Remark. All of the circle action constructed in Section 4 are totally wild. In Section 3, a constructed action is totally wild if the original fixed-point-free action is free.

THEOREM 5.3. *If there is a free action of G on S^q, $q \geqslant 0$, $p \geqslant 3$, then there is a totally wild action of G on S^{p+q+1}.*

Proof. Let $\bar{a}$ be action constructed in the proof of Theorem 5.2, where a is taken to be free. Then, for each $1 \neq g \in G$, X is the fixed-point set of $\bar{a}_g$. If $\bar{a}_g$ were conjugate to a piecewise linear homeomorphism, then X would be homeomorphic to the fixed-point set of a piecewise linear map, which is impossible since X is not a polyhedron.

References

[1] J. J. Andrews and M. L. Curtis, *n-Space modulo an arc*, Ann. of Math. 75 (1962), pp. 1–7.
[2] R. H. Bing, *Inequivalent families of periodic homeomorphisms of E^3*, ibid. 80 (1964), pp. 78–93.
[3] W. A. Blankinship, *Generalization of a construction of Antoine*, ibid. 53 (1951), pp. 276–297.
[4] M. Brown, *Locally flat imbeddings of topological manifolds*, ibid. 75 (1962), pp. 331–341.
[5] — *Wild cells and spheres in higher dimensions*, Mich. Math. J. 14 (1967), pp. 219–224.
[6] J. L. Bryant, *Euclidean space modulo a cell*, Fund. Math. 63 (1968), pp. 43–51.
[7] J. L. Bryant, and C. L. Seebeck, *Locally nice embeddings in codimension three*, Bull. Amer. Math. Soc. 74 (1968), pp. 378–380.
[8] R. C. Lacher, *Cell-like spaces*, Proc. Amer. Math. Soc. 20 (1969).
[9] L. L. Lininger, *Actions on S^n*, Topology (to appear).
[10] — *On topological transformation groups*, Proc. Amer. Math. Soc. 20 (1969), pp. 191–192.
[11] D. R. McMillan, *A criterion for cellularity in a manifold*, Ann. of Math. 79 (1964), pp. 327–337.
[12] D. V. Meyer, *More decompositions of E^n which are factors of E^{n+1}*, Fund. Math. (to appear).
[13] J. Milnor, *Groups which act on S^n without fixed points*, Amer. J. Math. 79 (1957), pp. 623–630.
[14] D. Montgomery and L. Zippin, *Examples of transformation groups*, Proc. Amer. Math. Soc. 5 (1954), pp. 460–465.
[15] M. H. A. Newman, *A theorem on periodic transformations of spaces*, Quart. J. Math. (Oxford Series) 2 (1931), pp. 1–9.
[16] R. H. Rosen, *Examples of non-orthogonal involutions of euclidean space*, Ann. of Math. 78 (1963), pp. 560–566.
[17] P. A. Smith, *Transformations of finite period*, ibid. (1939), pp. 690–711.
[18] J. Stallings, *On topologically unknotted spheres*, ibid. 78 (1963), pp. 501–526.

INSTITUTE FOR ADVANCED STUDY
Princeton, New Jersey
FLORIDA STATE UNIVERSITY
Tallahassee, Florida

Reçu par la Rédaction le 23. 4. 1968

About an imbedding conjecture for k-independent sets

by

A. B. Németh (Cluj)

Following [1] we say that a subset X of the n-dimensional real Euclidean space R^n is *k-independent* $(0 \leqslant k \leqslant n-1)$ if any $k+2$ distinct points of that subset are linearly independent. [1]

In what follows the homeomorphic image of the set $\{(x^1, \ldots, x^m)\colon \sum (x^i)^2 < 1\}$ in R^m will be said to be an *open m-cell*; the homeomorphic image of the set $\{(x^1, \ldots, x^m)\colon \sum (x^i)^2 = 1\}$ will be said to be an *$m-1$-sphere*.

K. Borsuk [1] has proved the following imbedding theorem concerning k-independent sets:

If X is a compact k-independent set in R^n and if N is an open subset in X containing k distinct points, then $X \setminus N$ is homeomorphic with a subset of R^{n-k}.

In [6], p. 503 and in [4], another notion of k-independence is applied, which is useful in applications in the approximation theory and which will be called in the sequel *k-vectorial-independence.*

The subset X of R^n will be said to be *k-vectorial-independent* if for any k of its distinct points $x_1, \ldots, x_k$ the vectors $\overrightarrow{Ox_1}, \ldots, \overrightarrow{Ox_k}$, where O is the origin in R^n, are linearly independent.

OBSERVATION 1. *A k-vectorial-independent subset X in R^n is $k-2$-independent in the sense of* [1].

Indeed, if $x_1, \ldots, x_k$ are k distinct points in X, then they cannot be contained in any $k-2$-dimensional hyperplane H^{k-2}, because such a hyperplane generates a $k-1$-dimensional subspace (i.e. a $k-1$-dimensional hyperplane passing through the origin), and if $x_1, \ldots, x_k$ were in H^{k-2}, the vectors $\overrightarrow{Ox_1}, \ldots, \overrightarrow{Ox_k}$ would be linearly dependent, being in R^{k-1}.

OBSERVATION 2. *If X is a k-independent subset in R^n, then it may be considered a $k+2$-vectorial-independent subset in R^{n+1} if we consider R^n as a hyperplane H^n in R^{n+1} not passing through the origin.*

---

(1) For the sake of simplicity, the affine space and the vectorial Euclidean space of dimension n are denoted by the same symbol R^n.

Indeed, consider $k+2$ distinct points $x_1, \ldots, x_{k+2}$ in X in H^n. The vectors $\overrightarrow{Ox_1}, \ldots, \overrightarrow{Ox_{k+2}}$ in R^{n+1} are linearly independent. If they were linearly dependent, there would exist a subspace R^{k+1} in R^{n+1} containing them, which would intersect H^n in a k-dimensional hyperplane H^k containing the points $x_1, \ldots, x_{k+2}$, which is a contradiction.

It was conjectured by A. M. Gleason (see [8]) that the k-independent compact subset X in R^n is homeomorphic with a subset of S^{n-k}, the $n-k$-sphere. Investigations about this imbedding conjecture were announced by C. T. Yang in [8], but we have not been able to obtain any information about his results.

In Theorem 2 of [4] it was proved that if X is a compact k-vectorial-independent set in R^n, N is open in X and contains $k-2$ distinct points, then $X \setminus N$ is homeomorphic with a subset of R^{n-k+1}, which is an analogue of the imbedding theorem of K. Borsuk for k-vectorial-independent sets. By a similar reformulation of the conjecture of A. M. Gleason, we obtain:

If X is a k-vectorial-independent compact subset of R^n, then it is homeomorphic with a subset of S^{n-k+1}.

Making use of our Observation 2 above, we can see that this conjecture implies the conjecture of Gleason. Our conjecture for $k = n$ is the well-known theorem of J. Mairhuber [3] in the approximation theory. For $k=1$ it is obviously true, and for $k=2$ an imbedding of X into a proper subset of S^{n-1} may be realised by the radial projection with respect to the origin of R^n into the geometrical sphere with its centre at the origin.

Let X be a compact k-independent set in R^n containing an m-cell. Then, as has been proved by S. S. Ryškov [5], the following inequality is valid: (²)

$$\left[\frac{k+2}{2}\right]m+\left[\frac{k+1}{2}\right] \leqslant n .$$

Suppose now that the compact k-vectorial-independent subset X in R^n is of dimension m. Then there exists a closed subset X_0 of X of the same dimension m and an $n-1$-hyperplane H^{n-1} which has the property of separating strictly the origin O and the subset X_0. Denote by X_0' the radial projection with respect to O of X_0 into H^{n-1}. Obviously, X_0' is a k-vectorial-independent homeomorphic image of X_0, and therefore

(²) In [5], Ryškov defines the so-called k-regular sets as being in fact k-independent in the sense of [1], and has announced his inequality for these sets. But all the reasonings in the text are valid for $k-1$-independent sets. In Uspehi Mat. Nauk 15 (6) (1960), pp. 125–132, the definition of the k-regular sets is changed in this sense. Our inequality follows from the inequality of Ryškov applied to k-independent sets.

from Observation 1 it follows that X_0' is a $k-2$-independent subset of H^{n-1}. Applying the inequality of S. S. Ryškov we conclude that

$$(*) \qquad \left[\frac{k}{2}\right]m+\left[\frac{k-1}{2}\right] \leqslant n-1 .$$

The present note aims at giving a proof of our conjecture in the particular case where the k-vectorial-independent set X in R^n contains an $n-k+1$-cell. More precisely, we shall prove the following.

Theorem. *Let X be a compact subset of R^n, $n \geqslant 2$, which is k-vectorial-independent and contains an $n-k+1$-cell. Then X is homeomorphic with a subset of S^{n-k+1}.*

The inequality $(*)$ restricts k in this case to $k \leqslant 3$ or $k = n$. As we have observed above, in the case of $k=2$ the proof is simple, and in the case of $k=n$ it is known. Therefore only the case $k=3$ will be considered, and to justify our theorem in this case, we observe that each geometrical sphere S^{n-2} in a hyperplane H^{n-1} in R^n not passing through the origin is a 3-vectorial-independent set in R^n containing $n-2$-cells.

In the proof given here we apply a method utilised by I. J. Schoenberg and C. T. Yang in [7] for proving the theorem of J. Mairhuber. An important moment in the proof is the employing of the following theorem of M. Brown [2]:

If h is a homeomorphic imbedding of $S^{n-1} \times I$ into S^n, then the closure of either complementary domain of $h(S^{n-1} \times \{1/2\})$ in S^n is a closed n-cell. (Here $I = [0, 1]$.)

We begin with a lemma:

Lemma. *Let X be a compact Hausdorff space having the following properties:*

(i) *X contains an open n-cell Q as an open subset;*

(ii) *if N is a non-empty open subset of X, then $X \setminus N$ may be imbedded in a proper subset of S^n.*

Then X is homeomorphic with a subset of S^n.

Proof. If X is not connected, the proof is immediate.

Suppose that X is connected. Let A be an annulus, that is to say the homeomorphic image of the set $S^{n-1} \times I$, which is contained in the n-cell Q. Since Q is open in X, A separates X and so does σ^{n-1}, the image in A of $S^{n-1} \times \{1/2\}$, i.e. $X \setminus A = Y_1 \cup Y_2$, $X \setminus \sigma^{n-1} = V_1 \cup V_2$, where Y_1, Y_2, and respectively V_1, V_2 are non-empty open disjoint subsets of X. Suppose that $Y_1 \subset V_1$, $Y_2 \subset V_2$ and introduce the notations: $B_1 = V_1 \cup \sigma^{n-1}$, $B_2 = V_2 \cup \sigma^{n-1}$. The sets B_1 and B_2 are both connected and are not separated by σ^{n-1}. Denote by f and by g the homeomorphisms of $X \setminus Y_2$ and, respectively, of $X \setminus Y_1$ in S^n, which exist according to (ii). Since the above sets both contain A, from the theorem of M. Brown [2] it follows that the complementary domains of $f\sigma^{n-1}$ and $g\sigma^{n-1}$ in S^n are open n-cells. Thus we may suppose that f and g are homeomorphisms which both transform σ^{n-1} in the equator E of S^n and by which B_1 is mapped into the north hemisphere and B_2 into the south hemisphere of S^n (suppose that S^n is a geometrical sphere). Consider the following homeomorphism of E onto itself: $l = g\circ f^{-1}|E$. Let h be an extension of the homeomorphism l to a homeomorphism of the whole north hemisphere onto itself. Then $h\circ f$ will be a homeomorphism of B_1 into the north hemisphere carrying σ^{n-1} onto E. Consider the mapping

$$\varphi x = \begin{cases} h\circ fx & \text{for} \quad x \text{ in } B_1, \\ gx & \text{for} \quad x \text{ in } B_2. \end{cases}$$

φ is a well-defined mapping which is one-to-one and continuous. To prove its continuity, let $U\subset\varphi X$ be open. If $U\cap E = \emptyset$, then $\varphi^{-1}U$ is open according to the continuity of $h\circ f$ and g. Suppose $U\cap E\neq\emptyset$. Then the sets $h\circ fB_1\cap U$ and $gB_2\cap U$ are open in the relative topology of $h\circ fB_1$ and gB_2 respectively. Therefore the sets

$$W_1 = \varphi^{-1}(h\circ fB_1\cap U) = (h\circ f)^{-1}(h\circ fB_1\cap U)$$

and

$$W_2 = \varphi^{-1}(gB_2\cap U) = g^{-1}(gB_2\cap U)$$

are open in the relative topology of B_1 and B_2, respectively. Let G_1 and G_2 be open sets in X such that $G_1\cap B_1 = W_1$, $G_2\cap B_2 = W_2$. Then the sets $G_1\cup V_2$ and $G_2\cup V_1$ are open in X and

$$W_1\cup W_2 = (G_1\cup V_2)\cap(G_2\cup V_1).$$

But $W_1\cup W_2 = \varphi^{-1}U$, which completes the proof of the continuity of φ. From the compactness of X it follows that φ is a homeomorphic imbedding of X into S^n.

Proof of the theorem. If X in the theorem contains an $n-2$-cell (remember that only the case $k = 3$ is considered), then it contains an open $n-2$-cell as an open set. Indeed, suppose that Q is an open $n-2$-cell in X such that $X\setminus\bar{Q}\neq\emptyset$. According to Theorem 2 in [4], Q is open in any closed proper subset in X in which it is contained. Then Q is open in X according to the normality of this space. From Theorem 2 in [4] it also follows that $X\setminus N$ may be topologically imbedded into a proper subset of S^{n-2} for any non-empty, open subset N in X. It follows that all the conditions of the lemma are satisfied and therefore X is homeomorphic with a subset of S^{n-2}.

References

[1] K. Borsuk, *On the k-independent sets of the Euclidean space and of the Hilbert space*, Bull. Acad. Sci. Pol. 5 (1957), pp. 351–356.

[2] M. Brown, *A proof of the generalised Schoenflies theorem*, Bull. Amer. Math. Soc. 66 (1960), pp. 74–76.

[3] J. C. Mairhuber, *On Haar's theorem concerning Chebyshev approximation problems having unique solutions*, Proc. Amer. Math. Soc. 7 (1956), pp. 609–615.

[4] A. B. Németh, *Homeomorphe projections of k-independent sets and Chebyshev subspaces of finite dimensional Chebyshev spaces*, Mathematica 9 (1967), pp. 325–333.

[5] S. S. Ryškov (С. С. Рышков), *О к-регулярных вложениях*, Докл. Акад. Наук. СССР 127 (1959), pp. 272–273.

[6] Yu. A. Šaškin (Ю. А. Шашкин), *Систем Коровкина в пространствах непрерывных функций*, Изв. Акад. Наук. СССР 26 (1962), pp. 495–512.

[7] I. J. Schoenberg and C. T. Yang, *On the unicity of solutions of problems of best approximation*, Ann. Math. Pura Appl. 54, pp. 1–12.

[8] C. T. Yang, *On non-orientable closed surfaces in Euclidean spaces*, Canad, J. Math. 14 (1962), pp. 660–668.

THE COMPUTING INSTITUTE OF THE ACADEMY OF ROUMANIA
Cluj

Reçu par la Rédaction le 5. 5. 1968