains at most countable set of orthogonal idempotents with supports consisting of n elements and this proves the lemma.

LEMMA 4. *Let R be a Noetherian ring. If for any maximal ideal $\mathfrak{m}\subset R$ we have $R\text{-}\dim R(\Pi)=\bar{R}\text{-}\dim\bar{R}(\Pi)$ with $\bar{R}=R/\mathfrak{m}$, then*

$$\text{(1)}\qquad \text{gl.}\dim R(\Pi)=\text{gl.}\dim R+\dim R(\Pi)\,.$$

Proof. Let us denote $s=\text{gl.}\dim R$, $n=\dim R(\Pi)$. By results of [5], p. 74, it follows that

$$\text{(2)}\qquad \text{gl.}\dim R(\Pi)\leqslant s+n\,.$$

In either case $s=0$, $s=\infty$, $n=0$, $n=\infty$, formula (1) is obvious. Let us assume $0<s<\infty$ and $0<n<\infty$. Let

$$P\colon\ 0\longrightarrow P_n\longrightarrow\ldots\longrightarrow P_0\longrightarrow R\longrightarrow 0_R$$

be a free $R(\Pi)$-resolution of R and $P_0=R(\Pi)$. There exists a finitely generated R-module A such that $\dim_R A=s$; then, there exists a finitely generated projective resolution

$$0\longrightarrow Q_s\longrightarrow\ldots\longrightarrow Q_0\longrightarrow A\longrightarrow 0\,.$$

To prove the lemma it is sufficient to find a projective R-module Q such that

$$\dim_{R(\Pi)}(A\oplus Q)=s+n\,.$$

If we add to all modules $Q_s,\ldots,Q_0$ appropriate finitely generated projective modules, we obtain a finitely generated free resolution

$$\text{(3)}\qquad 0\longrightarrow F'_s\longrightarrow\ldots\longrightarrow F'_0\longrightarrow A\oplus Q'\longrightarrow 0$$

of $A\oplus Q'$ with projective Q'. Among all projective modules Q' such that there exists finitely generated free resolution (3) of $A\oplus Q'$, let us consider a Q such that the rank of F'_s is minimal. Then we have a free resolution

$$F\colon\ 0\longrightarrow F_s\xrightarrow{d_s}\ldots\longrightarrow F_0\longrightarrow A\oplus Q\longrightarrow 0$$

and let $v_1,\ldots,v_r$ be free generators of F_s. By Lemma 2 of [2] it follows that there exist proper ideals $\mathfrak{a}_1,\ldots,\mathfrak{a}_r\subset R$ such that $d_sv_i\subset\mathfrak{a}_iF_{s-1}$, $i=1,\ldots,r$. Let $S=P\otimes_R F$; then S is an acyclic free $R(\Pi)$-complex and $H_0(S)=A\oplus Q$ and thus S is a free resolution of $R(\Pi)$-module $A\oplus Q$. Let $\{y_\gamma\}$, $\gamma\in\Gamma$, be a set of free generators of P_n, $z_\gamma=y_\gamma\otimes v_1\in S_{s+n}$, and let

$$\pi\colon\ S_{s+n}\longrightarrow P_n\otimes Rv_1=W\,,\qquad \sigma\colon\ S_{s+n-1}\longrightarrow P_{n-1}\otimes Rv_1=V$$

be projections on direct summands. By Lemma 1 of [2] it is sufficient to prove that $\text{Im}(S_{s+n}\xrightarrow{d}S_{s+n-1})$ is not a direct summand or that there does not exist a homomorphism $\varrho\colon S_{s+n-1}\to S_{s+n}$ such that $\varrho d=1$. If such homomorphism were exist, then

$$\begin{aligned} z_\gamma=\pi z_\gamma&=\pi\varrho dz_\gamma=\pi\varrho d(y_\gamma\otimes v_1)\\ &=\pi\varrho(d_ny_\gamma\otimes v_1)+(-1)^n\pi\varrho(y_\gamma\otimes d_sv_1)\,.\end{aligned}$$

Let $\mathfrak{m}\subset R$ be a maximal ideal which contains $\mathfrak{a}_1$. Then $d_sv_1\in\mathfrak{m}F_{s-1}$ and $\pi\varrho(y_\gamma\otimes d_sv_1)\in\pi\mathfrak{m}S_{s+n}=\mathfrak{m}W$. Let $\partial\colon W\to V$ be defined by $\partial(y_\gamma\otimes v_1)=d_ny_\gamma\otimes v_1$; then $w-\pi\varrho\partial w\in\mathfrak{m}W$ for all $w\in W$. If $\bar{W}=W/\mathfrak{m}W$, $\bar{V}=V/\mathfrak{m}V$, and $\alpha\colon W\to\bar{W}$, $\beta\colon V\to\bar{V}$ are canonical homomorphisms, then $\alpha w=\alpha\pi\varrho\partial w=\alpha\pi\varrho\iota\partial w$ where $\iota\colon V\to S_{s+n-1}$ is the embedding. In the diagram

$$\begin{array}{ccc} S_{s+n} & \underset{d}{\overset{\varrho}{\rightleftarrows}} & S_{s+n-1}\\ {\scriptstyle\pi}\downarrow & & {\scriptstyle\sigma}\downarrow\uparrow{\scriptstyle\iota}\\ W & \xrightarrow{\partial} & V\\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta}\\ \bar{W} & \xrightarrow{\bar{\partial}} & \bar{V}\\ {\scriptstyle\approx}\downarrow & & \downarrow{\scriptstyle\approx}\\ P_n\otimes_R\bar{R} & \longrightarrow & P_{n-1}\otimes_R\bar{R} \end{array}$$

both squares commute. If $\tau=\alpha\pi\varrho\iota$ then $\tau\partial=\alpha$ and $\tau(\mathfrak{m}W)=0$, and hence there exists an induced homomorphism $\bar{\tau}\colon\bar{V}\to\bar{W}$ and $\bar{\tau}\bar{\partial}=1$. Thus $\text{Im}(P_n\otimes\bar{R}\to P_{n-1}\otimes\bar{R})$ is a direct summand of $P_{n-1}\otimes\bar{R}$ and consequently $\dim_{\bar{R}(\pi)}\bar{R}<n$ since $P\otimes_R\bar{R}$ is a projective resolution of $\bar{R}(\Pi)$-module $\bar{R}$. We get a contradiction and the lemma follows.

2. In this section we compute $\dim R(T)$ of a group ring $R(T)$ of an abelian torsion group T assuming that T is a direct sum of cyclic groups $T=\bigoplus_{j\in J}\{t_j\}$, R is a field, and $mR=R$ if m is an order of an element in T. This results are used in Section 3.

Let $T=\bigoplus_{j\in J}\{t_j\}$ be a direct sum of finite cyclic groups; let us assume that R is a commutative ring and $mR=R$ if m is an order of an element in T. We write $K = R(T)$, m_j is the order of t_j for $j \in J$ and

$$\varepsilon_j = \frac{1}{m_j}(1+t_j+\ldots+t_j^{m_j-1}), \qquad \delta_j = 1-\varepsilon_j .$$

Then $\varepsilon_j^2 = \varepsilon_j$, $\delta_j^2 = \delta_j$, $\varepsilon_j\delta_j = 0$ and we have a free acyclic complex

$$X_j\colon \ \ldots \longrightarrow Ke_j^{(3)} \xrightarrow{\delta_j} Ke_j^{(2)} \xrightarrow{\varepsilon_j} Ke_j^{(1)} \xrightarrow{\delta_j} K \longrightarrow 0$$

where $e_j^{(p)}$ are free generators and $H_0(X_j) = K/K\delta_j$. If $j_1, \ldots, j_m \in J$ are distinct, then the complex $X_{j_1,\ldots,j_m} = X_{j_1} \otimes_K \ldots \otimes_K X_{j_m}$ is free and $H_0(X_{j_1,\ldots,j_m}) = K/K(\delta_{j_1}, \ldots, \delta_{j_m})$. The complex $X_{j_1,\ldots,j_m}$ is acyclic: for $m = 1$ it is obvious; then, let us assume that it is true for $m-1$. We consider the complex $X' \otimes X_{j_m}$ with $X' = X_{j_1} \otimes \ldots \otimes X_{j_{m-1}}$. We have a spectral sequence such that $E^0_{p,q} = X'_q \otimes Ke_{j_m}^{(p)}$. By induction hypothesis,

$$E^1_{p,q} = \begin{cases} H_0(X') \otimes Ke_{j_m}^{(p)} & \text{for} \quad q = 0, \\ 0 & \text{for} \quad q > 0, \end{cases}$$

and $d^1_{p,0}$ is induced by the differential of X_{j_m}. Then $E^2_{p,q} = 0$ for $q > 0$ and $E^2_{p,0}$ is a pth homology module of the complex

$$\ldots \longrightarrow H_0(X') \xrightarrow{\delta_{j_m}} H_0(X') \xrightarrow{\varepsilon_{j_m}} H_0(X') \xrightarrow{\delta_{j_m}} H_0(X') \longrightarrow 0$$

Since $H_0(X') = K/K(\delta_{j_1}, \ldots, \delta_{j_{m-1}})$ and the following conditions hold [1]

$$(\delta_{j_1}, \ldots, \delta_{j_{m-1}})\colon (\delta_{j_m}) = (\delta_{j_1}, \ldots, \delta_{j_{m-1}}, \varepsilon_{j_m}),$$
$$(\delta_{j_1}, \ldots, \delta_{j_{m-1}})\colon (\varepsilon_{j_m}) = (\delta_{j_1}, \ldots, \delta_{j_{m-1}}, \delta_{j_m}),$$

we have that $E^2_{p,0} = 0$ for $p > 0$ and the complex $X_{j_1,\ldots,j_m}$ is acyclic.

Let $\leqslant$ be a well-ordering relation of type ω_s on J. We denote by X the direct limit of complexes $X_{j_1} \otimes \ldots \otimes X_{j_m}$, $j_1 < \ldots < j_m$, over the system of all finite subsets of J, with respect to an obvious embedding of complexes. The complex X is acyclic and is freely generated by elements $e_{j_1}^{(p_1)} \otimes \ldots \otimes e_{j_m}^{(p_m)}$ ($j_1 < \ldots < j_m$ and $p_1, \ldots, p_m > 0$) of degree $p_1 + \ldots + p_m$ and by element 1 of degree 0. Moreover, $H_0(X) = K/K\{\delta_j\} = R$, thus X is a free resolution of the trivial $R(T)$-module R.

In the same way we find a projective resolution of R starting with tensor products $Y_{j_1} \otimes \ldots \otimes Y_{j_m}$ of complexes

$$Y_j\colon \ 0 \longrightarrow K\delta_j e_j^{(1)} \longrightarrow K \longrightarrow 0 .$$

Since Y_j is a direct summand of X_j, the direct limit Y of complexes $Y_{j_1} \otimes \ldots \otimes Y_{j_m}$, $j_1 < \ldots < j_m$, is also a direct summand of X. Consequently,

(1) Since $K\varepsilon_{j_m}$ is the annihilator of $K\delta_{j_m}$, and conversely, we see that this property may be regarded as a generalized normality of a sequence.

Y is projective and acyclic, and it is easy to see that $H_0(Y) = R$. For any finite subset $a = \{j_1, \ldots, j_m\}$, $j_1 < \ldots < j_m$, of J we write

$$e(a) = e(j_1, \ldots, j_m) = e_{j_1}^{(1)} \otimes \ldots \otimes e_{j_m}^{(1)},$$
$$\delta(a) = \delta_{j_1} \ldots \delta_{j_m} .$$

The complex Y is generated by the elements $\delta(a)e(a)$ and by 1. Differentials of Y are defined by the formulas

$$d_n\big(\delta_{j_1} \ldots \delta_{j_m} e(j_1, \ldots, j_m)\big) = \sum_{i=1}^{m} (-1)^{i-1} \delta_{j_1} \ldots \delta_{j_m} e(j_1, \ldots, \hat{j_i}, \ldots, j_m) \quad \text{for} \quad n > 1,$$
$$d_1\big(\delta_{j_1} e(j_1)\big) = \delta_{j_1} .$$

LEMMA 5. *Let an abelian group T be a direct sum of finite cyclic groups, let R be a field, and let $mR = R$ if m is an order of an element in T. If $|T| = \aleph_s$ then* $\dim R(T) = s+1$.

Proof. If $s = -1$ then T is a finite group, R is projective and the lemma follows.

Let us assume $s \geqslant 0$ and $T = \bigoplus_{j \in J} \{t_j\}$. We preserve the previous notation. Let I be the ideal of $K = R(T)$ generated by all elements δ_j, $j \in J$. By Lemma 2 of [3] it follows that $\dim_K I \leqslant s$; then

$$\dim R(T) = \dim K = \dim_K K/I \leqslant s+1 .$$

Since R is K-projective for finite T only, it is sufficient to prove that $\dim_K I \geqslant s$ if $0 < s < \infty$. We have a projective resolution of I

$$\ldots \longrightarrow Y_{s+1} \longrightarrow Y_s \xrightarrow{d_s} Y_{s-1} \longrightarrow \ldots \longrightarrow Y_1 \longrightarrow I \longrightarrow 0$$

and we shall prove that the module $U = \operatorname{Im} d_s$ is not projective.

Let us assume that U is projective. The ring K is regular and all elements $u(a) = d_s\big(\delta(a)e(a)\big)$ generate U if a runs over the family J_s of all s-element subsets of J. By Theorem 3.1 of [6] it follows that there exist idempotents $f(a) \in K$ such that

$$U = \bigoplus_{a \in J_s} Kf(a)u(a) . \tag{4}$$

Since $\delta(a)u(a) = u(a)$, we can assume that $f(a)\delta(a) = f(a)$. Similarly as in [6] we prove the following properties of idempotents $f(a)$:

(i) *$f(a) \neq 0$ for all $a \in J_s$,* *(ii) if $j_{10} < j_{11} < j_{20} < j_{21} < \ldots < \ldots < j_{s0} < j_{s1}$ is a sequence of elements of J, and for any map $q\colon \{1, 2, \ldots, s\} \to \{0, 1\}$ we denote $a(q) = \{j_{1q(1)}, \ldots, j_{sq(s)}\}$, then*

$$\prod_q f\big(a(q)\big) = 0$$

where the product is taken over all q.

To prove (i) let us write $u(a)$ as

$$u(a) = kf(a)u(a) + \sum_{m=1}^{n} k_m f(a_m)u(a_m)$$

with distinct $a, a_1, \ldots, a_n \in J_s$. Then there exist $j_1, \ldots, j_n \in J$ such that $j_m \notin a$, $j_m \in a_m$, $m = 1, \ldots, n$, and we have $\varepsilon_{j_m} u(a_m) = (1 - \delta_{j_m})u(a_m) = u(a_m) - \delta_{j_m} u(a_m) = 0$ because $\delta_{j_m} u(a_m) = \delta_{j_m}(a_m)u(a_m) = \delta(a_m)u(a_m) = u(a_m)$. Consequently, $\varepsilon_{j_1} \ldots \varepsilon_{j_n} u(a) = \varepsilon_{j_1} \ldots \varepsilon_{j_n} kf(a)u(a)$ and if $a = \{j'_1, \ldots, j'_s\}$ then

$$\varepsilon_{j_1} \ldots \varepsilon_{j_n} kf(a)u(a) = \varepsilon_{j_1} \ldots \varepsilon_{j_n} d_s\big(\delta(a)e(a)\big)$$
$$= \varepsilon_{j_1} \ldots \varepsilon_{j_n} \sum_{i=1}^{s} (-1)^{i-1}\delta(a)e(j'_1, \ldots, \hat{j}'_i, \ldots, j'_s) \neq 0,$$

because $\varepsilon_{j_1} \ldots \varepsilon_{j_n}\delta(a) \neq 0$; we have proved (i).

To prove (ii), let us write $c = \prod_q f\big(a(q)\big)$; then $cf\big(a(q)\big) = c$ for all q. If $p\colon \{0, 1, \ldots, s-1\} \to \{0, 1\}$ is a map, then we write

$$a'(p) = \{j_{1,p(1)}, \ldots, j_{s-1,p(s-1)}, j_{s,0}, j_{s,1}\},$$

and let an element φ of Y_{s+1} be defined as follows

$$\varphi = c\sum_p (-1)^{\sigma(p)}\delta\big(a'(p)\big)e\big(a'(p)\big)$$

where the sum is taken over all p and $\sigma(p) = p(0) + \ldots + p(s-1)$. We have

$$d_{s+1}\varphi = c\sum_p \sum_{i=1}^{s-1} (-1)^{\sigma(p)+i-1}\delta\big(a'(p)\big)e(j_{1,p(1)}, \ldots, \hat{j}_{i,p(i)}, \ldots, j_{s-1,p(s-1)}, j_{s,0}, j_{s,1}) +$$
$$+ c\sum_p (-1)^{\sigma(p)+s-1}\delta\big(a'(p)\big)e(j_{1,p(1)}, \ldots, j_{s-1,p(s-1)}, j_{s,1}) +$$
$$+ c\sum_p (-1)^{\sigma(p)+s}\delta\big(a'(p)\big)e(j_{1,p(1)}, \ldots, j_{s-1,p(s-1)}, j_{s,0}).$$

In the first sum, for fixed i the terms corresponding to the maps p, p_1, which differ only on i, cancel. Since $c\delta_{j_{it}} = c$, we have $c\delta\big(a'(p)\big) = c$ and $c\delta\big(a(q)\big) = c$, thus

$$d_{s+1}\varphi = (-1)^s \sum_p (-1)^{\sigma(p)+1} ce(j_{1,p(1)}, \ldots, j_{s-1,p(s-1)}, j_{s,1}) +$$
$$+ (-1)^s \sum_p (-1)^{\sigma(p)} ce(j_{1,p(1)}, \ldots, j_{s-1,p(s-1)}, j_{s,0})$$
$$= (-1)^s c\sum_q (-1)^{\sigma(q)}\delta\big(a(q)\big)e\big(a(q)\big).$$

Consequently, we get

$$0 = d_s d_{s+1}\varphi = (-1)^s \sum_q (-1)^{\sigma(q)} cd_s\Big(\delta\big(a(q)\big)e\big(a(q)\big)\Big)$$
$$= (-1)^s \sum_q (-1)^{\sigma(q)} cf\big(a(q)\big)u\big(a(q)\big).$$

By (4) it follows that $cf\big(a(q)\big)u\big(a(q)\big) = 0$ for all q. Each element

$$u\big(a(q)\big) = d_s\Big(\delta\big(a(q)\big)e\big(a(q)\big)\Big) = \delta\big(a(q)\big)d_s\Big(e\big(a(q)\big)\Big)$$

belongs to Y_{s-1} and is a linear form on free generators $e(a'')$, $a'' \in J_{s-1}$, of X_{s-1} with coefficients $\pm\delta\big(a(q)\big)$. Since $cu\big(a(q)\big) = cf\big(a(q)\big)u\big(a(q)\big)$, we have $c\delta\big(a(q)\big) = 0$ and $c = cf\big(a(q)\big) = c\delta\big(a(q)\big)f\big(a(q)\big) = 0$; we have proved (ii).

Let us denote by $\bar{f}$ the function $\bar{f}\colon J^s \to K$ defined as follows

$$\bar{f}(j_1, \ldots, j_s) = \begin{cases} f(\{j_1, \ldots, j_s\}) & \text{if all } j_1, \ldots, j_s \text{ are distinct},\\ 1 & \text{in opposite case}.\end{cases}$$

By (ii) it follows that if $\bar{f}$ satisfies assumptions of Lemma 5.3 of [6], then there exists distinct elements $j_1, \ldots, j_s$ such that $f(\{j_1, \ldots, j_s\}) = 0$ contrary to (i).

3. In this section we prove main theorems.

LEMMA 6. *Let T be a torsion abelian group, $|T| = \aleph_s$ ($s \geqslant -1$), let R be a commutative Noetherian ring, and let $mR = R$ if m is an order of an element in T. We have*

$$\dim R(T) = s+1.$$

Proof. If $s = -1$ then R is a projective $R(T)$-module and $\dim R(T) = 0$.

Let us asume $s \geqslant 0$. By Lemma 2, it is sufficient to prove the lemma for $s < \infty$. By Lemma 2 of [3] it follows that

$$(5)\qquad \dim_{R(T)} I \leqslant s$$

then

$$(6)\qquad \dim R(T) \leqslant s+1 .$$

By Lemma 1, for any maximal ideal $\mathfrak{m} \subset R$ we have

$$(7)\qquad \bar{R}\text{-}\dim \bar{R}(T) \leqslant \dim R(T)$$

where $\bar{R} = R/\mathfrak{m}$. The group T contains a direct sum of cyclic subgroups T_0 with $|T_0| = \aleph_s$ or $s = 0$, and T contains a generalized cyclic (*Prüfer*) group T_1. In the first case, by Lemmas 2 and 5 we have

$$(8)\qquad \dim \bar{R}(T) \geqslant \dim \bar{R}(T_0) = s+1$$

and the lemma follows by (6), (7), (8).

In the second case, $R(T)$-module R is not projective then by (5) it follows that $\dim R(T) = 1$ and the lemma is proved.

THEOREM 1. *Let Π be an abelian group, T the maximal torsion subgroup of Π and $|T| = \aleph_s$ $(s \geqslant -1)$. Let R be a commutative ring and $mR = R$ if m is an order of an element in T; then*

$$\mathrm{gl.\,dim}\, R(\Pi) = \mathrm{gl.\,dim}\, R + r(\Pi) + s + 1 + \varepsilon$$

where $\varepsilon = 1$ if $s = -1$ and Π is not finitely generated, and $\varepsilon = 0$ in opposite case [1] ($r(\Pi) = \mathrm{rank}(\Pi)$).

Proof. If $s = -1$ then $\Pi = T \oplus \overline{\Pi}$ and $R(T)$ is Noetherian. By the equality $\mathrm{gl.\,dim}\, R(T) = \mathrm{gl.\,dim}\, R$ and by Theorem 1 of [2] it follows that

$$\mathrm{gl.\,dim}\, R(\Pi) = \mathrm{gl.\,dim}\big(R(T)\big)(\overline{\Pi}) = \mathrm{gl.\,dim}\, R + r(\Pi) + \varepsilon .$$

Let us assume $s \geqslant 0$; it is sufficient to consider only the case $s < \infty$. If $r(\Pi) \geqslant \aleph_0$ then by Lemma 2 and by the first part of the proof it follows that $\mathrm{gl.\,dim}\, R(\Pi) = \infty$. Thus we assume $r(\Pi) < \aleph_0$. There exists a subgroup $\Pi_0 \subset \Pi$ such that $\Pi_0 = T \oplus \Pi_1$, $r(\Pi) = r(\Pi_1)$ and Π_1 is a free abelian group. We have $R(\Pi_0) = \big(R(\Pi_1)\big)(T)$ and $R(\Pi_1)$ is Noetherian. By Lemmas 4 and 6 and Theorem 1 of [2] we get

$$(9)\qquad \mathrm{gl.\,dim}\, R(\Pi_0) = \mathrm{gl.\,dim}\, R(\Pi_1) + s + 1 = \mathrm{gl.\,dim}\, R + r(\Pi) + s + 1 .$$

(1) If $mR \neq R$ for some m, then $\mathrm{gl.dim}\ R(\Pi) = \infty$.

By Theorem 2 of [3] we have $\mathrm{gl.\,dim}\, R(\Pi) \leqslant \mathrm{gl.\,dim}\, R + r(\Pi) + s + 1$ and the theorem follows by (9) and Lemma 2.

THEOREM 2. *Under assumptions and notation of Theorem 1 we have*

$$\dim R(\Pi) = r(\Pi) + s + 1 + \varepsilon .$$

Proof. It is proved in [5] that

$$(10)\qquad H^p\big(\Gamma, H^q(\Lambda, A)\big) \underset{p}{\Rightarrow} H^n(\Lambda \otimes_R \Gamma, A)$$

for all R-algebras Λ, Γ and all $\Lambda^e - \Gamma^e$-bimodules A. Then we get (see [5])

$$(11)\qquad \dim(\Lambda \otimes \Gamma) \leqslant \dim \Lambda + \dim \Gamma .$$

We keep notation of the proof of Theorem 1.

If $s = -1$ then $\dim R(T) = 0$ and by (11) and Lemma 2 it follows

$$\dim R(\overline{\Pi}) = \dim R(T) + \dim R(\overline{\Pi}) \geqslant \dim R(T) \otimes R(\overline{\Pi}) = \dim R(\Pi) \geqslant \dim R(\overline{\Pi}) .$$

By Theorem 2 of [2] we get

$$\dim R(\Pi) = \dim R(\overline{\Pi}) = r(\overline{\Pi}) + \varepsilon = r(\Pi) + \varepsilon .$$

If F is a free abelian group of rank 1 then $\dim R(F) = 1$ and it is easy to check that for any trivial $R(F)$-module A we have $H^1\big(R(F), A\big) = A$. By (10) it follows that if $\dim \Gamma = m$ and $H^m(\Gamma, A) \neq 0$, then $H^{m+1}\big(R(F) \otimes \Gamma, A\big) \neq 0$ and consequently $\dim R(F) \otimes \Gamma = \dim \Gamma + 1$. Trivial induction argument show that

$$(12)\qquad \dim R(F) \otimes \Gamma = \dim \Gamma + r(F)$$

for any free abelian group F.

Let us assume $s \geqslant 0$; if $r(\Pi) \geqslant \aleph_0$, then by Lemma 2 and by the first part of the proof it follows that $\dim R(\Pi) = \infty$. Thus we can assume $r(\Pi) < \infty$. It is proved in [2] that $\dim R(\Pi) \leqslant r(\Pi) + s + 1$ and by (12) and Lemmas 2 and 6 we have

$$\dim R(\Pi) \geqslant \dim R(\Pi_0) = \dim R(T) + r(\Pi_1) = r(\Pi) + s + 1 ,$$

thus the theorem is proved.

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Coanalytic sets that are not Blackwell spaces

by

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A σ-algebra $\boldsymbol{B}$ of subsets of a set X will be said to be *separable* if (i) $\boldsymbol{B}$ is countably generated and (ii) $\{x\} \in \boldsymbol{B}$ for all $x \in X$. Call a measurable space $(X, \boldsymbol{B})$ separable if $\boldsymbol{B}$ is a separable σ-algebra. We shall say that a measurable space $(X, \boldsymbol{B})$ is a *Blackwell space* if (i) $(X, \boldsymbol{B})$ is separable, and (ii) for every separable σ-algebra $\boldsymbol{C} \subset \boldsymbol{B}$, $\boldsymbol{C} = \boldsymbol{B}$. Say that a separable metric space X is a *Blackwell space* if $(X, \boldsymbol{B}_X)$ is a Blackwell space (here, and in the sequel, whenever Z is a metric space, $\boldsymbol{B}_Z$ will denote the σ-algebra of Borel sets of Z). In [1], Blackwell proved that every analytic subset of a Polish space is a Blackwell space. The question then arises if every separable metric space which is a Blackwell space is analytic in its completion (and, consequently, analytic in any Polish space in which it can be imbedded). I do not know the answer to this question. The aim of this note is to exhibit complementary analytic (referred to, hereafter, as coanalytic) subsets of Polish spaces which are not Blackwell spaces. Again, I do not know if every coanalytic non-analytic subset of a Polish space fails to be a Blackwell space.

We first characterise Blackwell spaces. We shall say that g is a *measurable mapping* from a measurable space $(X, \boldsymbol{B})$ to a measurable space $(Y, \boldsymbol{C})$ if g is a function from X into Y and $g^{-1}(\boldsymbol{C}) \subset \boldsymbol{B}$. Call a mapping g an *isomorphism* between $(X, \boldsymbol{B})$ and $(Y, \boldsymbol{C})$ if g is one-to-one on X onto Y and both g and g^{-1} are measurable. In case the range of a mapping is a metric space, the relevant σ-algebra will always be the Borel σ-algebra and it will not be mentioned explicitly.

PROPOSITION. *Let $(X, \boldsymbol{B})$ be a separable measurable space. Then the following conditions are equivalent:*

(a) *$(X, \boldsymbol{B})$ is a Blackwell space.*

(b) *If $(Y, \boldsymbol{C})$ is any separable measurable space and f is a one-to-one measurable mapping from $(X, \boldsymbol{B})$ onto $(Y, \boldsymbol{C})$, then f is an isomorphism between $(X, \boldsymbol{B})$ and $(Y, \boldsymbol{C})$.*

(c) *For every one-to-one $\boldsymbol{B}$-measurable mapping f from $(X, \boldsymbol{B})$ to a Polish space, f is an isomorphism between $(X, \boldsymbol{B})$ and $\big(f(X), \boldsymbol{B}_{f(X)}\big)$.*