It follows from [4], p. 57, and the fact that there is an order preserving homeomorphism between the irrationals in $I$ and the non-end points of $P$ that there is a universal null set $N$ in $P$ such that there is a continuous map $g$ of $N$ onto $P$. Denote the graph of $g$ by $G$.

There is a standard way of mapping $I$ into $I 	imes I$ by a homeomorphism $f$ such that $f$ maps $P$ onto $P 	imes P$ and, moreover, the equation $g(x) = y$ has only a finite number of solutions if $y \notin I - P$, where $y$ is defined on $I$ by $f(t) = (x(t), x(t))$. Beginning with a diagram, a sketch of a construction of such a function $f$ follows.

![Graph of the range of $f$]

Then $f$, maps $I_n$ linearly onto the noted diagonal of $I_{1n} \times I_{1n}$, $O_n$ linearly onto the noted diagonal of $I_{1n} \times O_{1n}$. To define $f$, let $f_1$ be patterned after $f_n$ on the sets $I_{1n}$, $I_n$ and $f_1 = f_n$ on the rest of $I$. Iterate this process, and let $f(s) = \lim f_n(s)$.

Because $G$ is a subset of $P$ $\times$ $P$, the set $E = f^{-1}(G)$ is a subset of $P$ which is homeomorphic to $G$. Hence $E$ is a universal null set. (In fact, since $G$ is a continuous, $E$ is also homeomorphic to $N$.) Moreover, $E$ is a continuous map of $I$ onto $I$, $v(E) = P$, and the equation $v(x) = y$ has only finitely many solutions if $y \notin I - P$. Hence it follows from [1], Theorem III, p. 633, that there exists a strictly increasing continuous function $f$ and a CBV function $h$ such that $v = f \circ h$. If $h(E)$ were a universal null set, then it would follow that $P = \varphi[h(E)]$ has measure zero. Hence $h(E)$ is not a universal null set.

References


Some remarks concerning the shape of pointed compacta

By Karol Borsuk (Warszawa)

By $Q$ we denote the Hilbert cube, that is, the subset of the Hilbert space consisting of all points $(x_1, x_2, ...)$ with $0 \leq x_n \leq 1/n$ for $n = 1, 2, ...$

Two pointed compacta $(X, a)$, $(Y, b)$ are said to be fundamentally equivalent (notation: $(X, a) < (Y, b)$) if there exist in $Q$ two pointed compacta $(X', a')$ and $(Y', b')$ homeomorphic to $(X, a)$ and $(Y, b)$ respectively and two fundamental sequences (see [1], p. 225)

$$f = (f_n, (X', a'), (Y', b'))$$ and $$g = (g_n, (X', b'), (X', a'))$$

such that $f_n \approx (x_n, a')$, $(X', b')$, where $g_n$ denotes the identity fundamental sequence $(x, (Z, a), (Z, a))$.

If we assume only that the second relation $g_n \approx (h_n, a')$ holds true, then we say that $(X, a)$ is fundamentally dominated by $(Y, b)$ and we write $(X, a) \lesssim (Y, b)$.

The collection of all pointed compacta $(Y, b)$ fundamentally equivalent to a given pointed compactum $(X, a)$ is called the shape of $(X, a)$ (see [3]); it is denoted by $Sh(X, a)$. Thus the relation $Sh(X, a) = Sh(Y, b)$ means that $(X, a) \lesssim (Y, b)$. If $(X, a) \lesssim (Y, b)$, then $Sh(X, a)$ is said to be less than or equal to $Sh(Y, b)$ and we write $Sh(X, a) \leq Sh(Y, b)$.

The aim of this note is to establish a condition under which $Sh(X, a)$ does not depend on the choice of the point $a$, and to study the operations of addition and multiplication of shapes of pointed compacta.

I wish to thank A. Lelek, who read the manuscript of this note, for his penetrating remarks.

1. A lemma on isotopy. By a map we understand here always a continuous function. A map

$$\varphi: X \times ([u, v], e) \to Y,$$

where $u, v$ are numbers with $u < v$,

is said to be a homotopy in a set $Z$ if all values of $\varphi$ belong to $Z$. If $a \in X$, $b \in Y$ and if $\varphi: X \times ([u, v], e) \to Y$ is a homotopy satisfying the condition

$$\varphi(a, t) = b$$

for every $u \leq t \leq v$,
then \( \psi \) is said to be a homotopy of \((X, a)\) in \((Y, b)\), and we write

\[
\psi: (X, a) \times ([0, 1]) \rightarrow (Y, b).
\]

If, for every \( t \in [0, 1] \), the map \( \psi_t: X \rightarrow Y \) defined by the formula

\[
\psi_t(a) = \psi(a, t) \quad \text{for every point } a \in X
\]

is a homeomorphism, then the homotopy \( \psi \) is said to be an isotopy.

Let us prove the following

(1.1) LEMMA. Let \( a \) be a point of an open subset \( G \) of the Hilbert cube \( Q \), and let \( u_0, v_0 \) be two numbers with \( u_0 < v_0 \) and let \( \beta: (u_0, v_0) \rightarrow G \) be a map with \( \beta(u_0) = a \), \( \beta(v_0) = b \). Then there exists an isotopy

\[
\varphi: Q \times (u_0, v_0) \rightarrow G
\]

satisfying the following conditions:

1. \( \varphi(s, u_0) = x \) for every point \( x \in Q \),
2. \( \varphi(s, t) = x \) for every \( (s, t) \in (Q, Q) \times (u_0, v_0) \),
3. \( \varphi(s, t) = \beta(t) \) for every \( t \in (u_0, v_0) \).

Proof. First, let us consider the special case when the values of the map \( \beta \) belong to the interior \( K^c \) of a ball \( K \subset G \) in the space \( Q \) with center \( e \) and radius \( r \). Let us set

\[
\begin{align*}
\varphi(s, t) &= s + r - g(s, a) \frac{1}{r} (\beta(t) - a) \quad \text{for every } (s, t) \in K \times (u_0, v_0), \\
\varphi(s, t) &= s \quad \text{for every } (s, t) \in (Q \setminus K) \times (u_0, v_0).
\end{align*}
\]

If \( a \in (Q \setminus K) \cap K \), both formulas (1.2) give \( \varphi(s, t) = s \). It follows that \( \varphi: Q \times (u_0, v_0) \rightarrow K \) is a homotopy. Moreover, if \( s \in K \), then

\[
\varphi(s, t, a) \leq \varphi(s, t, x) + g(s, a) = s + r - g(s, a) \frac{1}{r} (\beta(t) - a) + g(s, a) \leq r - g(s, a) + g(s, a) = r.
\]

Hence \( \varphi(K, t) \subset K \). Consequently, in order to prove that \( \varphi \) is an isotopy, it suffices to show that for every \( y \in K \) the equality \( \bar{\varphi}(s, t) = \varphi(y, t) \) implies \( x = y \). In fact, in this case

\[
x - y = g(s, a) - g(s, y) \frac{1}{r} (\beta(t) - a),
\]

whence

\[
\varphi(s, y) = \frac{g(\beta(t), a)}{r} \left[ g(s, a) - g(s, y) \right].
\]

Since \( g(\beta(t), a) < 1 \) we infer that \( x \neq y \) implies

\[
\varphi(s, y) \leq |g(s, a) - g(s, y)|,
\]

which contradicts the triangle inequality.

Thus \( \varphi \) is an isotopy and one can see by (1.2) that it satisfies the conditions (1), (2) and (3).

In order to finish the proof of Lemma (1.1), let us observe that there exists a finite sequence of numbers

\[
u_0 < u_1 < \ldots < u_n = r\]

such that for every \( i = 0, 1, \ldots, n \) all values \( \beta(t) \) for \( u_i \leq t \leq u_{i+1} \) lie in the interior \( K^c \) of a ball \( K \subset G \). By the special case just considered, there exists an isotopy

\[
\varphi: Q \times (u_i, u_{i+1}) \rightarrow G
\]

satisfying the conditions:

1. \( \varphi(s, u_0) = x \) for every point \( x \in Q \),
2. \( \varphi(s, u_{i+1}) = x \) for every \( (s, u_{i+1}) \in (Q, Q) \times (u_i, u_{i+1}) \),
3. \( \varphi(s, t) = \beta(t) \) for \( t \in (u_i, u_{i+1}) \).

It suffices to set

\[
f_j(x) = \varphi(x, u_{i+1}) \quad \text{for } j = 0, 1, \ldots, n-1 \]

and

\[
\varphi(x, t) = \varphi(x, u_{i+1}) \quad \text{for } u_{i} \leq t \leq u_{i+1},
\]

\[
\varphi(x, t) = \varphi(f_{i-1}, f_{i-2}, \ldots, f_1, x, t) \quad \text{for } u_{i} \leq t \leq u_{i+1} \quad \text{and} \quad i = 1, 2, \ldots, n,
\]

in order to obtain an isotopy satisfying the required conditions.

2. Movable pointed compacta. A pointed compactum \((X, x_0) \subset (Q, q_0)\) is said to be movable (compare [2], p. 157) if for every neighborhood \( U \) of \( X \) there exists a neighborhood \( U_0 \) of \( X \) such that, for every neighborhood \( V \) of \( X \), there is a homotopy

\[
\varphi: U_0 \times (0, 1) \rightarrow U
\]

such that

\[
\varphi(x, 0) = x \quad \text{and} \quad \varphi(x, 1) \in V \quad \text{for every point } x \in U_0,
\]

\[
\varphi(x_0, t) = x_0 \quad \text{for every } 0 \leq t \leq 1.
\]

By a slight modification of the arguments used in [2], one shows that:

(2.3) If \((X, x_0)\) is movable and \(Sh(Y, y_0) \leq Sh(X, x_0)\), then \((Y, y_0)\) is movable (compare [2], p. 144).

(2.4) If \(a_0 \in X \times ANR,\) then \((X, a_0)\) is movable (compare [2], p. 137).

(2.5) Every pointed plane compactum is movable (compare [2], p. 145).
(2.6) The pointed solenoids of van Dantzig ([5], p. 166) are not movable (compare [3], p. 138).

(2.7) If \((X_1, x_0), (Y, y_0)\) are movable, then \(X \times Y\) pointed by \((x_1, y_0)\) is movable.

Let us prove the following

(2.8) Theorem. If \((X_1, x_0), (X_2, x_2) \subset (Q, x_0)\) are movable, and if \(X_1 \cap X_2 = (x_0)\), then \(X_1 \cup X_2, (x_0)\) is movable.

Proof. Let \(U\) be a neighborhood (in \(Q\)) of the set \(X = X_1 \cup X_2\). Since \((X_1, x_0)\) and \((X_2, x_2)\) are movable, there exist two neighborhoods: \(U_1\) of \(X_1\) and \(U_2\) of \(X_2\) such that for every neighborhood \(V\) of \(X\) there are two homotopies

\[ \varphi_t : U_1 \times (0, 1) \to U, \quad t = 0, 1, \]

such that \(\varphi_t(x_0, 0) = x, \varphi_t(x_1, 1) \in V\) for every point \(x \in U\), and \(\varphi_t(x_0, t) = x_0\) for every \(0 \leq t \leq 1\).

Let \(B_\xi\) denote the ball (in \(Q\)) with center \(a_\xi\) and radius \(\eta > 0\). Since \(U_1 \cap U_2\) is a neighborhood of \(x_0\), there exists a positive number \(\epsilon\) such that \(B_\epsilon \subset U_1 \cap U_2\). It is clear that there exist a closed neighborhood \(\hat{U}_1 \subset U_1\) of \(x_1\) and a closed neighborhood \(\hat{U}_2 \subset U_2\) of \(x_2\) such that

\[ \hat{U}_1 \cap \hat{U}_2 \subset B_\epsilon. \]

The set

\[ U_\epsilon = \hat{U}_1 \cup \hat{U}_2 \]

is a neighborhood of \(X\). Let us define a map \(a : U_\epsilon \times (0, 1) \to U\) setting:

\[ a(x, t) = x \quad \text{if} \quad \varphi_t(x_0, 0) \leq \epsilon \quad \text{and} \quad 0 \leq t \leq 1, \]

\[ a(x, t) = \frac{2 \epsilon - \varphi_t(x_0, t) + \varphi_t(x_0, 1)}{\epsilon} \quad \text{if} \quad \epsilon \leq \varphi_t(x_0, 0) < 2 \epsilon \quad \text{and} \quad 0 \leq t \leq 1, \]

\[ a(x, t) = x \quad \text{if} \quad \varphi_t(x_0, t) \geq 2 \epsilon \quad \text{and} \quad 0 \leq t \leq 1. \]

Since for \(\varphi_t(x_0, \epsilon) = \epsilon\) the two formulas (2.10) and (2.11) coincide, and for \(\varphi_t(x_0, 2\epsilon) = 2\epsilon\) the same holds also for formulas (2.11) and (2.12), we infer that \(a\) is a map of \(U_\epsilon \times (0, 1)\) into \(U\).

Moreover, let us observe that

\[ a(x, 0) = x \quad \text{for every point} \quad x \in U_\epsilon, \]

\[ a(x, 1) = x_0 \quad \text{for every point} \quad x \in B_\epsilon, \]

and, since \(B_\epsilon \subset U_1 \cup U_2\),

\[ a(x, 1) \in U \quad \text{if} \quad x \in \hat{U}_1, \]

It follows by (2.9), (2.13), (2.14), and (2.15) that setting

\[ a(x, t) = \varphi_t(x_0, 0) \]

for \((x, t) \in U_\epsilon \times (0, 1)\),

\[ a(x, t) = \varphi_t(x_0, 1) \]

for \((x, t) \in (\hat{U}_1 \cup \hat{U}_2) \times (0, 1)\),

one gets a map \(a : U_\epsilon \times (0, 1) \to U\) such that \(a(x, 0) = x, a(x, 1) \in V\) for every point \(x \in U_\epsilon\) and \(a(x_0, t) = x_0\) for every \(0 \leq t \leq 1\). Hence \((X, x_1)\) is movable and the proof of Theorem (2.8) is finished.

Remark. Since the solenoid of Van Dantzig can be represented as the union of two compacta \(A, B\) homeomorphic to the Cartesian product of the Cantor set \(Q\) by a segment with \(A \cap B\) homeomorphic to \(Q\), we see that the union of two movable compacta with the movable common part is not necessarily movable. Even the question remains open whether the union of two movable compacta having only one point in common is necessarily movable.

3. A lemma on pointed movable compacta. Let us prove the following

(3.1) Lemma. Let \(a, b\) be two points belonging to one component of a compactum \(X \subset Q\). If \((X, a)\) is movable, then for every neighborhood \(U\) of \(X\) there exists a neighborhood \(U_a, U_b\) of \(X\) such that for every neighborhood \(W\) of \(X\) there is a homotopy \(a : U_a \times (0, 1) \to U\) satisfying the following conditions:

\[ a(x, 0) = x, \quad a(x, 1) \in W \quad \text{for every point} \quad x \in U_a, \]

\[ a(x, t) = a, \quad a(0, t) = b \quad \text{for every} \quad 0 \leq t \leq 1. \]

Proof. Since \((X, a)\) is movable, there exists a neighborhood \(U_a \subset U\) such that for every neighborhood \(W\) of \(X\) there is a homotopy

\[ \tilde{a} : U_a \times (0, 1) \to U, \]

such that

\[ \tilde{a}(x, 0) = x, \quad \tilde{a}(x, 1) \in W \quad \text{for every point} \quad x \in U_a, \]

\[ \tilde{a}(x, t) = a \quad \text{for every} \quad 0 \leq t \leq 1. \]

We can assume that \(U\) and \(W\) are open in \(Q\) and that \(a \neq b\). Let \(W\) denote the component of \(W\) containing \(a\). Then there is an arc \(L \subset U_a \cap W\) with endpoints \(a, b\). Setting \(\lambda(x) = \tilde{a}(x, 1)\) for every point \(x \in L\), we have a map \(\lambda : L \to W\) for which the map \(\tilde{\lambda} : \tilde{a} : U_a \to W\) given by the formula \(\tilde{\lambda}(x) = \tilde{a}(x, 1)\) is an extension. But all maps of \((L, a)\) into \((W, a)\) are homotopic, whence \(\lambda\) is homotopic to the inclusion map \(\beta : L \to W\). Since \(W\), as an open subset of \(Q\), is an absolute neighborhood retract for metric spaces, we can infer from the homotopy extension theorem that \(\tilde{\lambda}\) is homotopic to \(\tilde{a}\) in \((W, a)\) to a map \(\tilde{\lambda} : U_a \to W\) such that \(\tilde{\lambda}(L) = \beta\). It follows that we can...
assume from the beginning that the given homotopy \( \hat{\alpha} : U_4 \times \langle 0, 1 \rangle \to U \) satisfies the condition

\[
\hat{\alpha}(x, 0) = x \quad \text{for every point } x \in L.
\]

Consider now the disk

\[
D = L \times \langle 0, 1 \rangle \subset U_4 \times \langle 0, 1 \rangle.
\]

There exists a homotopy

\[
\chi : D \times \langle 0, 1 \rangle \to D
\]

joining the identity map \( i : D \to D \) with a map \( r \) retracting \( D \) to the set

\[
Z = [\{ 0 \times \langle 0, 1 \rangle \}] \cup \{ \langle 0 \times 0 \rangle \} \cup [L \times \{ 1 \}]
\]

and satisfying the condition \( \chi(x, t) = (x, t) \) for every \( (x, t) \in Z \).

Let \( \beta \) denote the projection of \( D \) onto \( L \) given by the formula

\[
\beta(x, t) = x \quad \text{for every } (x, t) \in L \times \langle 0, 1 \rangle.
\]

Then

\[
\beta(x, t) = \hat{\alpha}(x, t) = x \quad \text{for every } (x, t) \in Z.
\]

Setting

\[
\phi_1(x, t) = \hat{\alpha}(x, t) = x \quad \text{for every } (x, t) \in D \text{ and } 0 \leq t \leq \frac{1}{2},
\]

\[
\phi_2(x, t) = \beta(x, t) = x \quad \text{for every } (x, t) \in D \text{ and } \frac{1}{2} \leq t \leq 1,
\]

\[
\phi_3(x, t) = \hat{\alpha}(x, t) \quad \text{for every } (x, t) \in [U_4 \times \{ 0 \}] \cup [U_4 \times \{ 1 \}],
\]

we get a homotopy joining in \( U \) the map \( \phi_3 = \hat{\alpha} : U_4 \times \langle 0, 1 \rangle \to D \) to \( [U_4 \times \{ 1 \}] \) with the map \( \phi_1 : [U_4 \times \{ 0 \}] \cup D \to U \) given by the formulas:

\[
\phi_1(x, t) = x \quad \text{for every } (x, t) \in D,
\]

\[
\phi_3(x, t) = \hat{\alpha}(x, t) \quad \text{for every } (x, t) \in [U_4 \times \{ 0 \}] \cup [U_4 \times \{ 1 \}].
\]

Since \( \hat{\alpha} \) is an extension of the map \( \phi_1 \) with values in \( U \) and since \( U \) is an open subset of \( Q \), an absolute neighborhood retract for metric spaces, we infer that the map \( \phi_3 \) can be extended to a map \( \alpha : U_4 \times \langle 0, 1 \rangle \to U \). It is clear that \( \alpha \) satisfies all the required conditions. Thus the proof of Theorem 3.1 is finished.

As an immediate consequence of Lemma 3.1, we get the following

(3.2) THEOREM. If \( a, b \) are two points belonging to one component of a compactum \( X \subset Q \), then the movability of \( (X, a) \) implies the movability of \( (X, b) \).

4. Shape of pointed movable compacta. Let us prove the following

(4.1) THEOREM. If \( a, b \) are points belonging to one component of a compactum \( X \subset Q \) and if \( (X, a) \) is movable, then \( \text{Sh}(X, a) = \text{Sh}(X, b) \).

Proof. It follows by Lemma 3.1 that there exists a decreasing sequence \( U_1, U_2, \ldots \) of open neighborhoods of \( X \) (in \( Q \)) such that every neighborhood of \( X \) contains almost all \( U_n \) and that for every \( n = 1, 2, \ldots \) there is a homotopy

\[
a_n : U_{n+1} \times \langle 0, 1 \rangle \to U_n
\]

satisfying the following conditions:

\[(4.2) \ a_n(x, 0) = x \quad \text{and } \ a_n(x, 1) \in U_{n+2} \quad \text{for every point } x \in U_{n+1}, \]

\[(4.3) \ a_n(x, t) = x \quad \text{and } \ a_n(b, t) = b \quad \text{for every } 0 \leq t \leq 1.\]

Since the points \( a, b \) belong to one component of \( X \), there exists a map

\[
\beta : \langle 0, 1 \rangle \to U_n
\]

such that \( \beta(0) = a, \beta(1) = b \). Let us assume that for an \( n > 1 \) a map

\[
\beta_{n-1} : \langle 0, 1 \rangle \to U_n
\]

is already defined, such that \( \beta_{n-1}(0) = a, \beta_{n-1}(1) = b \), and let us set

\[
\beta_n(t) = a_{n-1}(\beta_{n-1}(t), 1) \quad \text{for every } 0 \leq t \leq 1.
\]

It is clear that this formula defines a map

\[(4.4) \ \beta_n : \langle 0, 1 \rangle \to U_{n+1} \]

such that

\[(4.5) \ \beta_n(0) = a \quad \text{and } \ \beta_n(1) = b .\]

By Lemma 1.1 there exists, for \( n = 1, 2, \ldots \), an isotopy

\[
\varphi_n : Q \times \langle 0, 1 \rangle \to Q
\]

such that

\[(4.6) \ \varphi_n(x, 0) = x \quad \text{for every point } x \in Q ,\]

\[(4.7) \ \varphi_n(x, t) \in U_n \quad \text{for every } (x, t) \in U_n \times \langle 0, 1 \rangle ,\]

\[(4.8) \ \varphi_n(x, t) = x \quad \text{for every } (x, t) \in (Q \setminus U_n) \times \langle 0, 1 \rangle ,\]

\[(4.9) \ \varphi_n(x, t) = \beta_n(t) \quad \text{for every } 0 \leq t \leq 1 .\]

Now let us set

\[(4.10) \ \varphi_n(x, 1) = \varphi_n(x, 1) \quad \text{for every point } x \in Q ,\]

and let us show that the maps \( \varphi_n : (Q, a) \to (Q, b) \) constitute a fundamental sequence \( f = \{ \varphi_n(\{ Q \setminus U_n \times \langle 0, 1 \rangle ) \}. \]

Consider an open neighborhood \( V \) of \( X \) (in \( Q \)) and an index \( n_0 \) such that \( U_n \subset V \). By (4.7) and (4.8), the isotopy \( \varphi_n \) satisfies the condition

\[
\varphi_n(U_{n+1}, 1) \subset U_n \quad \text{for every } n \geq n_0 .
\]

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Hence the restriction \( y_0 \mid U_n \times (0,1) \) joins in \( U_n \subset V \) the map \( f_n \mid U_n \) with the identity map. Similarly the restriction \( y_0 \mid (U_n \times (0,1)) \) joins in \( V \) the map \( f_{n+1} \mid U_n \) with the identity map. Setting

\[
y_n(x,t) = y_n(x, 1 - 2t) \quad \text{for} \quad 0 \leq t \leq \frac{1}{2},
\]

\[
y_n(x,t) = y_{n+1}(x, 2t-1) \quad \text{for} \quad \frac{1}{2} \leq t \leq 1,
\]

we get a homotopy \( \psi_n : U_n \times [0,1] \rightarrow V \). Joining the map \( f_n \mid U_n \) with the map \( f_{n+1} \mid U_n \), because

\[
y_n(x,0) = y_0(x,1) = f_n(x), \quad y_n(x,1) = y_{n+1}(x,1) = f_{n+1}(x)
\]

for every point \( x \in U_n \).

Moreover,

\[
y_n(x,t) = y_n(x, 1 - 2t) = y_n(x, 1 - 2t) = f_n(x), \quad \text{for} \quad 0 \leq t \leq \frac{1}{2},
\]

\[
y_n(x,t) = y_{n+1}(x, 2t-1) = y_{n+1}(x, 2t-1) = f_{n+1}(x), \quad \text{for} \quad \frac{1}{2} \leq t \leq 1.
\]

Now let us observe that the map assigning to every \( 0 \leq t \leq \frac{1}{2} \) the point \( \beta_n(1-2t) \) is a path in \( U_n \) with the initial point \( \beta_n(1) = b \) and the terminal point \( \beta_n(0) = a \), and the map assigning to every \( \frac{1}{2} \leq t \leq 1 \) the point \( \beta_{n+1}(2t-1) \) is a path in \( U_{n+1} \) with the initial point \( \beta_{n+1}(0) = a \) and the terminal point \( \beta_{n+1}(1) = b \). The map \( \beta_n(t) \) is a loop \( \lambda_n \) in \( U_n \) with the basic point \( b \).

It is clear that this loop is homotopic (in \( U_n \subset V \)) with the loop \( \lambda_n \) which we obtain if we run first by the path \( \beta_n : (0,1) \rightarrow U_n \) given by the formula \( \beta_n(t) = \beta_n(1-t) \) and then by the path \( \beta_{n+1} : (0,1) \rightarrow U_n \).

But this last loop is homotopic to a constant in \( V \), because setting

\[
\chi_n(x,t) = a_0(\beta_n(t), s),
\]

one gets a homotopy in the set \( U_n \subset U_n \subset V \) joining \( \beta_n \) with \( \beta_{n+1} \) and satisfying (by (4.3)) the conditions

\[
\chi_n(x,0) = a_0(\beta_n(0), s) = a_0(a, s) = a,
\]

\[
\chi_n(x,1) = a_0(\beta_n(1), s) = a_0(b, s) = b
\]

for every \( x \in (0,1) \).

It follows that there exists a homotopy contracting in \( V \) the loop \( \lambda_n \) to the point \( b \) and keeping this basic point fixed; that is, there exists a family of maps \( \theta_n \) depending continuously on \( s \in (0,1) \), with values in \( Y \), joining the map \( \theta_0 = y_0(0,1) \) with the constant map \( \theta_n = b \) and such that \( \theta_n(a,0) = \theta_n(a,1) = b \) for every \( s \in (0,1) \). Setting

\[
A = [U_n \times (0,1)] \cup ([a] \times [0,1]) \cup [U_{n+1} \times (1)],
\]

we infer that the map \( y_0(A) \) is homotopic in \( V \) with the map \( y_0 \) which coincides with \( y_0 \) in the set \( [U_n \times (0,1)] \cup [U_{n+1} \times (1)] \) and is constant in the set \( [a] \times [0,1] \). By the homotopy extension theorem we infer that \( y_0(A) \) can be extended to a map \( \tilde{\psi}_n : U_n \times (0,1) \rightarrow V \). This map \( \tilde{\psi}_n \) is a homotopy joining in \( V \) the map \( y_0(0,1) = f_0 U_n \) with the map \( y_0(U_{n+1} \times (1)) = f_{n+1} U_{n+1} \), and it satisfies the condition \( \tilde{\psi}_n(a,t) = b \) for every \( 0 \leq t \leq 1 \).

Hence \( f_n(0,1) = f_{n+1} U_n \) in \( V \), for \( n \geq n_0 \). It follows that \( f = (f_0, (X,a), (X,b)) \) is a fundamental sequence.

By an analogous argument, one shows that setting

\[
g_n = f_{n+1} : (Q, b) \rightarrow (Q, a),
\]

one gets a fundamental sequence \( g = (g_0, (X,b), (X,a)) \).

Finally, the relation \( f_n g_{n}(x) = f_n f_{n}(x) = x \) for every point \( x \in X \) implies that the fundamental sequences \( f_n g_{n}(x) \) are generated by the identity maps \( i_{X,a}, \ldots, i_{X,b} \) and \( i_{X,b} \). Thus \( X, a \) is a fundamental sequence of \( X, b \) and the proof of Theorem (4.1) is concluded.

5. An example. Let us show that there exist continua \( X \) such that the shape \( S(X,a) \) depends on the choice of the point \( a \in X \).

Consider the circular disk \( D \) with the center \( a = (\frac{1}{2}, 0, 1) \) and the radius \( \frac{1}{2} \) lying in the plane \( \mathbb{R}^2 \) given in the Euclidean 3-space \( \mathbb{R}^3 \) by the equation \( x_3 = 1 \). Let \( A_0 \) denote the anchor ring which we obtain by revolving the disk \( D \) about the straight line \( L \) given by the equations \( x_1 = 0, x_2 = 1 \). Then \( A_0 \subset \mathbb{R}^2 \) and \( A_0 \cap D \) is the union of two disjoint circular disks \( D \) with the center \( a \) and \( D \) with the center \( a \) lying symmetrically with respect to \( L \).

The circle \( C_0 \) obtained by the rotation of the point \( a \) is said to be the core of \( A_0 \). Let us give it a fixed orientation.

Now let us assign to every point \( p \in \mathbb{R}^3 \) the point \( s(p) \) in which the half-plane \( H_p \) passing through \( p \) and having \( L \) as its edge intersects the circle \( C_0 \). Then to every path \( s \) lying in \( \mathbb{R}^3 \) corresponds a number \( s(\sigma) \) defined as the oriented angle described by the vector with the beginning \( (\frac{1}{2}, 0, 1) \) and the end \( s(p) \) when \( p \) runs through the path \( \sigma \). The fixed orientation of \( C_0 \) determines the sign of \( s(\sigma) \). If \( s \) is a path in \( \mathbb{R}^3 \) from \( p \) to \( q \) and \( \tau \) is a path in \( \mathbb{R}^3 \) from \( q \) to \( r \), then \( s(\sigma) = s(\tau) + s(\tau) \). If \( \gamma \) is a loop in \( \mathbb{R}^3 \), then \( s(\gamma) = 2k\pi \), where \( k \) is an integer. In particular, for the loop \( \gamma_0 \) constituted by the oriented curve \( C_0 \) with the beginning \( a \), the number \( s(\gamma_0) \) is equal to \( 2\pi \).

By a regular anchor ring of degree \( m \) (and the radius \( r \)) we understand every set \( A \subset A_0 \) homeomorphic to \( A_0 \) and such that for every point \( p \times C_0 \), the set \( A \cap H_p \) is the union of \( m \) disjoint circular disks with
Let $A_n \cap D'$ consist of $3^n$ disjoint disks $D_{a_1}, D_{a_2}, \ldots, D_{a_3^n}$, and we may assume that their order is such that the loop $\gamma_n$ meets them consecutively. Then the point $a'_n$ satisfying (5.3) is the center of a disk $D_{a'_n}$ with $3^{-n+1} < 3^{-n} < 2 - 3^{-n}$. Now let us observe that the loop $\gamma_{n+1}$ meets consecutively the disks

$$D_{a_1}, D_{a_2}, \ldots, D_{a_{2^n}}, D_{a_1}, D_{a_2}, \ldots, D_{a_{2^n}}, D_{a_1}, D_{a_2}, \ldots, D_{a_{3^n}},$$

containing the disks

$$D_{a_1}, D_{a_1}, \ldots, D_{a_{3^n}}$$

respectively. It follows that there is an index $k_{n+1}$ with $3^{-n} < k_{n+1} < 2 - 3^{-n}$ such that $D_{a_{k_{n+1}}}, C_{a_{k_{n+1}}}, \ldots, D_{a_{3^n}}$ is a sequence of points $a'_{n+1}$ equal to the center of $D_{a_{k_{n+1}}}, a_{k_{n+1}}, \ldots, a_{3^n}$, in order to satisfy (5.3) and (5.2). It follows by (5.3) that the sequence of points $a'_{n}$ converges to a point $b \in S$ such that

$$\text{The segment } a'_{n}, b \text{ lies in } A_n \text{ for every } n = 0, 1, 2, \ldots$$

Moreover, let us observe that inequality (5.4) implies that

(5.6) For every $N > 0$ there exists an index $n_0$ such that for every $n > n_0$ all numbers of the form

$$[2 - 3^n k - \nu(\lambda_n)]$$

where $k$ is an integer, are greater than $N$.

It is clear that $D \cap S$ is a subset of the diameter $R$ of the disk $D$. Consequently there exists in the half-plane $H_a$ a circle $K$ tangent to $E$ at the point $a$ and such that $K \cap D \neq \emptyset$. Let us observe that

$$X = K \cup S,$$

let us show that

(5.8) $\text{Sh}(X, a) \neq \text{Sh}(X, b)$.

Proof. We shall show more, namely that $(X, a)$ is not fundamentally dominated by $(X, b)$. Otherwise, there exist two pointed fundamental sequences

(5.9) $f = \{f(x, X, a), (X, b)\}$ and $g = \{g(x, X, b), (X, a)\}$ such that

$$gf = \{g(x, X, a)\}.$$
By a double anchor ring we shall understand a set homeomorphic to the Cartesian product of the interval (0, 1) by the set $M$ given in the Euclidean 2-space $E^2$ by the formula

$$M = \{ \xi = (x_1, x_2), \phi(x, (0, 0)) \leq 4, \phi(x, (2, 0)) > 1, \phi(x, (-2, 0)) > 1 \}.$$ 

One can easily see, by virtue of (5.7), that for every $n = 0, 1, 2, \ldots$ there exists a positive number $c_n$ such that $c_n < 1/(n+1)$ and that the set $B_n$, being the union of $A_n$ and of the anchor ring $T_n$ defined as the union of all balls in $E^2$ with radius $c_n$ and centers belonging to $K$, is a double anchor ring. Moreover, $B_n$ is a closed neighborhood of $X$ (in $E^n$) and

$$X = \bigcap_{n=0}^{\infty} B_n.$$ 

Let us observe that

(5.11) None of the curves $C_n$ and $K$ is contractible in $B_n$.

Now, consider the retraction $r: \mathbb{Q} \rightarrow \mathbb{Q}^2$ by the formula

$$r(x) = (x_1, x_2, 0, 0, \ldots) \quad \text{for every point } x = (x_1, x_2, \ldots) \in \mathbb{Q}.$$ 

Let $P_n$ denote the set consisting of all points $x$ belonging to $Q$ such that $g(x, r(x)) \leq 1/(n+1)$. It is clear that setting

(5.12) $U_n = r^{-1}(B_n) \cap P_n$ for $n = 0, 1, 2, \ldots$, we get a descending sequence of neighborhoods of $X$ (in $Q$) such that

$$X = \bigcap_{n=0}^{\infty} U_n, \quad B_n = U_n \cap \mathbb{Q}^2 \quad \text{for} \quad n = 0, 1, 2, \ldots$$

and $B_n$ is a deformation retract of $U_n$ for $n = 0, 1, 2, \ldots$.

It follows by (5.11) and (5.12) that

(5.13) Neither $C_n$ nor $K$ is contractible in $U_n$.

Now let us denote, for every $n = 0, 1, 2, \ldots$ by $e_n$ the path consisting of the oriented segment from $b$ to $a_n$, and by $x$ the loop with the beginning $a$ which we obtain if we give to the curve $K$ a fixed orientation. Let us observe that setting

(5.14) $\eta = a_n^2 a_n^{-1}$ and $\zeta = a_n^2 a_n^{-1}$,

we get two loops in $(B_n, b)$ which are representatives of two generators of the free group $\pi_n(B_n, b)$. Let $a_n$ denote the subgroup of $\pi_n(B_n, b)$ generated by all elements which have as a representative a loop lying in the set $B_n$. The group $\pi_n(B_n, b)$ is a free group with two generators. It is easy to see that we can select as one of them the element for which the loop $a_n^2 a_n^{-1}$ lying by (5.3) in $A_n \subset B_n$ is a representative, and as the other the element with the representative of the form $a_n^2 a_n^{-1} a_n^2 a_n^{-1}$. By (5.1) the loop $\eta_n$ is homotopic in $B_n$, and we infer that the first of those loops is homotopic in $(B_n, b)$ to the loop $\zeta = a_n a_n^2 a_n^{-1}$ and the second is homotopic in $(B_n, b)$ to the loop

$$(a_n^2 a_n^{-1} a_n^2 a_n^{-1}) (a_n^2 a_n^{-1} a_n^2 a_n^{-1}) = \frac{a_n}{a_n^2 a_n^{-1} a_n^2 a_n^{-1}}$$

where $a_n^2 a_n^{-1} a_n^2 a_n^{-1}$ is a loop in $(A_n, b)$. Since $r(a_n) = 0$, we infer that

$$\eta = \eta_n, \quad \zeta = \zeta_n,$$

Since $a_n a_n^2 a_n^{-1}$ is a loop, the number $r(a_n a_n^2 a_n^{-1})$ may be written in the form $2 \pi m_1$, where $m_1$ is an integer. Consequently $r(a_n a_n^2 a_n^{-1}) = r(a_n)^2$, whence

$$a_n^2 a_n^{-1} a_n^2 a_n^{-1} = a_n^2 a_n^{-1} a_n^2 a_n^{-1}$$

in $(B_n, b)$.

Moreover, it follows by (5.6) that $m_1$ satisfies the following condition

(5.15) For every $N > 0$ there exists an index $n_0$ such that $n > n_0$ implies that $|3^n - m_1| > N$ for every integer $k$.

Thus we have shown that every element of the group $\pi_0$ has a representative which is a product of the potences of two loops: of $e_n^2$ and of $\eta^2 \zeta^2 a_n^2 a_n^{-1}$. Let us observe that every such product (if it is not trivial) is of the form

$$e_n^2 \eta \zeta^2 a_n^2 a_n^{-1} \eta \zeta^2 a_n^2 a_n^{-1} \ldots \eta \zeta^2 a_n^2 a_n^{-1}$$

Consider now the element of the group $\pi_n(B_n, b)$ with the representative $f_n(a)$. Recall that $f$ and $g$ are pointed fundamental sequences. It follows by (5.9), (5.10) and (5.12) that there exists an increasing sequence of natural numbers $f_1, f_2, \ldots$, such that

$$f_1([B_n, a]) = f_1([B_n, a]) \quad \text{in} \quad (B_n, b)$$

and

$$g_1 f_1([B_n, a]) = g_1([B_n, a]) \quad \text{in} \quad (B_n, b)$$

for every $i = 1, 2, \ldots$.

It follows that the loop $\zeta = f_1(a)$ is a representative of an element belonging to the group $\pi_0$ for every $n = 1, 2, \ldots$. This element is not trivial, because otherwise the element $g_1 f_1(a) \approx \approx \approx \approx$ would be trivial in $(B_n, a)$, contrary to (5.13). Consequently it has a representative of the form (5.16) for every $n$. But this is incompatible with (5.15), because the elements with the representatives $\eta$ and $\zeta$ are independent in the group $\pi_n(B_n, b)$.
Thus the supposition that \((X, a)\) is fundamentally dominated by \((X, b)\) leads to a contradiction. Hence the proof of proposition \((5.8)\) is finished.

6. Sum of shapes of pointed compacta. Let \((X, a_0)\) and \((Y, a_1)\) be two pointed compacta. It is clear that there exists a pointed compactum \((Z, a_2)\) such that \(Z = Z' \cup Z''\), where \(Z' \cap Z'' = \{a_2\}\) and there exists a homeomorphism \(g: (X, a_0) \to (Z', a_2)\) and a homeomorphism \(h: (Y, a_1) \to (Z'', a_2)\). Manifoldly the topological properties of \((Z, a_2)\) depend only on the topological properties of \((X, a_0)\) and of \((Y, a_1)\). Thus we can say that the topological type of \((Z, a_2)\) is the sum of the topological types of \((X, a_0)\) and of \((Y, a_1)\). We write shortly: \((Z, a_2) = (X, a_0) \boxplus (Y, a_1)\).

Let us prove the following

\section{Theorem.} If \((Z, a_2) = (X, a_0) \boxplus (Y, a_1)\), then \(\text{Sh}(Z, a_2)\) depends only on \(\text{Sh}(X, a_0)\) and \(\text{Sh}(Y, a_1)\).

\section{Proof.} Setting

\(P = \{(a_1, a_2, \ldots) \in \mathbb{Q}; 0 \leq a_i \leq \frac{1}{2}\}, \quad \mathcal{Q} = \{(a_1, a_2, \ldots) \in \mathbb{Q}; \frac{1}{2} \leq a_i \leq 1\}\),

we get a decomposition of \(\mathcal{Q}\) into two sets \(P\) and \(\mathcal{Q}\) homeomorphic to \(\mathcal{Q}\). Now let

\(\left(\begin{array}{ll}
(X, a_0) & \cong (X', a'_0) \\
(Y, a_1) & \cong (Y', a'_1)
\end{array}\right)\).

In order to prove that the shape of \((Z', a'_2) = (X', a'_0) \cup (Y', a'_1)\) is the same as the shape of \((Z, a_2)\), we can assume that \(a_0 = a'_0 = a'_1\).

\(\mathcal{Q} = \mathbb{R}^n \cap P \cap \mathbb{R}^n \cap \mathbb{R}^n\).

This hypothesis implies that there exist two maps

\(a, b: (X, a_0) \to (X', a'_0)\)

\(\beta: (Y, a_1) \to (Y', a'_1)\)

such that

\(a(a) = a\) for every point \(a \in X \cap X'\), \(a(a) = a_0\) for every point \(a \in X \cap X'\), \(\beta(a) = a_1\) for every point \(a \in Y \cap Y'\).\n
It follows by \((6.2)\) that there exist four fundamental sequences

\(f = (f, (X, a_0), (X', a'_0)), \quad f' = (f, (X, a_0), (X', a'_0))\)

\(g = (g, (Y, a_1), (Y', a'_1)), \quad g' = (g, (Y, a_1), (Y', a'_1))\)

such that

\(f \cong g, f' \cong g'\).

Setting

\(f_k = f_k a, \quad f_k = f_k a, \quad g_k = g_k a, \quad g_k = g_k a\),

we get for every \(k = 1, 2, \ldots\) four maps of \((Q, x_0)\) into \((Q, x_1)\).

Let us observe that

\(\tilde{f} = (\tilde{f}, (X, x_0), (X', x'_0), (Y, x_1), (Y', x'_1))\),

\(\tilde{g} = (\tilde{g}, (Y, x_1), (Y', x'_1), (X, x_0), (X', x'_0))\).

are fundamental sequences homotopic to \(f, g, f', g'\) respectively.

In fact, if \(U'\) is a neighborhood of \(X'\), there is a neighborhood \(U, x_0, x_1\) \(U\) of \(X, Y, Y'\) such that for every point \(x \in U\) the segment \(x\) lies in \(U\).

\(\chi(x, t) = f_k \cdot (x + (1 - t) a_0)\) for every \((x, t) \in U_0 \times (0, 1)\).

we get a homotopy joining in \(U'\) the map \(\tilde{f}_k U_0\) with the map \(\tilde{f}_k U_0\). Hence \(\tilde{f} \cong f\). By a similar argument one proves that \(\tilde{g} \cong g, \tilde{f}' \cong f'\) and \(\tilde{g}' \cong g'\).

It follows by \((6.4)\) that

\(f \cong f', g \cong g\).

Since

\(\tilde{f}_k(a) = \tilde{g}_k(a) = \tilde{f}_k(a) = \tilde{g}_k(a) = x_0\) for every point \(a \in P \cap R\),

we infer that setting

\(\mathcal{Q} = \{(a, x_0, x_1) \to (Q, x_0)\}, \quad \mathcal{Q} = \{(a, x_0, x_1) \to (Q, x_0)\}, \quad \mathcal{Q} = \{(a, x_0, x_1) \to (Q, x_0)\}, \quad \mathcal{Q} = \{(a, x_0, x_1) \to (Q, x_0)\}, \quad \mathcal{Q} = \{(a, x_0, x_1) \to (Q, x_0)\}, \quad \mathcal{Q} = \{(a, x_0, x_1) \to (Q, x_0)\}, \quad \mathcal{Q} = \{(a, x_0, x_1) \to (Q, x_0)\}

we get, for every \(k = 1, 2, \ldots\), two maps

\(\mathcal{Q} = \{(a, x_0, x_1) \to (Q, x_0)\}, \quad \mathcal{Q} = \{(a, x_0, x_1) \to (Q, x_0)\}, \quad \mathcal{Q} = \{(a, x_0, x_1) \to (Q, x_0)\}, \quad \mathcal{Q} = \{(a, x_0, x_1) \to (Q, x_0)\}, \quad \mathcal{Q} = \{(a, x_0, x_1) \to (Q, x_0)\}, \quad \mathcal{Q} = \{(a, x_0, x_1) \to (Q, x_0)\}, \quad \mathcal{Q} = \{(a, x_0, x_1) \to (Q, x_0)\}

Let us observe that \(a = \chi(\mathcal{Q}, (X \cup X', a_0), (X' \cup Y', a'_1))\) and \(\omega = \chi(\mathcal{Q}, (X \cup Y, a_0), (X' \cup Y, a'_1))\) are fundamental sequences.

In fact, if \(U'\) is a neighborhood of \(X' \cup Y',\) then there exist an open neighborhood \(U\) of \(X\) and an open neighborhood \(V\) of \(Y\) such that for almost all \(k\)

\(\mathcal{Q} = \chi(U, x_0) \cong \chi(U', x_0) \cap W'\) and \(\mathcal{Q} = \chi(U, x_0) \cong \chi(U', x_0) \cap W'\).

It means that there exist two homotopies

\(h_k: U \times (0, 1) \to W, \quad k: V \times (0, 1) \to W\).
satisfying the conditions
\begin{align*}
\lambda(x, 0) &= f(x), \\
\lambda(x, 1) &= f_{\lambda+1}(x) \\
\mu_0(x, 0) &= g_0(x), \\
\mu_0(x, 1) &= g_{\mu+1}(x) \\
\lambda(x, t) &= \mu(x, t) = x_0 \quad \text{for every } 0 \leq t \leq 1.
\end{align*}
Consider the sets
\begin{align*}
\tilde{U} &= a^{-1}(U), \\
\tilde{V} &= b^{-1}(V),
\end{align*}
which are open neighborhoods of \(X\) and \(Y\) respectively. Setting
\begin{align*}
\tilde{\lambda}(x, 0) &= \lambda_0(a(x), t) \quad \text{for every point } x \in \tilde{U} \quad \text{and for } 0 \leq t \leq 1, \\
\tilde{\mu}(x, t) &= \mu_0(b(x), t) \quad \text{for every point } x \in \tilde{V} \quad \text{and for } 0 \leq t \leq 1,
\end{align*}
one gets two homotopies,
\begin{align*}
\tilde{\lambda}_0: \tilde{U} \times \langle 0, 1 \rangle \to \tilde{W}_0, \\
\tilde{\mu}_0: \tilde{V} \times \langle 0, 1 \rangle \to \tilde{W},
\end{align*}
such that \(\tilde{\lambda}_0\) joins in \((W, x_0)\) the map \(f(a(x)) = f_{\lambda+1}(x)\) with the map \(f_{\lambda+1}(y) = f_{\lambda+1}(x)\) and \(\tilde{\mu}_0\) joins in \((W, x_0)\) the map \(g_0(y) = b_{\mu+1}(x)\) with the map \(g_{\mu+1}(y) = b_{\mu+1}(x)\). Moreover, since \(a(x) = b(x) = x_0\) for every point \(x \in P \cap R\),
we infer that the formulas
\begin{align*}
\tilde{\theta}_0(x, 0) &= \tilde{\lambda}_0(x, t) \quad \text{for } (x, t) \in \tilde{U} \cap \tilde{P} \times \langle 0, 1 \rangle, \\
\tilde{\theta}_0(x, t) &= \tilde{\mu}_0(x, t) \quad \text{for } (x, t) \in \tilde{V} \cap \tilde{R} \times \langle 0, 1 \rangle
\end{align*}
give a homotopy \(\tilde{\theta}_0: [(\tilde{U} \cap \tilde{P}) \cup (\tilde{V} \cap \tilde{R})] \times \langle 0, 1 \rangle \to \tilde{W}\) joining \(\tilde{\omega}\) with \(\tilde{\omega}_0\) in \((\tilde{W}, x_0)\). Since the set \(\tilde{W}_0 = (\tilde{U} \cap \tilde{P}) \cup (\tilde{V} \cap \tilde{R})\) is a neighborhood of the set \(X \cup Y\), we infer that \(\tilde{\omega}\) is a fundamental sequence. By an analogous argument we show that \(\tilde{\omega}\) is also a fundamental sequence.

Let us show that
\[\tilde{\omega}^n = \tilde{\omega}_0 \circ \tilde{\omega}^{n-1}.\]
Consider a neighborhood \(W\) of the set \(X \cup Y\). By (6.4) there are a neighborhood \(U_0\) of \(X\) and a neighborhood \(V_0\) of \(Y\) such that for almost all \(k\) there exist homotopies
\begin{align*}
\varphi_{\omega}: U_0 \times \langle 0, 1 \rangle &\to W, \\
\varphi_{\omega}: V_0 \times \langle 0, 1 \rangle &\to W
\end{align*}
such that
\begin{align*}
\varphi_{\omega}(x, 0) &= f_0(x) \quad \text{for every } x \in U_0, \\
\varphi_{\omega}(x, 0) &= g_0(x) \quad \text{for every } x \in V_0, \\
\varphi_{\omega}(x, 1) &= x \quad \text{for every } x \in U_0, \\
\varphi_{\omega}(x, 1) &= x \quad \text{for every } x \in V_0, \\
\varphi_{\omega}(x_0, t) &= x_0 \quad \text{for every } 0 \leq t \leq 1.
\end{align*}
Setting
\begin{align*}
\tilde{\varphi}_{\omega}(x, t) &= \varphi_{\omega}(a(x), t) \quad \text{for every } (x, t) \in (U_0 \cap \tilde{P}) \times \langle 0, 1 \rangle, \\
\tilde{\varphi}_{\omega}(x, t) &= \varphi_{\omega}(b(x), t) \quad \text{for every } (x, t) \in (V_0 \cap \tilde{R}) \times \langle 0, 1 \rangle,
\end{align*}
we get a homotopy
\[\tilde{\varphi}_{\omega}: W_0 \times \langle 0, 1 \rangle \to W,
\]
satisfying the condition \(\tilde{\varphi}_{\omega}(x, t) = x_0\) for every \(0 \leq t \leq 1\), because \(a(x) = b(x) = x_0\) for every point \(x \in P \cap R\). The homotopy \(\tilde{\varphi}_{\omega}\) joins in \((W, x_0)\) the map \(\varphi_{\omega}(U_0, x_0) \to \tilde{W}_0\) with the map \(\varphi_{\omega}(V_0, x_0) \to \tilde{W}\) given by the formula \(\varphi_{\omega}(x) = \varphi_{\omega}(a(x), 1)\) for every point \(x \in W_0\).
It follows that \(\varphi = [\varphi_{\omega}(X \cup Y, x_0), (X \cup Y, x_0)]\) is a fundamental sequence mapping \(\tilde{\omega}\). Moreover, if \(x \in X\) then
\[\varphi(x) = \varphi_{\omega}(x, 1) = \varphi_{\omega}(x_0, 1) = \varphi_{\omega}(x, 1) = x.
\]
By an analogous argument one shows that \(\varphi_{\omega}(x) = x\) for every point \(x \in Y\). This implies that the fundamental sequence \(\varphi_{\omega}\) is homotopic to the fundamental identity sequence \(\tilde{\omega}^n \circ \tilde{\omega}^{-n}\). Hence \(\tilde{\omega}^n \cong \tilde{\omega}^{-n}\).

By an analogous argument one proves that \(\tilde{\omega}^n \cong \tilde{\omega}^{-n}\). Thus the proof of Theorem (6.1) is finished.

Remark. Let us observe that if one replaces hypothesis (6.4) by the weaker one that \(f \circ f \approx \tilde{\omega}_0 \circ \tilde{\omega}_0\) and \(g \circ g \approx \tilde{\omega}_0 \circ \tilde{\omega}_0\) (that is, the hypothesis that \(X(x_0) \approx X(x_0)\) and \(Y(y_0) \approx Y(y_0)\) by the hypothesis that \(X(x_0) \approx X(x_0)\) and \(Y(y_0) \approx Y(y_0)\), then in the same way we obtain the following proposition:
\[\text{(6.6) If Sh}(X, x_0) \leq \text{Sh}(X', x_0') \quad \text{and Sh}(Y, y_0) \leq \text{Sh}(Y', y_0'), \quad \text{then}
\text{Sh}(X, x_0) \oplus \text{Sh}(Y, y_0) \leq \text{Sh}(X', x_0') \oplus \text{Sh}(Y', y_0').
\]

7. Cartesian product of shapes of pointed compacta. By the Cartesian product of two pointed compacta \((X, x_0)\) and \((Y, y_0)\) one understands the pointed compactum \(\{X \times Y, (x_0, y_0)\}\) and one writes
\[(X, x_0) \times (Y, y_0) = \{X \times Y, (x_0, y_0)\}.
\]
Let us prove the following
\[\text{(7.1) Theorem. Sh}(X_0 \times Y, (x_0, y_0)) \text{ depends only on Sh}(X, x_0) \quad \text{and Sh}(Y, y_0).
\]
Proof. It suffices to show that if \(X, X', Y, Y'\) are subsets of \(Q\), and if \(\text{Sh}(X, x_0) = \text{Sh}(X', x'_0)\) and \(\text{Sh}(Y, y_0) = \text{Sh}(Y', y'_0)\), then
\[ Sh(X \times Y, (a_k, y_k)) = Sh(X' \times Y', (a'_k, y'_k)) \]. By our hypotheses, there exist four fundamental sequences,
\[ f = (f_1(X, a_1), (X', a'_1)), f' = (f'_1(X', a'_1), (X, a_1)), \]
\[ g = (g_2(X, y_2), (Y, y_2)), g' = (g'_2(Y, y_2), (X, y_2)), \]
that such that
\[ f \mapsto f = f_1 \times f_2, f' \mapsto f'_1 \times f'_2, \]
\[ g \mapsto g = g_1 \times g_2, g' \mapsto g'_1 \times g'_2. \]
Setting
\[ f_k(x, y) = (f_k(x), g_k(y)), f'_k(x', y') = (f'_k(x'), g'_k(y')) \]
for every \( x, y \in Q \)
and for \( k = 1, 2, \ldots, \)
we get the maps
\[ f_k, f'_k : (Q \times Q, (a_k, y_k)) \to (Q \times Q, (a'_k, y'_k)) \]
for every \( k = 1, 2, \ldots, \)
Consider now the set \( Q \times Q \) homeomorphic to \( Q \), containing the two sets \( X \times Y \) and \( X' \times Y' \). Let \( W \) be a neighborhood of \( X' \times Y' \) in \( Q \times Q \).
Then there exist a neighborhood \( U' \) of \( X' \) in \( Q \) and a neighborhood \( V' \) of \( Y' \) in \( Q \) such that
\[ U' \times V' \subseteq W. \]
Since \( f \) and \( g \) are fundamental sequences, there exist a neighborhood \( U \) of \( X \) in \( Q \) and a neighborhood \( V \) of \( Y \) in \( Q \) such that the homotopies
\[ f_k(U, a_k) \simeq f_{k+1}(U, a_k) \] \( (U, a_k), g_k(V, y_k) \simeq g_{k+1}(V, y_k) \] \( (V, y_k) \)
both hold true for almost all \( k \). It follows that
\[ f_k(U \times V, (a_k, y_k)) \simeq f_{k+1}(U \times V, (a_k, y_k)) \] \( (U \times V', (a'_k, y'_k)) \)
for almost all \( k \). Since \( W = U \times V \) is a neighborhood of \( X \times Y \) in \( Q \times Q \), we infer by (7.3) that
\[ f_k(W, (a_k, y_k)) \simeq f_{k+1}(W, (a_k, y_k)) \] \( (W', (a'_k, y'_k)) \)
for almost all \( k \), and consequently \( \tilde{f} = \{ f_k, (X \times Y, (a_k, y_k)), (X' \times Y', (a'_k, y'_k)) \} \]
is a fundamental sequence.
By an analogous argument one shows that
\[ \tilde{f}' = \{ f'_k, (X' \times Y', (a'_k, y'_k)), (X \times Y, (a_k, y_k)) \} \]
is a fundamental sequence.

Since
\[ \tilde{f}_k(x, y) = \{ f_k(x), g_k(y) \} \]
for every \( (x, y) \in Q \times Q \) and \( k = 1, 2, \ldots, \)
one shows, in the same way, that the fundamental sequence \( \tilde{f}' \)
\[ = \{ f'_k(x), g'_k(y) \} \]
is homotopic to \( \{ f_k(x), g_k(y) \} \), and the fundamental sequence \( \tilde{f}' \)
\[ = \{ f'_k(x), g'_k(y) \} \]
is homotopic to \( \{ f_k(x), g_k(y) \} \). It follows that \( Sh(X \times Y, (a_k, y_k)) \)
and the proof of Theorem (7.3) is finished.

Let us observe that if we replace the hypotheses that \( Sh(X, a_k) = Sh(X', a'_k) \) and \( Sh(Y, y_k) = Sh(Y', y'_k) \) by the weaker ones that \( Sh(X, a_k) \preceq Sh(X', a'_k) \) and \( Sh(Y, y_k) \preceq Sh(Y', y'_k) \), then we get in the same way the following proposition:

\[ (7.4) \text{ If } Sh(X, a_k) \preceq Sh(X', a'_k) \text{ and } Sh(Y, y_k) \preceq Sh(Y', y'_k) \text{ then } \]
\[ Sh(X \times Y, (a_k, y_k)) \preceq Sh(X' \times Y', (a'_k, y'_k)) \).

8. Simple and prime pointed shapes. Some problems. It follows by Theorem (6.1) and Theorem (7.3) that the formulas
\[ Sh(X, a_k) + Sh(Y, y_k) = Sh((X, a_k) + (Y, y_k)) \]
and
\[ Sh(X, a_k) \times Sh(Y, y_k) = Sh((X \times Y, (a_k, y_k)) \]
define uniquely two commutative operations, called addition and multiplication, respectively, assigning to two pointed shapes a pointed shape.
Since \( (X, a_k) \) is an \( r \)-image (see [4], p. 8) of \( (X, a_k) \preceq (Y, y_k) \) and also of \( (X \times Y, (a_k, y_k)) \), and since the shape of an \( r \)-image \( (X, a_k) \) of \( (Z, a_k) \) is less than or equal to the shape of \( (Z, a_k) \), we infer that
\[ (8.1) \text{ Sh}(X, a_k) \preceq \text{ Sh}(X, a_k) + \text{ Sh}(Y, y_k) \]
and
\[ \text{ Sh}(X, a_k) \times \text{ Sh}(X, a_k) \preceq \text{ Sh}(X, a_k) \times \text{ Sh}(Y, y_k) \]
for every pointed shapes \( Sh(X, a_k) \) and \( Sh(Y, y_k) \).

It is clear that the trivial pointed shape (that is the shape \( Sh(X, a_k) \)) where \( X \) consists of only one point \( a_k \) is the identity element for both operations, the addition and the multiplication; that is:
\[ Sh((x_k, a_k) \times Sh(Y, y_k) = Sh(Y, y_k), \]
\[ Sh((x_k, a_k) \times Sh(Y, y_k) = Sh(Y, y_k) \]
for every pointed shape \( Sh(Y, y_k) \).
The global dimension of the group rings of abelian groups III
by S. Balcerzyk (Toruń)

This paper is a continuation of papers [1], [2] and is concerned with the computation of the global dimension of the group ring of arbitrary abelian group with commutative Noetherian coefficient ring. Also the dimension of those rings as algebras is computed.

In this paper all rings and groups are assumed to be commutative.

For any R-algebra A, we denote by dim R A or R-dim A the projective dimension of A as R A module. If A = R(II) is a group ring, then it is known (see [4]) that dim R(II) = dimR(II) R where II operates trivially on R.

1. In this section we prove some preliminary lemmas.

Lemma 1. Let II be a group and a ∈ R be an ideal of a ring R. If R = R(II), then R-dim R(II) ≥ R-dim R(II)

Proof. If P is a projective resolution of R(II)-module R, then P ⊗ R R is a R(II)-projective complex. Since R(II) ⊗ R R = Tor^R(II) R, R, then P ⊗ R R is a projective resolution of R and the lemma follows.

Lemma 2. If II is a subgroup of a group II, then

\[ \text{gl. dim } R(II) \geq \text{gl. dim } R(II) \]
\[ \text{dim } R(II) \geq \text{dim } R(II) \]

Proof. It is easy to prove the formula

\[ \text{dim } R(II) = \text{dim } R(II) \oplus R(II) \]

for any R(II)-module A and this implies the first inequality. The second one follows by the fact that any R(II)-projective module is R(II)-projective.

Lemma 3. If R is a field and mR = R if m is an order of an element in a group II, then in the group ring R(II) any set of orthogonal idempotents is at most countable.

Proof. It is easy to see that all idempotents of R(II) belong to the subring R(T) where T is the maximal torsion subgroup of II. The group T

References


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