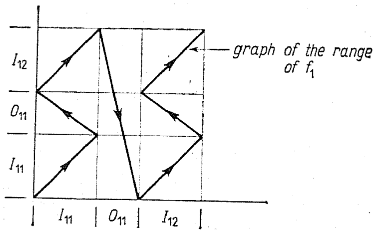
It follows from [4], p. 57, and the fact that there is an order preserving homeomorphism between the irrationals in I and the non-end points of P that there is a universal null set N in P such that there is a continuous map g of N onto P. Denote the graph of g by G.

There is a standard way of mapping I into $I\times I$ by a homeomorphism f such that f maps P onto $P\times P$ and, moreover, the equation $v(x)=y$ has only a finite number of solutions if $y\in I-P$, where v is defined on I by $f(t)=\big(u(t),v(t)\big)$. Beginning with a diagram, a sketch of a construction of such a function f follows.



Then f_1 maps I_{21} linearly onto the noted diagonal of $I_{11}\times I_{11}$, O_{21} linearly onto the noted diagonal of $I_{11}\times O_{11}$, ... To define f_2, let f_2 be patterned after f_1 on the sets I_{2i}, $i\leqslant 4$, and $f_2=f_1$ on the rest of I. Iterate this process, and let $f(x)=\lim f_n(x)$.

Because G is a subset of $P\times P$, the set $E=f^{-1}(G)$ is a subset of P which is homeomorphic to G. Hence E is a universal null set. (In fact, since g is continuous, E is also homeomorphic to N.) Moreover, v is a continuous map of I onto I, $v(E)=P$, and the equation $v(x)=y$ has only finitely many solutions if $y\in I-P$. Hence it follows from [1], Theorem III, p. 635, that there exists a strictly increasing continuous function φ and a CBV function h such that $v=\varphi\circ h$. If $h(E)$ were a universal null set, then it would follow that $P=\varphi\big(h(E)\big)$ has measure zero. Hence $h(E)$ is not a universal null set.

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Some remarks concerning the shape of pointed compacta

by

Karol Borsuk (Warszawa)

By Q we denote the Hilbert cube, that is, the subset of the Hilbert space consisting of all points $(x_1, x_2, \ldots)$ with $0\leqslant x_n\leqslant 1/n$ for $n=1,2,\ldots$ Two pointed compacta (X,a), (Y,b) are said to be *fundamentally equivalent* (notation: $(X,a)\underset{F}{\simeq}(Y,b)$) if there exist in Q two pointed compacta (X',a') and (Y',b') homeomorphic to (X,a) and (Y,b) respectively and two fundamental sequences (see [1], p. 225)

$$\underline{f}=\{f_k,(X',a'),(Y',b')\}\quad\text{and}\quad \underline{g}=\{g_k,(Y',b'),(X',a')\}$$

such that $\underline{f}\underline{g}\simeq\underline{i}_{(Y',b')}$ and $\underline{g}\underline{f}\simeq\underline{i}_{(X',a')}$, where $\underline{i}_{(Z,c)}$ denotes the identity fundamental sequence $\{i,(Z,c),(Z,c)\}$.

If we assume only that the second relation $\underline{g}\underline{f}\simeq\underline{i}_{(X',a')}$ holds true, then we say that (X,a) is *fundamentally dominated by* (Y,b) and we write $(X,a)\underset{F}{\leqslant}(Y,b)$.

The collection of all pointed compacta (Y,b) fundamentally equivalent to a given pointed compactum (X,a) is called the *shape* of (X,a) (see [3]); it is denoted by $\mathrm{Sh}(X,a)$. Thus the relation $\mathrm{Sh}(X,a)=\mathrm{Sh}(Y,b)$ means that $(X,a)\underset{F}{\simeq}(Y,b)$. If $(X,a)\underset{F}{\leqslant}(Y,b)$, then $\mathrm{Sh}(X,a)$ is said to be *less than or equal to* $\mathrm{Sh}(Y,b)$ and we write $\mathrm{Sh}(X,a)\leqslant\mathrm{Sh}(Y,b)$.

The aim of this note is to establish a condition under which $\mathrm{Sh}(X,a)$ does not depend on the choice of the point a, and to study the operations of addition and multiplication of shapes of pointed compacta.

I wish to thank A. Lelek, who read the manuscript of this note, for his penetrating remarks.

1. A lemma on isotopy. By a *map* we understand here always a continuous function. A map

$$\varphi\colon\ X\times\langle u,v\rangle\to Y,\quad\text{where } u,v \text{ are numbers with } u<v,$$

is said to be a *homotopy in a set* Z if all values of φ belong to Z. If $a\in X$, $b\in Y$ and if $\varphi\colon X\times\langle u,v\rangle\to Y$ is a homotopy satisfying the condition

$$\varphi(a,t)=b\quad\text{for every } u\leqslant t\leqslant v,$$

then φ is said to be a *homotopy of* (X, a) *in* (Y, b), and we write

$$\varphi\colon (X, a)\times\langle u, v\rangle \to (Y, b).$$

If, for every $t\in\langle u, v\rangle$, the map $\varphi_t\colon X\to Y$ defined by the formula

$$\varphi_t(x) = \varphi(x, t) \quad \text{for every point } x\in X$$

is a homeomorphism, then the homotopy φ is said to be an *isotopy*.

Let us prove the following

(1.1) Lemma. *Let a be a point of an open subset G of the Hilbert cube Q, and let u_0, v_0 be two numbers with $u_0 < v_0$ and let $\beta\colon \langle u_0, v_0\rangle\to G$ be a map with $\beta(u_0) = a$, $\beta(v_0) = b$. Then there exists an isotopy*

$$\varphi\colon Q\times\langle u_0, v_0\rangle\to Q$$

satisfying the following conditions:

(1) $\varphi(x, u_0) = x$ *for every point* $x\in Q$,

(2) $\varphi(x, t) = x$ *for every* $(x, t)\in(Q\setminus G)\times\langle u_0, v_0\rangle$,

(3) $\varphi(a, t) = \beta(t)$ *for every* $t\in\langle u_0, v_0\rangle$.

Proof. First, let us consider the special case when the values of the map β belong to the interior K° of a ball $K\subset G$ in the space Q with center a and radius r. Let us set

$$\text{(1.2)}\qquad \begin{aligned} &\varphi(x, t) = x + \frac{r-\varrho(a, x)}{r}\big(\beta(t) - a\big) && \text{for every } (x, t)\in K\times\langle u_0, v_0\rangle,\\ &\varphi(x, t) = x && \text{for every } (x, t)\in (Q\setminus K^\circ)\times\langle u_0, v_0\rangle. \end{aligned}$$

If $x\in(Q\setminus K^\circ)\cap K$, both formulas (1.2) give $\varphi(x, t) = x$. It follows that $\varphi\colon Q\times\langle u_0, v_0\rangle\to Q$ is a homotopy. Moreover, if $x\in K$, then

$$\begin{aligned}\varphi[\varphi(x, t), a] &\leqslant \varrho[\varphi(x, t), x] + \varrho(a, x) = \frac{r-\varrho(a, x)}{r}|\beta(t) - a| + \varrho(a, x)\\ &\leqslant r - \varrho(a, x) + \varrho(a, x) = r.\end{aligned}$$

Hence $\varphi(K, t)\subset K$. Consequently, in order to prove that φ is an isotopy, it suffices to show that for $x, y\in K$ the equality $\varphi(x, t) = \varphi(y, t)$ implies $x = y$. In fact, in this case

$$x - y = \frac{\varrho(a, x) - \varrho(a, y)}{r}\big(\beta(t) - a\big),$$

whence

$$\varrho(x, y) = \frac{\varrho\big(\beta(t), a\big)}{r}\cdot|\varrho(a, x) - \varrho(a, y)|.$$

Since $\dfrac{\varrho\big(\beta(t), a\big)}{r} < 1$, we infer that $x\neq y$ implies

$$\varrho(x, y) < |\varrho(a, x) - \varrho(a, y)|,$$

which contradicts the triangle inequality.

Thus φ is an isotopy and one can see by (1.2) that it satisfies the conditions (1), (2) and (3). In order to finish the proof of Lemma (1.1), let us observe that there exists a finite sequence of numbers

$$u_0 < u_1 < \ldots < u_n < u_{n+1} = v_0$$

such that for every $i = 0, 1, \ldots, n$ all values $\beta(t)$ for $u_i\leqslant t\leqslant u_{i+1}$ lie in the interior K_i° of a ball $K_i\subset G$. By the special case just considered, there exists an isotopy

$$\varphi_i\colon Q\times\langle u_i, u_{i+1}\rangle\to Q$$

satisfying the conditions:

(1) $\varphi_i(x, u_i) = x$ for every point $x\in Q$,

(2) $\varphi_i(x, t) = x$ for $(x, t)\in(Q\setminus G)\times\langle u_i, u_{i+1}\rangle$,

(3) $\varphi_i\big(\beta(u_i), t\big) = \beta(t)$ for $t\in\langle u_i, u_{i+1}\rangle$.

It suffices to set

$$f_j(x) = \varphi_j(x, u_{j+1}) \quad \text{for} \quad j = 0, 1, \ldots, n-1$$

and

$$\begin{aligned} &\varphi(x, t) = \varphi_0(x, t) && \text{for } u_0\leqslant t\leqslant u_1,\\ &\varphi(x, t) = \varphi_i\big(f_{i-1}f_{i-2}\ldots f_0(x), t\big) && \text{for } u_i\leqslant t\leqslant u_{i+1} \text{ and } i = 1, 2, \ldots, n,\end{aligned}$$

in order to obtain an isotopy satisfying the required conditions.

2. Movable pointed compacta. A pointed compactum $(X, x_0)\subset(Q, x_0)$ is said to be *movable* (compare [2], p. 137) if for every neighborhood U of X there exists a neighborhood U_0 of X such that, for every neighborhood V of X, there is a homotopy

$$\varphi\colon U_0\times\langle 0, 1\rangle\to U$$

such that

(2.1) $\quad\varphi(x, 0) = x \quad$ and $\quad\varphi(x, 1)\in V \quad$ for every point $x\in U_0$,

(2.2) $\quad\varphi(x_0, t) = x_0 \quad$ for every $0\leqslant t\leqslant 1$.

By a slight modification of the arguments used in [2], one shows that:

(2.3) *If* (X, x_0) *is movable and* $\mathrm{Sh}(Y, y_0)\leqslant\mathrm{Sh}(X, x_0)$, *then* (Y, y_0) *is movable* (compare [2], p. 140).

(2.4) *If* $x_0\in X\in\mathrm{ANR}$, *then* (X, x_0) *is movable* (compare [2], p. 137).

(2.5) *Every pointed plane compactum is movable* (compare [2], p. 145).

(2.6) *The pointed solenoids of van Dantzig* ([5], p. 106) *are not movable* (compare [2], p. 138).

(2.7) *If* (X, x_0), (Y, y_0) *are movable, then* $X \times Y$ *pointed by* (x_0, y_0) *is movable.*

Let us prove the following

(2.8) THEOREM. *If* (X_1, x_0), $(X_2, x_0) \subset (Q, x_0)$ *are movable, and if* $X_1 \cap X_2 = (x_0)$, *then* $(X_1 \cup X_2, (x_0))$ *is movable.*

Proof. Let U be a neighborhood (in Q) of the set $X = X_1 \cup X_2$. Since (X_1, x_0) and (X_2, x_0) are movable, there exist two neighborhoods: U_1 of X_1 and U_2 of X_2 such that for every neighborhood V of X there are two homotopies

$$\varphi_\nu\colon U_\nu \times \langle 0, 1\rangle \to U, \qquad \nu = 1, 2,$$

such that $\varphi_\nu(x, 0) = x$, $\varphi_\nu(x, 1) \in V$ for every point $x \in U_\nu$ and $\varphi_\nu(x_0, t) = x_0$ for every $0 \leqslant t \leqslant 1$.

Let B_η denote the ball (in Q) with center x_0 and radius $\eta > 0$. Since $U_1 \cap U_2$ is a neighborhood of x_0, there exists a positive number ε such that $B_{2\varepsilon} \subset U_1 \cap U_2$. It is clear that there exist a closed neighborhood $\hat{U}_1 \subset U_1$ of X_1 and a closed neighborhood $\hat{U}_2 \subset U_2$ of X_2 such that

(2.9) $$\hat{U}_1 \cap \hat{U}_2 \subset B_\varepsilon.$$

The set

$$U_0 = \hat{U}_1 \cup \hat{U}_2 \cup B_\varepsilon$$

is a neighborhood of X. Let us define a map $\alpha\colon U_0 \times \langle 0, 1\rangle \to U$ setting:

(2.10) $$\alpha(x, t) = t\cdot x_0 + (1-t)\cdot x \quad \text{if} \quad \varrho(x, x_0) \leqslant \varepsilon \text{ and } 0 \leqslant t \leqslant 1,$$

(2.11) $$\alpha(x, t) = \frac{2\varepsilon - \varrho(x, x_0)}{\varepsilon}\cdot[t\cdot x_0 + (1-t)\cdot x] + \frac{\varrho(x, x_0) - \varepsilon}{\varepsilon}\cdot x \quad \text{if} \quad \varepsilon \leqslant \varrho(x, x_0) \leqslant 2\varepsilon \text{ and } 0 \leqslant t \leqslant 1,$$

(2.12) $$\alpha(x, t) = x \quad \text{if} \quad \varrho(x, x_0) \geqslant 2\varepsilon \text{ and } 0 \leqslant t \leqslant 1.$$

Since for $\varrho(x, x_0) = \varepsilon$ the two formulas (2.10) and (2.11) coincide, and for $\varrho(x, x_0) = 2\varepsilon$ the same holds also for formulas (2.11) and (2.12), we infer that α is a map of $U_0 \times \langle 0, 1\rangle$ into U.

Moreover, let us observe that

(2.13) $$\alpha(x, 0) = x \quad \text{for every point } x \in U_0,$$

(2.14) $$\alpha(x, 1) = x_0 \quad \text{for every point } x \in B_\varepsilon$$

and, since $B_{2\varepsilon} \subset U_1 \cap U_2$,

(2.15) $$\alpha(x, 1) \in U \quad \text{if } x \in \hat{U}_\nu.$$

It follows by (2.9), (2.13), (2.14), and (2.15) that setting

$$\varphi(x, t) = \alpha(x, 2t) \qquad \text{for } (x, t) \in U_0 \times \langle 0, \tfrac{1}{2}\rangle,$$

$$\varphi(x, t) = \varphi_\nu(\alpha(x, 1), 2t-1) \quad \text{for } (x, t) \in (\hat{U}_\nu \cup B_\varepsilon) \times \langle \tfrac{1}{2}, 1\rangle, \ \nu = 1, 2,$$

one gets a map $\varphi\colon U_0 \times \langle 0, 1\rangle \to U$ such that $\varphi(x, 0) = x$, $\varphi(x, 1) \in V$ for every point $x \in U_0$ and $\varphi(x_0, t) = x_0$ for every $0 \leqslant t \leqslant 1$. Hence (X, x_0) is movable and the proof of Theorem (2.8) is finished.

Remark. Since the solenoid of Van Dantzig can be represented as the union of two compacta A, B homeomorphic to the Cartesian product of the Cantor set C by a segment with $A \cap B$ homeomorphic to C, we see that the union of two movable compacta with the movable common part is not necessarily movable. Even the question remains open whether the union of two movable compacta having only one point in common is necessarily movable.

3. A lemma on pointed movable compacta. Let us prove the following

(3.1) LEMMA. *Let a, b be two points belonging to one component of a compactum $X \subset Q$. If (X, a) is movable, then for every neighborhood U of X there exists a neighborhood U_0 of X such that for every neighborhood W of X there is a homotopy $\alpha\colon U_0 \times \langle 0, 1\rangle \to U$ satisfying the following conditions:*

$$\alpha(x, 0) = x, \quad \alpha(x, 1) \in W \quad \textit{for every point } x \in U_0,$$

$$\alpha(a, t) = a, \quad \alpha(b, t) = b \quad \textit{for every } 0 \leqslant t \leqslant 1.$$

Proof. Since (X, a) is movable, there exists a neighborhood $U_0 \subset U$ such that for every neighborhood W of X there is a homotopy

$$\hat{\alpha}\colon U_0 \times \langle 0, 1\rangle \to U$$

such that

$$\hat{\alpha}(x, 0) = x, \quad \hat{\alpha}(x, 1) \in W \quad \text{for every point } x \in U_0,$$

$$\hat{\alpha}(a, t) = a \quad \text{for every } 0 \leqslant t \leqslant 1.$$

We can assume that U and W are open in Q and that $a \neq b$. Let W_0 denote the component of W containing a. Then there is an arc $L \subset U_0 \cap W_0$ with endpoints a and b. Setting $\lambda(x) = \hat{\alpha}(x, 1)$ for every point $x \in L$, we have a map $\lambda\colon L \to W$ for which the map $\bar{\lambda}\colon U_0 \to W$ given by the formula $\bar{\lambda}(x) = \hat{\alpha}(x, 1)$ is an extension. But all maps of (L, a) into (W, a) are homotopic, whence λ is homotopic to the inclusion map $j\colon L \to W$. Since W, as an open subset of Q, is an absolute neighborhood retract for metric spaces, we infer by the homotopy extension theorem that $\bar{\lambda}$ is homotopic in (W, a) to a map $\hat{\lambda}\colon U_0 \to W$ such that $\hat{\lambda}/L = j$. It follows that we can assume from the beginning that the given homotopy $\hat{\alpha}\colon U_0\times\langle 0,1\rangle\to U$ satisfies the condition

$$\hat{\alpha}(x,1)=x \quad \text{for every point } x\in L.$$

Consider now the disk

$$D=L\times\langle 0,1\rangle\subset U_0\times\langle 0,1\rangle.$$

There exists a homotopy

$$\chi\colon D\times\langle 0,1\rangle\to D$$

joining the identity map $i\colon D\to D$ with a map r retracting D to the set

$$Z=[L\times(0)]\cup[(a)\times\langle 0,1\rangle]\cup[L\times(1)]$$

and satisfying the condition $\chi(x,t)=(x,t)$ for every $(x,t)\in Z$.

Let β denote the projection of D onto L given by the formula

$$\beta(x,t)=x \quad \text{for every } (x,t)\in L\times\langle 0,1\rangle.$$

Then

$$\beta(x,t)=\hat{\alpha}(x,t)=x \quad \text{for every } (x,t)\in Z.$$

Setting

$$\vartheta_s(x,t)=\hat{\alpha}\chi[(x,t),2s] \quad \text{for every } (x,t)\in D \text{ and } 0\leqslant s\leqslant \tfrac{1}{2},$$
$$\vartheta_s(x,t)=\beta\chi[(x,t),2-2s] \quad \text{for every } (x,t)\in D \text{ and } \tfrac{1}{2}\leqslant s\leqslant 1,$$
$$\vartheta_s(x,t)=\hat{\alpha}(x,t) \quad \text{for every } (x,t)\in[U_0\times(0)]\cup[U_0\times(1)],$$

we get a homotopy joining in U the map $\vartheta_0=\hat{\alpha}/[U_0\times(0)]\cup D\cup[U_0\times(1)]$ with the map $\vartheta_1\colon [U_0\times(0)]\cup D\cup[U_0\times(1)]\to U$ given by the formulas:

$$\vartheta_1(x,t)=x \quad \text{for } (x,t)\in D,$$
$$\vartheta_1(x,t)=\hat{\alpha}(x,t) \quad \text{for } (x,t)\in[U_0\times(0)]\cup[U_0\times(1)].$$

Since $\hat{\alpha}$ is an extension of the map ϑ_0 with values in U and since U, as an open subset of Q, is an absolute neighborhood retract for metric spaces, we infer that the map ϑ_1 can be extended to a map $\alpha\colon U_0\times\langle 0,1\rangle\to U$. It is clear that α satisfies all the required conditions. Thus the proof of Lemma (3.1) is finished.

As an immediate consequence of Lemma (3.1), we get the following

(3.2) Theorem. *If a,b are two points belonging to one component of a compactum $X\subset Q$, then the movability of (X,a) implies the movability of (X,b).*

4. Shape of pointed movable compacta. Let us prove the following

(4.1) Theorem. *If a,b are points belonging to one component of a compactum $X\subset Q$ and if (X,a) is movable, then* $\mathrm{Sh}(X,a)=\mathrm{Sh}(X,b)$.

Proof. It follows by Lemma (3.1) that there exists a decreasing sequence $U_1,U_2,\ldots$ of open neighborhoods of X (in Q) such that every neighborhood of X contains almost all U_n and that for every $n=1,2,\ldots$ there is a homotopy

$$\alpha_n\colon U_{n+1}\times\langle 0,1\rangle\to U_n$$

satisfying the following conditions:

(4.2) $\alpha_n(x,0)=x$ and $\alpha_n(x,1)\in U_{n+2}$ for every point $x\in U_{n+1}$,

(4.3) $\alpha_n(a,t)=a$ and $\alpha_n(b,t)=b$ for every $0\leqslant t\leqslant 1$.

Since the points a,b belong to one component of X, there exists a map

$$\beta_1\colon\langle 0,1\rangle\to U_2$$

such that $\beta_1(0)=a$, $\beta_1(1)=b$. Let us assume that for an $n>1$ a map $\beta_{n-1}\colon\langle 0,1\rangle\to U_n$ is already defined, such that $\beta_{n-1}(0)=a$, $\beta_{n-1}(1)=b$, and let us set

$$\beta_n(t)=\alpha_{n-1}(\beta_{n-1}(t),1) \quad \text{for every } 0\leqslant t\leqslant 1.$$

It is clear that this formula defines a map

(4.4) $$\beta_n\colon\langle 0,1\rangle\to U_{n+1}$$

such that

(4.5) $$\beta_n(0)=a \quad\text{and}\quad \beta_n(1)=b.$$

By Lemma (1.1) there exists, for $n=1,2,\ldots$, an isotopy

$$\varphi_n\colon Q\times\langle 0,1\rangle\to Q$$

such that

(4.6) $\varphi_n(x,0)=x$ for every point $x\in Q$,

(4.7) $\varphi_n(x,t)\in U_n$ for every $(x,t)\in U_n\times\langle 0,1\rangle$,

(4.8) $\varphi_n(x,t)=x$ for every $(x,t)\in(Q\setminus U_n)\times\langle 0,1\rangle$,

(4.9) $\varphi_n(a,t)=\beta_n(t)$ for every $0\leqslant t\leqslant 1$.

Now let us set

(4.10) $f_n(x)=\varphi_n(x,1)$ for every point $x\in Q$,

and let us show that the maps $f_n\colon(Q,a)\to(Q,b)$ constitute a fundamental sequence $\underline{f}=\{f_k,(X,a),(X,b)\}$.

Consider an open neighborhood V of X (in Q) and an index n_0 such that $U_{n_0}\subset V$. By (4.7) and (4.8), the isotopy φ_n satisfies the condition

$$\varphi_n(U_{n_0},t)\subset U_{n_0} \quad \text{for every } n\geqslant n_0.$$

Hence the restriction $\varphi_n/(U_{n_0}\times\langle 0,1\rangle)$ joins in $U_{n_0}\subset V$ the map f_n/U_{n_0} with the identity map. Similarly the restriction $\varphi_{n+1}/(U_{n_0}\times\langle 0,1\rangle)$ joins in V the map f_{n+1}/U_{n_0} with the identity map. Setting

$$\psi_n(x,t)=\varphi_n(x,1-2t) \quad \text{for} \quad 0\leqslant t\leqslant \tfrac{1}{2},$$
$$\psi_n(x,t)=\varphi_{n+1}(x,2t-1) \quad \text{for} \quad \tfrac{1}{2}\leqslant t\leqslant 1,$$

we get a homotopy

$$\psi_n\colon\ U_{n_0}\times\langle 0,1\rangle\to V$$

joining the map f_n/U_{n_0} with the map f_{n+1}/U_{n_0}, because

$$\left.\begin{aligned}\psi_n(x,0)&=\varphi_n(x,1)=f_n(x),\\ \psi_n(x,1)&=\varphi_{n+1}(x,1)=f_{n+1}(x)\end{aligned}\right\}\quad \text{for every point } x\in U_{n_0}.$$

Moreover,

$$\psi_n(a,t)=\varphi_n(a,1-2t)=\beta_n(1-2t) \quad \text{for} \quad 0\leqslant t\leqslant \tfrac{1}{2},$$
$$\psi_n(a,t)=\varphi_{n+1}(a,2t-1)=\beta_{n+1}(2t-1) \quad \text{for} \quad \tfrac{1}{2}\leqslant t\leqslant 1.$$

Now let us observe that the map assigning to every $0\leqslant t\leqslant\frac{1}{2}$ the point $\beta_n(1-2t)$ is a path in U_{n_0} with the initial point $\beta_n(1)=b$ and the terminal point $\beta_n(0)=a$, and the map assigning to every $\frac{1}{2}\leqslant t\leqslant 1$ the point $\beta_{n+1}(2t-1)$ is a path in U_{n_0} with the initial point $\beta_{n+1}(0)=a$ and the terminal point $\beta_{n+1}(1)=b$. Hence the map assigning to every $0\leqslant t\leqslant 1$ the point $\psi_n(a,t)$ is a loop Λ_n in U_{n_0} with the basic point b.

It is clear that this loop is homotopic (in $U_{n_0}\subset V$) with the loop Λ_n' which we obtain if we run first by the path $\bar\beta_n\colon\langle 0,1\rangle\to U_{n_0}$ given by the formula $\bar\beta_n(t)=\beta_n(1-t)$ and then by the path $\beta_{n+1}\colon\langle 0,1\rangle\to U_{n_0}$. But this last loop is homotopic to a constant in V, because setting

$$\chi_n(s,t)=a_n\big(\beta_n(t),s\big),$$

one gets a homotopy in the set $U_n\subset U_{n_0}\subset V$ joining β_n with β_{n+1} and satisfying (by (4.3)) the conditions:

$$\chi_n(s,0)=a_n\big(\beta_n(0),s\big)=a_n(a,s)=a,$$
$$\chi_n(s,1)=a_n\big(\beta_n(1),s\big)=a_n(b,s)=b$$

for every $s\in\langle 0,1\rangle$.

It follows that there exists a homotopy contracting in V the loop Λ_n to the point b and keeping this basic point fixed; that is, there exists a family of maps ϑ_s depending continuously on $s\in\langle 0,1\rangle$, with values in V, joining the map $\vartheta_0=\psi_n/[(a)\times\langle 0,1\rangle]$ with the constant map $\vartheta_1=b$ and such that $\vartheta_s(a,0)=\vartheta_s(a,1)=b$ for every $s\in\langle 0,1\rangle$. Setting

$$A=[U_{n_0}\times(0)]\cup[(a)\times\langle 0,1\rangle]\cup[U_{n_0}\times(1)],$$

we infer that the map ψ_n/A is homotopic in V with the map ψ_n' which coincides with ψ_n in the set $[U_{n_0}\times(0)]\cup[U_{n_0}\times(1)]$ and is constant in the set $(a)\times\langle 0,1\rangle$. By the homotopy extension theorem we infer that ψ_n' can be extended to a map $\hat\varphi_n\colon U_{n_0}\times\langle 0,1\rangle\to V$. This map $\hat\varphi_n$ is a homotopy joining in V the map $\psi_n/[U_{n_0}\times(0)]=f_n/U_{n_0}$ with the map $\psi_n/[U_{n_0}\times(1)]=f_{n+1}/U_{n_0}$ and it satisfies the condition $\hat\varphi_n(a,t)=b$ for every $0\leqslant t\leqslant 1$. Hence $f_n/(U_{n_0},a)\simeq f_{n+1}/(U_{n_0},a)$ in (V,b) for $n\geqslant n_0$. It follows that $\underline{f}=\{f_k,(X,a),(X,b)\}$ is a fundamental sequence.

By an analogous argument, one shows that setting

$$g_n=f_n^{-1}\colon\ (Q,b)\to(Q,a),$$

one gets a fundamental sequence $\underline{g}=\{g_n,(X,b),(X,a)\}$.

Finally, the relation $f_ng_n(x)=g_nf_n(x)=x$ for every point $x\in X$ implies that the fundamental sequences $\underline{fg}=\{f_ng_n,(X,b),(X,b)\}$ and $\underline{gf}=\{g_nf_n,(X,a),(X,a)\}$ are generated by the identity maps $i_{(X,b)}\colon(X,b)\to(X,b)$ and $i_{(X,a)}\colon(X,a)\to(X,a)$ respectively. Hence $\underline{fg}\simeq\underline{i}_{(X,b)}$ and $\underline{gf}\simeq\underline{i}_{(X,a)}$. Thus $(X,a)\underset{F}{\simeq}(X,b)$ and the proof of Theorem (4.1) is concluded.

5. An example. Let us show that there exist continua X such that the shape $\operatorname{Sh}(X,a)$ depends on the choice of the point $a\in X$.

Consider the circular disk D with the center $a=(\frac{1}{2},\frac{1}{3},\frac{1}{6})$ and the radius $\frac{1}{13}$ lying in the plane P given in the Euclidean 3-space E^3 by the equation $x_1=\frac{1}{2}$. Let A_0 denote the anchor ring which we obtain by revolving the disk D about the stright line L given by the equations $x_1=\frac{1}{2}$, $x_2=\frac{1}{4}$. Then $A_0\subset E^3\cap Q$ and the set $A_0\cap P$ is the union of two disjoint circular disks D with the center a and D' with the center a' lying symmetrically with respect to L.

The circle C_0 obtained by the rotation of the point a is said to be the *core* of A_0. Let us give to it a fixed orientation.

Now let us assign to every point $p\in E^3\cap L$ the point $s(p)$ in which the half-plane H_p passing through p and having L as its edge intersects the circle C_0. Then to every path σ lying in $E^3\setminus L$ corresponds a number $\nu(\sigma)$ defined as the oriented angle described by the vector with the beginning $(\frac{1}{2},\frac{1}{4},\frac{1}{6})$ and the end $s(p)$ when p runs through the path σ. The fixed orientation of C_0 determines the sign of $\nu(\sigma)$. If σ is a path in $E^3\setminus L$ from p to q and τ is a path in $E^3\setminus L$ from q to r, then $\nu(\sigma\tau)=\nu(\sigma)+\nu(\tau)$. If γ is a loop in $E^3\setminus L$, then $\nu(\gamma)=2k\pi$, where k is an integer. In particular, for the loop γ_0 constituted by the oriented curve C_0 with the beginning a, the number $\nu(\gamma_0)$ is equal to 2π.

By a *regular anchor ring of degree* m (and the radius r) we understand every set $A\subset A_0$ homeomorphic to A_0 and such that for every point $p\in C_0$ the set $A\cap H_p$ is the union of m disjoint circular disks with radii r and such that for $p = a$ the centers of those disks lie on the diameter R of D parallel to the axis x_2. In particular, A_0 is a regular anchor ring of degree 1.

Assume as known the following elementary geometric facts:

(1) If A is a regular anchor ring of degree m, then the set of all centers of the circular disks being components of $A \cap H_p$ for $p \in C_0$ is a simple closed curve C; let us call it the *core* of the anchor ring A. One can give to C an orientation such that the loop γ obtained from C (with the beginning arbitrarily selected on C) satisfies the condition $\nu(\gamma) = 2\pi m$.

(2) Two loops γ and γ' lying in A_0 are homotopic in A_0 to one another if and only if $\nu(\gamma) = \nu(\gamma')$.

(3) If A is a regular anchor ring of degree m and if p belongs to the core of A, then there exists a regular anchor ring A' of degree $3m$ lying in the interior A° of A and such that p belongs to the core of A'.

It follows by (3) that there exists a sequence of regular anchor rings $A_0, A_1, A_2, \ldots$ with cores $C_0, C_1, C_2, \ldots$ such that A_{n+1} lies in the interior A_n° of A_n, the degree of A_n is 3^n and $a \in C_n$ for every $n = 0, 1, 2, \ldots$ It is known that the set

$$S = \bigcap_{n=0}^{\infty} A_n$$

is an indecomposable continuum, called the *3-adic solenoid of van Dantzig* ([5], p. 106).

By (1), we can assign to C_n an orientation such that the loop γ_n with the beginning a, obtained in this way from C_n, satisfies the condition

(5.1) $$\nu(\gamma_n) = 3^n 2\pi .$$

Let us show that we can assign to every $n = 0, 1, \ldots$ a point $a'_n \in C_n \cap D'$, so that a'_n and a decompose C_n into two oriented arcs: L_n (from a to a'_n) and L'_n (from a'_n to a) with the property that the corresponding paths λ_n and λ'_n satisfy the conditions:

(5.2)^n $$3^{n-1} 2\pi < \nu(\lambda_n) < 2 \cdot 3^{n-1} 2\pi ,$$

(5.3) *The segment* $\overline{a_n a_{n+1}}$ *lies in* A_n.

Since for $a'_0 = a'$ the path λ_0 is the oriented half-circumference of C_0, the equality

(5.4) $$\nu(\lambda_0) = \pi$$

holds true, and consequently condition (5.2)^0 is satisfied. Assume that for a given n condition (5.2)^n is satisfied and let us prove that there exists a point $a_{n+1} \in C_{n+1}$ satisfying (5.2)$^{n+1}$ and (5.3).

The set $A_n \cap D'$ consists of 3^n disjoint disks $D_{n,1}, D_{n,2}, \ldots, D_{n,3^n}$, and we may assume that their order is such that the loop γ_n meets them consecutively. Then the point a'_n satisfying (5.3)^n is the center of a disk D_{n,k_n} with $3^{n-1} \leqslant k_n < 2 \cdot 3^{n-1}$. Now let us observe that the loop γ_{n+1} meets consecutively the disks

$$D_{n,1}, D_{n,2}, \ldots, D_{n,3^n}, D_{n,1}, D_{n,2}, \ldots, D_{n,3^n}, D_{n,1}, D_{n,2}, \ldots, D_{n,3^n},$$

containing the disks

$$D_{n+1,1}, D_{n+1,2}, \ldots, D_{n+1,3^{n+1}}$$

respectively. It follows that there is an index k_{n+1} with $3^n \leqslant k_{n+1} < 2 \cdot 3^n$ such that $D_{n+1,k_{n+1}} \subset D_{n,k_n}$. It suffices to set a'_{n+1} equal to the center of $D_{n+1,k_{n+1}}$ in order to satisfy (5.3) and (5.2)$^{n+1}$.

It follows by (5.3) that the sequence of points a'_n converges to a point $b \in S$ such that

(5.5) *The segment* $\overline{a'_n b}$ *lies in* A_n *for every* $n = 1, 2, \ldots$

Moreover, let us observe that inequality (5.4) implies that

(5.6) *For every* $N > 0$ *there exists an index* n_0 *such that for every* $n > n_0$ *all numbers of the form*

$$|2 \cdot 3^n k\pi - \nu(\lambda_n)| ,$$

where k *is an integer, are greater than* N.

It is clear that $D \cap S$ is a subset of the diameter R of the disk D. Consequently there exists in the half-plane H_a a circle K tangent to R at the point a and such that $K \backslash D \neq \emptyset$. Let us observe that

(5.7) $K \cap A_n$ *is an arc for every* $n = 0, 1, 2, \ldots$

Setting

$$X = K \cup S ,$$

let us show that

(5.8) $$\mathrm{Sh}(X, a) \neq \mathrm{Sh}(X, b) .$$

Proof. We shall show more, namely that (X, a) is not fundamentally dominated by (X, b). Otherwise, there exist two pointed fundamental sequences

(5.9) $\underline{f} = \{f_k, (X, a), (X, b)\}$ and $\underline{g} = \{g_k, (X, b), (X, a)\}$

such that

(5.10) $$\underline{g}\,\underline{f} \simeq \underline{i}_{(X,a)} .$$

By a *double anchor ring* we shall understand a set homeomorphic to the Cartesian product of the interval $\langle 0,1\rangle$ by the set M given in the Euclidean 2-space E^2 by the formula

$$M=\{x=(x_1,x_2),\ \varrho(x,(0,0))\leqslant 4,\ \varrho(x,(2,0))\geqslant 1,\ \varrho(x,(-2,0))\geqslant 1\}.$$

One can easily see, by virtue of (5.7), that for every $n=0,1,2,\ldots$ there exists a positive number ε_n such that $\varepsilon_{n+1}<\varepsilon_n<1/(n+1)$ and that the set B_n, being the union of A_n and of the anchor ring T_n defined as the union of all balls in E^3 with radius ε_n and centers belonging to K, is a double anchor ring. Moreover, B_n is a closed neighborhood of X (in E^3) and

$$X=\bigcap_{n=0}^{\infty}B_n.$$

Let us observe that

(5.11) *None of the curves C_0 and K is contractible in B_0.*

Now, consider the retraction $r\colon Q\to Q^3=E^3\cap Q$ given by the formula

$$r(x)=(x_1,x_2,x_3,0,0,\ldots)\quad\text{for every point } x=(x_1,x_2,\ldots)\in Q.$$

Let F_n denote the set consisting of all points x belonging to Q such that $\varrho(x,r(x))\leqslant 1/(n+1)$. It is clear that setting

(5.12) $$U_n=r^{-1}(B_n)\cap F_n\quad\text{for}\quad n=0,1,2,\ldots,$$

we get a descending sequence of neighborhoods of X (in Q) such that

$$X=\bigcap_{n=0}^{\infty}U_n,\qquad B_n=U_n\cap Q^3\quad\text{for}\quad n=0,1,2,\ldots$$

and

B_n is a deformation retract of U_n for $n=0,1,2,\ldots$

It follows by (5.11) and (5.12) that

(5.13) *Neither C_0 nor K is contractible in U_0.*

Now let us denote, for every $n=0,1,2,\ldots$, by σ_n the path consisting of the oriented segment from b to a_n', and by $\varkappa$ the loop with the beginning a which we obtain if we give to the curve K a fixed orientation. Let us observe that setting

(5.14) $$\eta=\lambda_0^{-1}\varkappa\lambda_0\quad\text{and}\quad\zeta=\lambda_0^{-1}\gamma_0\lambda_0,$$

we get two loops in (B_0,b) which are representatives of two generators of the free group $\pi_1(B_0,b)$. Let π_n' denote the subgroup of $\pi_1(B_0,b)$ generated by all elements which have as a representative a loop lying in the set B_n. The group $\pi_1(B_n,b)$ is a free group with two generators. It is easy to see that we can select as one of them the element for which the loop $\sigma_n\gamma_n\sigma_n^{-1}$ (lying by (5.5) in $A_n\subset B_n$) is a representative, and as the other the element with the representative of the form $\sigma_n\lambda_n^{-1}\varkappa\lambda_n\sigma_n^{-1}$. By (5.1) the loop γ_n is homotopic in A_0 with the loop $\gamma_0^{3^n}$, and we infer that the first of those loops is homotopic in (B_0,b) to the loop $\zeta^{3^n}=\sigma_0\gamma_0^{3^n}\sigma_0^{-1}$ and the second is homotopic in (B_0,b) to the loop

$$(\sigma_n\lambda_n^{-1}\lambda_0)(\lambda_0^{-1}\varkappa\lambda_0)(\lambda_0^{-1}\lambda_n\sigma_n^{-1}),$$

where $\sigma_n\lambda_n^{-1}\lambda_0$ is a loop in (A_0,b). Since $\nu(\sigma_n)=0$, we infer that

$$\nu(\sigma_n\lambda_n^{-1}\lambda_0)=\nu(\sigma_n)+\nu(\lambda_n^{-1})+\nu(\lambda_0)=\pi-\nu(\lambda_n).$$

Since $\sigma_n\lambda_n^{-1}\lambda_0$ is a loop, the number $\nu(\sigma_n\lambda_n^{-1}\lambda_0)$ may be written in the form $2\pi m_n$, where m_n is an integer. Consequently $\nu(\sigma_n\lambda_n^{-1}\lambda_0)=\nu(\zeta^{m_n})$, whence

$$\sigma_n\lambda_n^{-1}\lambda_0\simeq\zeta^{m_n}\quad\text{in } (B_0,b).$$

Moreover, it follows by (5.6) that m_n satisfies the following condition

(5.15) *For every $N>0$ there exists an index n_0 such that $n>n_0$ implies that $|3^nk-m_n|>N$ for every integer k.*

Thus we have shown that every element of the group π_n' has a representative which is a product of the potences of two loops: of ζ^{3^n} and of $\zeta^{m_n}\eta\zeta^{-m_n}$. Let us observe that every such product (if it is not trivial) is of the form

(5.16) $$\zeta^{(3^nk+m_n)}\eta^{k_1}\zeta^{l_1}\eta^{k_2}\zeta^{l_2}\ldots\eta^{k_q}\zeta^{(3^nk'-m_n)}.$$

Consider now the element of the group $\pi_1(B_0,b)$ with the representative $f_1(\varkappa)$. Recall that $\underline{f}$ and $\underline{g}$ are pointed fundamental sequences. It follows by (5.9), (5.10) and (5.12) that there exists an increasing sequence of natural numbers $j_1,j_2,\ldots$ such that

$$f_{j_i}/(B_{j_i},a)\simeq f_{j_1}/(B_{j_1},a)\quad\text{in } (B_0,b)$$

and

$$g_{j_i}f_{j_i}/(B_{j_i},a)\simeq i/(B_{j_i},a)\quad\text{in } (B_0,b)$$

for every $i=1,2,\ldots$

It follows that the loop $\hat{\varkappa}=f_{j_1}(\varkappa)$ is a representative of an element belonging to the group π_n' for every $n=1,2,\ldots$ This element is not trivial, because otherwise the element $g_{j_1}f_{j_1}(\varkappa)\simeq\varkappa$ would be trivial in (B_0,a), contrary to (5.13). Consequently it has a representative of the form (5.16) for every n. But this is incompatible with (5.15), because the elements with the representatives η and ζ are independent in the group $\pi_1(B_0,b)$.

Thus the supposition that (X, a) is fundamentally dominated by (X, b) leads to a contradiction. Hence the proof of proposition (5.8) is finished.

6. Sum of shapes of pointed compacta. Let (X, x_0) and (Y, y_0) be two pointed compacta. It is clear that there exists a pointed compactum (Z, z_0) such that $Z = Z' \cup Z''$, where $Z' \cap Z'' = (z_0)$ and there exists a homeomorphism $g\colon (X, x_0) \to (Z', z_0)$ and a homeomorphism $h\colon (Y, y_0) \to (Z'', z_0)$. Manifestly the topological properties of (Z, z_0) depend only on the topological properties of (X, x_0) and of (Y, y_0). Thus we can say that *the topological type of* (Z, z_0) *is the sum of the topological types of* (X, x_0) *and of* (Y, y_0). We write shortly: $(Z, z_0) = (X, x_0) \underset{\text{top}}{+} (Y, y_0)$.

Let us prove the following

(6.1) THEOREM. *If* $(Z, z_0) = (X, x_0) \underset{\text{top}}{+} (Y, y_0)$, *then* $\mathrm{Sh}(Z, z_0)$ *depends only on* $\mathrm{Sh}(X, x_0)$ *and* $\mathrm{Sh}(Y, y_0)$.

Proof. Setting

$$P = \{(x_1, x_2, \ldots) \in Q;\ 0 \leqslant x_1 \leqslant \tfrac{1}{2}\}, \qquad R = \{(x_1, x_2, \ldots) \in Q;\ \tfrac{1}{2} \leqslant x_1 \leqslant 1\},$$

we get a decomposition of Q into two sets P and R homeomorphic to Q. Now let

$$(6.2) \qquad (X, x_0) \underset{\mathrm{F}}{\simeq} (X', x_0') \quad \text{and} \quad (Y, y_0) \underset{\mathrm{F}}{\simeq} (Y', y_0')\,.$$

In order to prove that the shape of $(Z', z_0') = (X', x_0') \underset{\text{top}}{+} (Y', y_0')$ is the same as the shape of (Z, z_0), we can assume that $x_0 = y_0 = x_0' = y_0' \in P \cap R$ and that

$$(X \cup X') \setminus (x_0) \subset P \setminus R \quad \text{and} \quad (Y \cup Y') \setminus (x_0) \subset R \setminus P\,.$$

This hypothesis implies that there exist two maps

$$\alpha, \beta\colon (Q, x_0) \to (Q, x_0)$$

such that

$$(6.3) \quad \begin{aligned} &\alpha(x) = x \text{ for every point } x \in X \cup X', \quad \alpha(x) = x_0 \text{ for every point } x \in R,\\ &\beta(x) = x \text{ for every point } x \in Y \cup Y', \quad \beta(x) = x_0 \text{ for every point } x \in P. \end{aligned}$$

It follows by (6.2) that there exist four fundamental sequences

$$\underline{f} = \{f_k, (X, x_0), (X', x_0)\}, \quad \underline{f}' = \{f_k', (X', x_0), (X, x_0)\},$$
$$\underline{g} = \{g_k, (Y, x_0), (Y', x_0)\}, \quad \underline{g}' = \{g_k', (Y', x_0), (Y, x_0)\}$$

such that

$$(6.4) \quad \begin{aligned} &\underline{f}'\underline{f} \simeq \underline{i}_{(X, x_0)}, \quad \underline{f}\underline{f}' \simeq \underline{i}_{(X', x_0)},\\ &\underline{g}'\underline{g} \simeq \underline{i}_{(Y, x_0)}, \quad \underline{g}\underline{g}' \simeq \underline{i}_{(Y', x_0)}\,. \end{aligned}$$

Setting

$$\hat{f}_k = f_k\alpha, \quad \hat{f}_k' = f_k'\alpha, \quad \hat{g}_k = g_k\beta, \quad \hat{g}_k' = g_k'\beta,$$

we get for every $k = 1, 2, \ldots$ four maps of (Q, x_0) into (Q, x_0).

Let us observe that

$$\hat{\underline{f}} = \{\hat{f}_k, (X, x_0), (X', x_0)\}, \quad \hat{\underline{g}} = \{g_k, (Y, x_0), (Y', x_0)\},$$
$$\hat{\underline{f}}' = \{\hat{f}_k', (X', x_0), (X, x_0)\}, \quad \hat{\underline{g}}' = \{\hat{g}_k', (Y', x_0), (Y, x_0)\}$$

are fundamental sequences homotopic to $\underline{f}, \underline{g}, \underline{f}', \underline{g}'$ respectively.

In fact, if U' is a neighborhood of X', then there is a neighborhood U_1 of X such that $f_k(U_1) \subset U'$ for almost all k. By (6.3) we can assign to U_1 a neighborhood U_0 of X such that for every point $x \in U_0$ the segment $\overline{x\alpha(x)}$ lies in U_1. Setting

$$\chi_k(x, t) = f_k\big(tx + (1-t)\alpha(x)\big) \quad \text{for every } (x, t) \in U_0 \times \langle 0, 1\rangle,$$

we get a homotopy joining in U' the map $\hat{f}_k/U_0$ with the map f_k/U_0. Hence $\hat{\underline{f}} \simeq \underline{f}$. By a similar argument one proves that $\hat{\underline{g}} \simeq \underline{g}$, $\hat{\underline{f}}' \simeq \underline{f}'$ and $\hat{\underline{g}}' \simeq \underline{g}'$.

It follows by (6.4) that

$$(6.5) \qquad \hat{\underline{f}}'\hat{\underline{f}} \simeq \underline{i}_{(X, x_0)}, \quad \hat{\underline{f}}\hat{\underline{f}}' \simeq \underline{i}_{(X', x_0)}\,.$$

Since

$$\hat{f}_k(x) = \hat{g}_k(x) = \hat{f}_k'(x) = \hat{g}_k'(x) = x_0 \quad \text{for every point } x \in P \cap R,$$

we infer that setting

$$\omega_k(x) = \begin{cases} \hat{f}_k(x) & \text{for every } x \in P,\\ \hat{g}_k(x) & \text{for every } x \in R, \end{cases} \qquad \omega_k'(x) = \begin{cases} \hat{f}_k'(x) & \text{for every } x \in P,\\ \hat{g}_k'(x) & \text{for every } x \in R, \end{cases}$$

we get, for every $k = 1, 2, \ldots$, two maps

$$\omega_k, \omega_k'\colon (Q, x_0) \to (Q, x_0)\,.$$

Let us observe that $\underline{\omega} = \{\omega_k, (X \cup Y, x_0), (X' \cup Y', x_0)\}$ and $\underline{\omega}' = \{\omega_k', (X' \cup Y', x_0), (X \cup Y, x_0)\}$ are fundamental sequences.

In fact, if W' is a neighborhood of the set $X' \cup Y'$, then there exist an open neighborhood U of X and an open neighborhood V of Y such that for almost all k

$$f_k/(U, x_0) \simeq f_{k+1}/(U, x_0) \text{ in } W' \quad \text{and} \quad g_k/(V, x_0) \simeq g_{k+1}/(V, x_0) \text{ in } W'.$$

It means that there exist two homotopies

$$\lambda_k\colon U \times \langle 0, 1\rangle \to W', \quad \mu_k\colon V \times \langle 0, 1\rangle \to W'$$

satisfying the conditions

$$\lambda_k(x,0)=f_k(x)\,,\quad \lambda_k(x,1)=f_{k+1}(x)\quad \text{for every point } x\in U\,,$$
$$\mu_k(x,0)=g_k(x)\,,\quad \mu_k(x,1)=g_{k+1}(x)\quad \text{for every point } x\in V\,,$$
$$\lambda_k(x_0,t)=\mu_k(x_0,t)=x_0\quad \text{for every } 0\leqslant t\leqslant 1.$$

Consider the sets

$$\hat{U}=\alpha^{-1}(U)\,,\quad \hat{V}=\beta^{-1}(V)\,,$$

which are open neighborhoods of X and Y respectively. Setting

$$\hat{\lambda}_k(x,t)=\lambda_k[\alpha(x),t]\quad \text{for every point } x\in\hat{U} \text{ and for } 0\leqslant t\leqslant 1\,,$$
$$\hat{\mu}_k(x,t)=\mu_k[\beta(x),t]\quad \text{for every point } x\in\hat{V} \text{ and for } 0\leqslant t\leqslant 1\,,$$

one gets two homotopies,

$$\hat{\lambda}_k\colon\ \hat{U}\times\langle 0,1\rangle\to W'\,,\quad \hat{\mu}_k\colon\ \hat{V}\times\langle 0,1\rangle\to W'\,,$$

such that $\hat{\lambda}_k$ joins in (W',x_0) the map $f_k\alpha/\hat{U}=\hat{f}_k/\hat{U}$ with the map $f_{k+1}\alpha/\hat{U}=\hat{f}_{k+1}/\hat{U}$ and $\hat{\mu}_k$ joins in (W',x_0) the map $g_k\beta/\hat{V}=\hat{g}_k/\hat{V}$ with the map $g_{k+1}\beta/\hat{V}=\hat{g}_{k+1}/\hat{V}$. Moreover, since $\alpha(x)=\beta(x)=x_0$ for every point $x\in P\cap R$, we infer that the formulas

$$\vartheta_k(x,t)=\hat{\lambda}_k(x,t)\quad \text{for}\quad (x,t)\in(\hat{U}\cap P)\times\langle 0,1\rangle\,,$$
$$\vartheta_k(x,t)=\hat{\mu}_k(x,t)\quad \text{for}\quad (x,t)\in(\hat{V}\cap R)\times\langle 0,1\rangle$$

give a homotopy $\vartheta_k\colon [(\hat{U}\cap P)\cup(\hat{V}\cap R)]\times\langle 0,1\rangle\to W'$ joining ω_k with ω_{k+1} in (W',x_0). Since the set $W_0=(\hat{U}\cap P)\cup(\hat{V}\cap R)$ is a neighborhood of the set $X\cup Y$, we infer that $\underline{\omega}$ is a fundamental sequence. By an analogous argument we show that $\underline{\omega}'$ is also a fundamental sequence.

Let us show that

$$\underline{\omega}'\underline{\omega}\simeq\underline{i}_{(X\cup Y,x_0)}\,.$$

Consider a neighborhood W of the set $X\cup Y$. By (6.4) there are a neighborhood U_0 of X and a neighborhood V_0 of Y such that for almost all k there exist homotopies

$$\varphi_k\colon\ U_0\times\langle 0,1\rangle\to W\quad \text{and}\quad \psi_k\colon\ V_0\times\langle 0,1\rangle\to W$$

such that

$$\varphi_k(x,0)=f'_kf_k(x)\ \text{for every } x\in U_0\,,\qquad \psi_k(x,0)=g'_kg_k(x)\ \text{for every } x\in V_0$$
$$\varphi_k(x,1)=x\ \text{for every } x\in U_0\qquad \psi_k(x,1)=x\ \text{for every } x\in V_0\,,$$
$$\varphi_k(x_0,t)=\psi_k(x_0,t)=x_0\quad \text{for every } 0\leqslant t\leqslant 1\,.$$

Setting

$$\hat{\chi}_k(x,t)=\varphi_k[\alpha(x),t]\quad \text{for every } (x,t)\in(U_0\cap P)\times\langle 0,1\rangle\,,$$
$$\hat{\chi}_k(x,t)=\psi_k[\beta(x),t]\quad \text{for every } (x,t)\in(V_0\cap R)\times\langle 0,1\rangle\,,$$
$$W_0=(U_0\cap P)\cup(V_0\cap R)\,,$$

we get a homotopy

$$\hat{\chi}_k\colon\ W_0\times\langle 0,1\rangle\to W\,,$$

satisfying the condition $\hat{\chi}_k(x_0,t)=x_0$ for every $0\leqslant t\leqslant 1$, because $\alpha(x)=\beta(x)=x_0$ for every point $x\in P\cap R$. The homotopy $\hat{\chi}_k$ joins in (W,x_0) the map $\omega'_k\omega_k/(W_0,x_0)$ with the map $\hat{\omega}_k\colon (W_0,x_0)\to(W,x_0)$ given by the formula $\hat{\omega}_k(x)=\hat{\chi}_k(x,1)$ for every point $x\in W_0$.

It follows that $\hat{\underline{\omega}}=\{\hat{\omega}_k,(X\cup Y,x_0),(X\cup Y,x_0)\}$ is a fundamental sequence homotopic to $\underline{\omega}'\underline{\omega}$. Moreover, if $x\in X$ then

$$\hat{\omega}_k(x)=\hat{\chi}_k(x,1)=\varphi_k[\alpha(x),1]=\varphi_k(x,1)=x\,.$$

By an analogous argument one shows that $\hat{\omega}_k(x)=x$ for every point $x\in Y$. This implies that the fundamental sequence $\hat{\underline{\omega}}$ is homotopic to the fundamental identity sequence $\underline{i}_{(X\cup Y,x_0)}$. Hence $\underline{\omega}'\underline{\omega}\simeq\underline{i}_{(X\cup Y,x_0)}$.

By an analogous argument one proves that $\underline{\omega}\underline{\omega}'\simeq\underline{i}_{(X'\cup Y',x_0)}$. Thus the proof of Theorem (6.1) is finished.

Remark. Let us observe that if one replaces hypothesis (6.4) by the weaker one that $\underline{f}'\underline{f}\simeq\underline{i}_{(X,x_0)}$ and $\underline{g}'\underline{g}\simeq\underline{i}_{(Y,y_0)}$ (that is, the hypothesis that $(X,x_0)\underset{F}{\simeq}(X',x'_0)$ and $(Y,y_0)\underset{F}{\simeq}(Y',y'_0)$ by the hypothesis that $(X,x_0)\underset{F}{\leqslant}(X',x'_0)$ and $(Y,y_0)\underset{F}{\leqslant}(Y',y'_0)$), then in the same way we obtain the following proposition:

(6.6) *If* $\mathrm{Sh}(X,x_0)\leqslant\mathrm{Sh}(X',x'_0)$ *and* $\mathrm{Sh}(Y,y_0)\leqslant\mathrm{Sh}(Y',y'_0)$, *then*

$$\mathrm{Sh}\big((X,x_0)\underset{\mathrm{top}}{+}(Y,y_0)\big)\leqslant\mathrm{Sh}\big((X',x'_0)\underset{\mathrm{top}}{+}(Y',y'_0)\big)\,.$$

7. Cartesian product of shapes of pointed compacta. By the Cartesian product of two pointed compacta (X,x_0) and (Y,y_0) one understands the pointed compactum $\big(X\times Y,(x_0,y_0)\big)$ and one writes

$$(X,x_0)\times(Y,y_0)=\big(X\times Y,(x_0,y_0)\big)\,.$$

Let us prove the following

(7.1) THEOREM. *$\mathrm{Sh}\big(X\times Y,(x_0,y_0)\big)$ depends only on $\mathrm{Sh}(X,x_0)$ and $\mathrm{Sh}(Y,y_0)$.*

Proof. It suffices to show that if X, X', Y, Y' are subsets of Q, and if $\mathrm{Sh}(X,x_0)=\mathrm{Sh}(X',x'_0)$ and $\mathrm{Sh}(Y,y_0)=\mathrm{Sh}(Y',y'_0)$, then

$\operatorname{Sh}(X\times Y,(x_0,y_0))=\operatorname{Sh}(X'\times Y',(x'_0,y'_0))$. By our hypotheses, there exist four fundamental sequences,

$$\underline{f}=\{f_k,(X,x_0),(X',x'_0)\},\qquad \underline{f}'=\{f'_k,(X',x'_0),(X,x_0)\},$$
$$\underline{g}=\{g_k,(Y,y_0),(Y',y'_0)\},\qquad \underline{g}'=\{g'_k,(Y',y'_0),(Y,y_0)\},$$

such that

$$\text{(7.2)}\qquad \underline{f}'\underline{f}\simeq\underline{i}_{(X,x_0)},\quad \underline{f}\underline{f}'\simeq\underline{i}_{(X',x'_0)},\qquad \underline{g}'\underline{g}\simeq\underline{i}_{(Y,y_0)},\quad \underline{g}\underline{g}'\simeq\underline{i}_{(Y',y'_0)}.$$

Setting

$$\hat{f}_k(x,y)=(f_k(x),g_k(y)),\qquad \hat{f}'_k(X,y)=(f'_k(x),g'_k(y))\qquad\text{for every } x,y\in Q$$

and for $k=1,2,\ldots$, we get the maps

$$\hat{f}_k,\hat{f}'_k\colon (Q\times Q,(x_0,y_0))\to(Q\times Q,(x'_0,y'_0))$$

for every $k=1,2,\ldots$

Consider now the set $Q\times Q$ homeomorphic to Q, containing the two sets $X\times Y$ and $X'\times Y'$. Let W' be a neighborhood of $X'\times Y'$ in $Q\times Q$. Then there exist a neighborhood U' of X' in Q and a neighborhood V' of Y' in Q such that

$$\text{(7.3)}\qquad U'\times V'\subset W'.$$

Since $\underline{f}$ and $\underline{g}$ are fundamental sequences, there exist a neighborhood U of X (in Q) and a neighborhood V of Y (in Q) such that the homotopies

$$f_k/(U,x_0)\simeq f_{k+1}/(U,x_0)\quad\text{in } (U',x'_0),$$
$$g_k/(V,y_0)\simeq g_{k+1}/(V,y_0)\quad\text{in } (V',y'_0)$$

both hold true for almost all k. It follows that

$$\hat{f}_k/(U\times V,(x_0,y_0))\simeq\hat{f}_{k+1}/(U\times V,(x_0,y_0))\quad\text{in } (U'\times V',(x'_0,y'_0))$$

for almost all k. Since $W=U\times V$ is a neighborhood of $X\times Y$ in $Q\times Q$, we infer by (7.3) that

$$\hat{f}_k/(W,(x_0,y_0))\simeq\hat{f}_{k+1}/(W,(x_0,y_0))\quad\text{in } (W',(x'_0,y'_0))$$

for almost all k, and consequently $\underline{\hat{f}}=\{\hat{f}_k,(X\times Y,(x_0,y_0)),(X'\times Y',(x'_0,y'_0))\}$ is a fundamental sequence.

By an analogous argument one shows that

$$\underline{\hat{f}}'=\{\hat{f}'_k,(X'\times Y',(x'_0,y'_0)),(X\times Y,(x_0,y_0))\}$$

is a fundamental sequence.

Since

$$\hat{f}'_k\hat{f}_k(x,y)=(f'_kf_k(x),g'_kg_k(y))\qquad\text{for every } (x,y)\in Q\times Q \text{ and } k=1,2,\ldots,$$

one shows, in the same way, that the fundamental sequence $\underline{\hat{f}}'\underline{\hat{f}}=\{\hat{f}'_k\hat{f}_k,(X\times Y,(x_0,y_0)),(X\times Y,(x_0,y_0))\}$ is homotopic to $\underline{i}_{(X\times Y,(x_0,y_0))}$, and the fundamental sequence $\underline{\hat{f}}\underline{\hat{f}}'=\{\hat{f}_k\hat{f}'_k,(X'\times Y',(x'_0,y'_0)),(X'\times Y',(x'_0,y'_0))\}$ is homotopic to $\underline{i}_{(X'\times Y',(x'_0,y'_0))}$. It follows that $\operatorname{Sh}(X\times Y,(x_0,y_0))$ is equal to $\operatorname{Sh}(X'\times Y',(x'_0,y'_0))$, and the proof of Theorem (7.1) is finished.

Let us observe that if we replace the hypotheses that $\operatorname{Sh}(X,x_0)=\operatorname{Sh}(X',x'_0)$ and $\operatorname{Sh}(Y,y_0)=\operatorname{Sh}(Y',y'_0)$ by the weaker ones that $\operatorname{Sh}(X,x_0)\leqslant\operatorname{Sh}(X',x'_0)$ and $\operatorname{Sh}(Y,y_0)\leqslant\operatorname{Sh}(Y',y'_0)$, then we get in the same way the following proposition:

(7.4) *If* $\operatorname{Sh}(X,x_0)\leqslant\operatorname{Sh}(X',x'_0)$ *and* $\operatorname{Sh}(Y,y_0)\leqslant\operatorname{Sh}(Y',y'_0)$ *then*

$$\operatorname{Sh}(X\times Y,(x_0,y))\leqslant\operatorname{Sh}(X'\times Y',(x'_0,y'_0)).$$

8. Simple and prime pointed shapes. Some problems. It follows by Theorem (6.1) and Theorem (7.1) that the formulas

$$\operatorname{Sh}(X,x_0)+\operatorname{Sh}(Y,y_0)=\operatorname{Sh}((X,x_0)\underset{\text{top}}{+}(Y,y_0))$$

and

$$\operatorname{Sh}(X,x_0)\times\operatorname{Sh}(Y,y_0)=\operatorname{Sh}((X\times Y),(x_0,y_0))$$

define uniquely two commutative operations, called *addition* and *multiplication*, respectively, assigning to two pointed shapes a pointed shape. Since (X,x_0) is an r-image (See [4], p. 8) of $(X,x_0)\underset{\text{top}}{+}(Y,y_0)$ and also of $(X\times Y,(x_0,y_0))$ and since the shape of an r-image (X,x_0) of (Z,z_0) is less than or equal to the shape of (Z,z_0), we infer that

$$\text{(8.1)}\qquad \operatorname{Sh}(X,x_0)\leqslant\operatorname{Sh}(X,x_0)+\operatorname{Sh}(Y,y_0)\quad\text{and}\quad \operatorname{Sh}(X,x_0)\leqslant\operatorname{Sh}(X,x_0)\times\operatorname{Sh}(Y,y)$$

for every pointed shapes $\operatorname{Sh}(X,x_0)$ and $\operatorname{Sh}(Y,y_0)$.

It is clear that the trivial pointed shape (that is the shape $\operatorname{Sh}(X,x_0)$ where X consists of only one point x_0) is the identity element for both operations, the addition and the multiplication; that is:

$$\operatorname{Sh}((x_0),x_0)+\operatorname{Sh}(Y,y_0)=\operatorname{Sh}(Y,y_0),$$
$$\operatorname{Sh}((x_0),x_0)\times\operatorname{Sh}(Y,y_0)=\operatorname{Sh}(Y,y_0)$$

for every pointed shape $\operatorname{Sh}(Y,y_0)$.

If $\mathrm{Sh}(Z, z_0) = \mathrm{Sh}(X, x_0) + \mathrm{Sh}(Y, y_0)$, then $\mathrm{Sh}(X, x_0)$ and $\mathrm{Sh}(Y, y_0)$ will be said to be *constituents* of $\mathrm{Sh}(Z, z_0)$. And if $\mathrm{Sh}(Z, z_0) = \mathrm{Sh}(X, x_0) \times \mathrm{Sh}(Y, y_0)$, then $\mathrm{Sh}(X, x_0)$ and $\mathrm{Sh}(Y, y_0)$ will be said to be *factores* of $\mathrm{Sh}(Z, z_0)$. Thus (8.1) implies that every constituent and also every factor of a pointed shape is less than or equal to that pointed shape.

Let us say that a pointed shape $\mathrm{Sh}(X, x_0)$ is *movable* if (X, x_0) is movable. It follows by (2.3) and (8.1) that all constituents and all factors of a movable pointed shape are movable.

A pointed shape $\mathrm{Sh}(X, x_0)$ is said to be *simple* if each of its constituents either is trivial or coincides with $\mathrm{Sh}(X, x_0)$. A pointed shape $\mathrm{Sh}(X, x_0)$ is said to be *prime* if it is non-trivial and each of its factors either is trivial or coincides with $\mathrm{Sh}(X, x_0)$.

Let us formulate some problems concerning those notions:

1. Is it true that every pointed non-trivial shape has at least one non-trivial simple constituent and at least one non-trivial prime factor?

2. Is it true that there is at most one decomposition of a pointed shape into a finite sum of simple pointed shapes?

3. Is it true that for every compact manifold X the shape $\mathrm{Sh}(X, x_0)$ is simple?

4. Is it true that the shape of every acyclic curve is trivial?

5. Is true that $\mathrm{Sh}(X, x_0) = \mathrm{Sh}(Y, y_0) + \mathrm{Sh}(Z, z_0)$ implies that the fundamental dimension $\mathrm{Fd}(X)$ of X is equal to $\mathrm{Max}\big(\mathrm{Fd}(Y), \mathrm{Fd}(Z)\big)$?

By the *fundamental dimension* of X we understand here the number $\mathrm{Fd}(X)$ given by the formula (compare [3])

$$\mathrm{Fd}(X) = \operatorname*{Min}_{\mathrm{Sh}(X) \leqslant \mathrm{Sh}(Y)} \dim Y .$$

6. Is it true that if $Z \in \mathrm{ANR}$ and $\mathrm{Sh}(Z, z_0) = \mathrm{Sh}(X, x_0) + \mathrm{Sh}(Y, y_0)$, then $\mathrm{Sh}(X, x_0)$ is determined by $\mathrm{Sh}(Y, y_0)$ and $\mathrm{Sh}(Z, z_0)$?

7. Is it true that for every ANR-set X the shape $\mathrm{Sh}(X, x_0)$ has only a finite number of simple constituents and prime factors?

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The global dimension of the group rings of abelian groups III

by

S. Balcerzyk (Toruń)

This paper is a continuation of papers [1], [2] and is concerned with computation of the global dimension of the group ring of arbitrary abelian group with commutative Noetherian coefficient ring. Also the dimension of those rings as algebras is computed.

In this paper all rings and groups are assumed to be commutative.

For any R-algebra Λ, we denote by $\dim \Lambda$ or R-$\dim \Lambda$ the *projective dimension of* Λ as Λ^e-module. If $\Lambda = R(\Pi)$ is a group ring, then it is known (see [4]) that $\dim R(\Pi) = \dim_{R(\Pi)} R$ where Π operates trivially on R.

1. In this section we prove some preliminary lemmas.

LEMMA 1. *Let Π be a group and $\mathfrak{a} \subset R$ be an ideal of a ring R. If $\bar{R} = R/\mathfrak{a}$, then R-$\dim R(\Pi) \geqslant \bar{R}$-$\dim \bar{R}(\Pi)$.*

Proof. If P is a projective resolution of $R(\Pi)$-module R, then $P \otimes_R \bar{R}$ is a $\bar{R}(\Pi)$-projective complex. Since $H_n(P \otimes_R \bar{R}) = \mathrm{Tor}_n^R(R, \bar{R})$, then $P \otimes_R \bar{R}$ is a projective resolution of $\bar{R}$ and the lemma follows.

LEMMA 2. *If Π_0 is a subgroup of a group Π, then*

$$\mathrm{gl.\,dim}\, R(\Pi) \geqslant \mathrm{gl.\,dim}\, R(\Pi_0) ,$$
$$\dim R(\Pi) \geqslant \dim R(\Pi_0) .$$

Proof. It is easy to prove the formula

$$\dim_{R(\Pi_0)} A = \dim_{R(\Pi)} A \otimes_{R(\Pi_0)} R(\Pi)$$

for any $R(\Pi_0)$-module A and this implies the first inequality. The second one follows by the fact that any $R(\Pi)$-projective module is $R(\Pi_0)$-projective.

LEMMA 3. *If R is a field and $mR = R$ if m is an order of an element in a group Π, then in the group ring $R(\Pi)$ any set of orthogonal idempotents is at most countable.*

Proof. It is easy to see that all idempotents of $R(\Pi)$ belong to the subring $R(T)$ where T is the maximal torsion subgroup of Π. The group T