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Reçu par la Rédaction le 22. 8. 1968

A CBV image of a universal null set need not be a universal null set

by

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In [3], the author very naively lists six properties which a subset E of the interval $I=[0,1]$ might possess. None of these properties is less stringent than its successor, and the sixth property is that of being a universal null set:

(6) *If μ is a non-negative non-atomic finite Baire measure on I, then $\mu(E)=0$.*

Several remarks about whether a listed property is equivalent to or stricter than its successor are given in [3], which was written while the author sought to answer the following question.

If E is a subset of I with property (6) *and h is a continuous function of bounded variation on I* (CBV), *then must $h(E)$ have property* (6)?

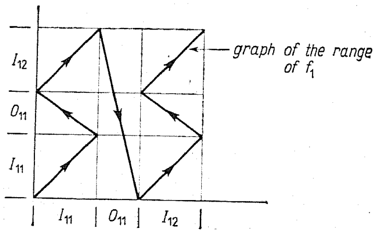
Unfortunately, the author was ignorant of a vast amount of pertinent information that was available. For example, A. S. Besicovitch ([2]) had shown that property (4) does not imply property (3) (cf. [3]). Before becoming aware of Besicovitch's result, the author found another proof of it; but, he was still unable to settle the original question or to show that property (5) does not imply property (4). However, W. Sierpiński ([4], p. 57) had shown that a continuous image of a universal null set need not be a universal null set, from which it follows that (5) does not imply (4).

The purpose of this note is to answer our question in the negative by modifying Sierpiński's construction so that a theorem of N. Bary can be applied.

Let P be the Cantor set in I of measure 1/2 which is obtained by taking out middle quarters. Let I_{ij}, $j\leqslant 2^i$, denote the ith stage intervals in the canonical representation of P as an intersection of finite unions of intervals, and let O_{ij}, $j\leqslant 2^{i-1}$, denote the corresponding segments which are removed at the ith stage.

It follows from [4], p. 57, and the fact that there is an order preserving homeomorphism between the irrationals in I and the non-end points of P that there is a universal null set N in P such that there is a continuous map g of N onto P. Denote the graph of g by G.

There is a standard way of mapping I into $I\times I$ by a homeomorphism f such that f maps P onto $P\times P$ and, moreover, the equation $v(x)=y$ has only a finite number of solutions if $y\in I-P$, where v is defined on I by $f(t)=\big(u(t),v(t)\big)$. Beginning with a diagram, a sketch of a construction of such a function f follows.



Then f_1 maps I_{21} linearly onto the noted diagonal of $I_{11}\times I_{11}$, O_{21} linearly onto the noted diagonal of $I_{11}\times O_{11}$, ... To define f_2, let f_2 be patterned after f_1 on the sets I_{2i}, $i\leqslant 4$, and $f_2=f_1$ on the rest of I. Iterate this process, and let $f(x)=\lim f_n(x)$.

Because G is a subset of $P\times P$, the set $E=f^{-1}(G)$ is a subset of P which is homeomorphic to G. Hence E is a universal null set. (In fact, since g is continuous, E is also homeomorphic to N.) Moreover, v is a continuous map of I onto I, $v(E)=P$, and the equation $v(x)=y$ has only finitely many solutions if $y\in I-P$. Hence it follows from [1], Theorem III, p. 635, that there exists a strictly increasing continuous function φ and a CBV function h such that $v=\varphi\circ h$. If $h(E)$ were a universal null set, then it would follow that $P=\varphi\big(h(E)\big)$ has measure zero. Hence $h(E)$ is not a universal null set.

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Reçu par la Rédaction le 28. 9. 1968

Some remarks concerning the shape of pointed compacta

by

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By Q we denote the Hilbert cube, that is, the subset of the Hilbert space consisting of all points $(x_1, x_2, \ldots)$ with $0\leqslant x_n\leqslant 1/n$ for $n=1,2,\ldots$ Two pointed compacta (X,a), (Y,b) are said to be *fundamentally equivalent* (notation: $(X,a)\underset{F}{\simeq}(Y,b)$) if there exist in Q two pointed compacta (X',a') and (Y',b') homeomorphic to (X,a) and (Y,b) respectively and two fundamental sequences (see [1], p. 225)

$$\underline{f}=\{f_k,(X',a'),(Y',b')\}\quad\text{and}\quad \underline{g}=\{g_k,(Y',b'),(X',a')\}$$

such that $\underline{f}\underline{g}\simeq \underline{i}_{(Y',b')}$ and $\underline{g}\underline{f}\simeq \underline{i}_{(X',a')}$, where $\underline{i}_{(Z,c)}$ denotes the identity fundamental sequence $\{i,(Z,c),(Z,c)\}$.

If we assume only that the second relation $\underline{g}\underline{f}\simeq \underline{i}_{(X',a')}$ holds true, then we say that (X,a) is *fundamentally dominated by* (Y,b) and we write $(X,a)\underset{F}{\leqslant}(Y,b)$.

The collection of all pointed compacta (Y,b) fundamentally equivalent to a given pointed compactum (X,a) is called the *shape* of (X,a) (see [3]); it is denoted by $\mathrm{Sh}(X,a)$. Thus the relation $\mathrm{Sh}(X,a)=\mathrm{Sh}(Y,b)$ means that $(X,a)\underset{F}{\simeq}(Y,b)$. If $(X,a)\underset{F}{\leqslant}(Y,b)$, then $\mathrm{Sh}(X,a)$ is said to be *less than or equal to* $\mathrm{Sh}(Y,b)$ and we write $\mathrm{Sh}(X,a)\leqslant\mathrm{Sh}(Y,b)$.

The aim of this note is to establish a condition under which $\mathrm{Sh}(X,a)$ does not depend on the choice of the point a, and to study the operations of addition and multiplication of shapes of pointed compacta.

I wish to thank A. Lelek, who read the manuscript of this note, for his penetrating remarks.

1. A lemma on isotopy. By a *map* we understand here always a continuous function. A map

$$\varphi\colon\ X\times\langle u,v\rangle\to Y,\qquad\text{where } u,v \text{ are numbers with } u<v,$$

is said to be a *homotopy in a set* Z if all values of φ belong to Z. If $a\in X$, $b\in Y$ and if $\varphi\colon X\times\langle u,v\rangle\to Y$ is a homotopy satisfying the condition

$$\varphi(a,t)=b\qquad\text{for every } u\leqslant t\leqslant v,$$