Some wild spheres and group actions

by

R. C. Lacher* (Tallahassee, Fla.)

We present here methods for constructing uncountably many topologically distinct q-spheres in S^{p+q}, provided $p \geqslant 3$ and $q \geqslant 1$. The methods allow us to construct actions of various groups on S^{p+q} having any one of the wild spheres as fixed-point set, so that we obtain uncountably many actions of certain groups on certain spheres. For G a finite group, we show that there are uncountably many topologically distinct actions of G on S^{p+q} each having a q-sphere for its fixed-point set, provided $p \geqslant 3$, $q \geqslant 1$, and S^{p-1} admits a fixed-point-free G-action. Since such p always exists for a particular finite group G, we obtain that, for some n depending on G, there exist uncountably many topologically distinct G-actions on S^n. Essentially the same result holds for circle actions: there are uncountably many topologically distinct circle actions on S^{p+q}, each rotating freely about a q-sphere of fixed-points, provided $p \geqslant 4$, $q \geqslant 1$, and p is even.

Other examples in the same spirit as ours may be found in [2], [9], [10], [14], and [16]; other references are found in the bibliographies of these articles.

Notation. $\boldsymbol{R}^n$ is used to denote the euclidean n-space, S^n the one-point compactification of $\boldsymbol{R}^n$. The symbol "$\approx$" means "is homeomorphic to".

1. Wild spheres. In most of our constructions we will need to consider decompositions of the following type.

Definition. Let X be a compact set in $\boldsymbol{R}^p$, and let Y be a closed set in $\boldsymbol{R}^q$. Then $\Gamma(X, Y)$ is defined to be that decomposition of $\boldsymbol{R}^{p+q} = \boldsymbol{R}^p \times \boldsymbol{R}^q$ whose non-degenerate elements are the sets of the form $X \times y$, $y \in Y$.

Recall that a decomposition Γ of a space S is *shrinkable by a pseudo-isotopy* if there is a pseudo-isotopy h_t $(0 \leqslant t \leqslant 1)$ of S such that $h_1 =$ identity and $\{h_0^{-1}(s) |\ s \in S\} = \Gamma$. A pseudo-isotopy of S is a homo-

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topy $h\colon S\times I\to S$ such that $h_1=$ identity, h_t is a homeomorphism of S onto itself for $0<t\leqslant 1$, and $h_0(S)=S$.

THEOREM 1.1. *Let the following be given:*

(a) *A compact set X in R^p such that the decomposition $\Gamma(X, R^1)$ is shrinkable by a pseudo-isotopy, and*

(b) *A closed subset Y of R^q for some $q\geqslant 1$.*

Then the following hold:

(1) *The decomposition $\Gamma=\Gamma(X, Y)$ is shrinkable by a pseudo-isotopy; in particular, $R^{p+q}/\Gamma\approx R^{p+q}$.*

(2) *$X\times 0$ is cellular in R^{p+1}.*

(3) *If $f\colon R^q\to R^{p+q}/\Gamma$ is defined by $f(y)=\varphi(x_0, y)$, $y\in R^q$, where φ is the quotient map and $x_0\in X$, then f is an embedding of R^q onto a closed subset of R^{p+q}.*

(4) *f is locally flat at each point of R^q-Y.*

(5) *If R^p-X is not simply connected and $p\geqslant 3$ then f is locally wild at each point of $Y^\circ=R^q-\overline{(R^q-Y)}$.*

(6) *If X is not cellular in R^p and $p\neq 4$ then f is locally wild at each point of Y°.*

(For the definition of "locally flat", see [4]. "Locally wild" means "not locally flat". For the definition of "cellular", see [11].)

Proof. Let g_t $(0\leqslant t\leqslant 1)$ be a pseudo-isotopy of R^{p+1} which shrinks $\Gamma(X, R^1)$ at time $t=0$. Define $\bar g$ on $R^p\times R^1\times R^{q-1}\times I$ by

$$\bar g(x, y, z, t)=\big(g_t(x, y), z\big)\,,\quad x\in R^p,\ y\in R^1,\ z\in R^{q-1},\ t\in I\,.$$

Clearly, $\bar g$ is a pseudo-isotopy shrinking $\Gamma(X, R^q)$ at time $t=0$.

Now, let $\varepsilon\colon R^q\to[0,1]$ be a continuous function such that $\varepsilon(y)=0$ if and only if $y\in Y$. Define h on $R^p\times R^q\times I$ by

$$h(x, y, t)=\bar g\big(x, y, \max(\varepsilon(y), t)\big)\,,\quad x\in R^p,\ y\in R^q,\ t\in I\,.$$

It is easily checked that h is a pseudo-isotopy of R^{p+q} shrinking $\Gamma(X, Y)$. This completes the proof of (1). Setting $Y=\{0\}$ for $0\in R^1$, we see that $X\times 0$ is point-like in R^{p+1}, so that (2) is proved.

Conclusions (3) and (4) are quite easy to prove and (5) follows from a well-known argument similar to the one in the following paragraphs.

We turn now to the proof of (6). Let y_0 be a point of Y°. Then there is a neighborhood W of $f(y_0)$ in R^{p+q} such that the triples $\big(W, W\cap f(R^q), f(y_0)\big)$ and $\big(R^{p+q}, f_0(R^q), f_0(0)\big)$ are homeomorphic, where f_0 is the embedding we get by setting $Y=R^q$. Hence we can assume $Y=R^q$ and $y_0=0$.

Suppose, under these assumptions, that f is locally flat at 0. Let U_0 be a neighborhood of X in R^p, and let $U=\varphi(U_0\times U^q)$, where U^q is the open unit ball in R^q. Then $\varphi^{-1}(U)=U_0\times U^q$. Now, we assume that $p\geqslant 3$ since otherwise X is cellular in R^p. (See [8]; X is cell-like by (2) above.) Therefore we can use local flatness of f at 0 to find a neighborhood V of $f(0)$ in R^{p+q} such that any loop in $V-f(R^q)$ is contractible to a point in $U-f(R^q)$. Let $V_0=\varphi^{-1}(V)\cap R^p$. Since $U-f(R^q)$ deforms onto $[U-f(R^q)]\cap\varphi(R^p)=\varphi(U_0-X)$, any loop in $\varphi(V_0-X)=\varphi(V_0)-f(R^q)$ is contractible in $\varphi(U_0-X)$. Finally, since φ is a homeomorphism on U_0-X, we see that any loop in V_0-X is contractible in U_0-X.

To summarize, we have shown that if f is locally flat at a point y_0 of Y°, then the inclusion $X\subset R^p$ satisfies McMillan's cellularity criterion [11]. However, X need not be a compact absolute retract, so we must appeal to the extension of McMillan's theorem [8] to see that X is cellular in R^p. (X is cell-like by (2).)

THEOREM 1.2. *If $n\geqslant 1$ and $k\geqslant 3$ there are uncountably many closed embeddings of R^n into R^{n+k}, no two being setwise equivalent.*

(A *closed* embedding is one whose image is a closed set. Two embeddings $f, g\colon X\to Y$ are *setwise equivalent* if there is a homeomorphism of Y which carries $f(X)$ onto $g(X)$.)

Proof. First set $p=k$ and $n=q$. Let A be an arc in R^p such that R^p-A is not simply connected. (See [3].) Hypothesis (1.1)(a) with $X=A$ is satisfied by [1]. Let Y and Y' be closures of open sets in R^q, and let f and f' be the embeddings of R^q obtained from $\Gamma(A, Y)$ and $\Gamma(A, Y')$ as in (1.1) (3). Then, by (3), (4), and (5), Y is the wild set of f and Y' is the wild set of f'. Consequently, if f and f' are setwise equivalent, then $Y\approx Y'$. We will show in (1.4) that there are uncountably many possibilities for Y.

COROLLARY 1.3. *If $n\geqslant 1$ and $k\geqslant 3$, there are uncountably many embeddings of S^n into S^{n+k} no two of which are setwise equivalent.*

To complete the proof of (1.2) we need the following cardinality result.

LEMMA 1.4. *Let $\mathcal{K}_q$ be the set of compact subsets of R^q which are closures of open sets in R^q. If $q\geqslant 1$ then $\mathcal{K}_q$ contains uncountably many topological types.*

Proof. First assume that $q\geqslant 2$. Let $\mathfrak{L}$ be the set of compact subsets of R^{q-1}. Since $q\geqslant 2$, $\mathfrak{L}$ has uncountably many topological types. We will construct for each $L\in\mathfrak{L}$ an element $\widetilde L\in\mathcal{K}_q$ such that $L\times 0\subset\widetilde L$ and $\widetilde L$ fails to be a manifold precisely along the set $L\times 0$. It follows then from invariance of domain that $\widetilde L_1\approx\widetilde L_2\Rightarrow L_1\approx L_2$, and hence that $\mathcal{K}_q$ contains uncountably many topological types.

To construct $\widetilde L$, let $l_1, l_2, \ldots$ be a dense sequence in L. Let S_1 be a small circle in R^q centered at $l_1\times 1$. Assuming that S_α has been defined for each sequence α of the form $\alpha=(j_1, \ldots, j_n)$, $1\leqslant j_i\leqslant i$, let $S_{\alpha 1}, \ldots, S_{\alpha(n+1)}$

be very small circles, S_{ai} centered at $l_i \times (1/(n+1))$. Make sure that the collection $\{S_a\}_a$ is disjoint and that $\operatorname{diam} S_a \leqslant [2(\text{length of } a)]^{-1}$ for each a. Let S be the union of the S_a. Finally, let $\tilde{L}$ be the closure of a tapering regular neighborhood of S in $\boldsymbol{R}^q$.

If $q = 1$, we can give a similar argument using the fact that there are uncountably many topological types of initial segments of countable ordinals.

2. Definitions. Let G be a topological group, X a topological space. An *action* of G on X is a mapping $a\colon G \times X \to X$ with the following properties:

For each $g \in G$, the equation $a_g(x) = a(g, x)$ defines a homeomorphism a_g of X onto itself.

a_1 is the identity map on X.

$a_h \circ a_g = a_{hg}$ for all $g, h \in G$.

If $g \neq h$ then $a_g \neq a_h$.

Two actions a, a' of G on X are (*topologically*) *equivalent* if there is a homeomorphism h of X onto itself such that $a'_g = h a_g h^{-1}$ for all $g \in G$.

If a is an action of G on X, we define the *fixed-point set* $F(a)$ to be the set $\{x \in X \mid a_g(x) = x \text{ for all } g \in G\}$. a is *fixed-point-free* if $F(a) = \emptyset$. More generally, if H is any subset of G, we let $a|H$ denote the map $a|(H \times X)$ and $F(a|H) = \{x \in X \mid a_h(x) = x \text{ for all } h \in H\}$. The action a is said to be *free* when $F(a|H) = \emptyset$ for all subsets H of G, $H \neq \{1\}$.

The following observation is useful in distinguishing non-free group actions:

If a and a' are equivalent actions of G on X, then $(X, F(a|H)) \approx (X, F(a'|H))$ for all subsets H of G

3. Wild finite group actions.

THEOREM 3.1. *Suppose that there exists a fixed-point-free action of the finite group G on the sphere S^{p-1}, $p \geqslant 3$. Let Y be the closure of an open set in $\boldsymbol{R}^q$, $q \geqslant 1$. Then there exist an action a of G on $\boldsymbol{R}^{p+q}$ and an embedding $f\colon R^q \to \boldsymbol{R}^{p+q}$ such that*

(1) *the fixed-point set of a is $f(\boldsymbol{R}^q)$, and*

(2) *the wild set of f is Y.*

Proof. Let γ be a fixed-point-free action of G on the sphere S of radius one, center 0, in $\boldsymbol{R}^p$. Extend γ radially to an action β on $\boldsymbol{R}^p$ which takes each sphere $\|x\| = r$ onto itself, $r \geqslant 0$, and whose fixed-point set is $\{0\}$.

Now, for each $g \in G$, γ_g is a homeomorphism of S of finite period. Hence, by Newman's Theorem [15], the fixed-point set $F(g)$ of γ_g is closed and nowhere dense in S. Therefore, the set $N(g) = \{x \in S \mid \gamma_g(x) \neq x\}$ is open and dense in S. Clearly, then,

$$\bigcap_{g \in G} N(g) \neq \emptyset .$$

I.e., there exists a point z of S such that $\gamma_g(z) \neq \gamma_h(z)$ whenever $g \neq h$. It follows that there is an open neighborhood V of z in S such that $\gamma_g(V) \cap \gamma_h(V) = \emptyset$ whenever $g \neq h$. Let U be the union of all interiors of line segments joining $0 \in \boldsymbol{R}^p$ with points of V. U is an open set of $\boldsymbol{R}^p$ with the property:

$$\beta_g(U) \cap \beta_h(U) = \emptyset \quad \text{when} \quad g \neq h .$$

Using [3], we can easily construct an arc A in $U \cup \{0\}$, with 0 for one endpoint, such that $\boldsymbol{R}^p - A$ is not simply connected. Let X be the union of all the arcs $\beta_g(A)$, $g \in G$. Clearly X is a k-odd (where $k =$ order of G) with the properties:

$\boldsymbol{R}^p - X$ is not simply connected, and $\beta_g(X) = X$ for each $g \in G$.

Let $\Gamma = \Gamma(X, Y)$ be the decomposition of $\boldsymbol{R}^{p+q}$ defined by X and Y, as in § 1, and let $\varphi\colon \boldsymbol{R}^{p+q} \to \boldsymbol{R}^{p+q}/\Gamma$ be the quotient map. In [12], Meyer showed that $\Gamma(X, \boldsymbol{R}^1)$ is shrinkable by a pseudo-isotopy of $\boldsymbol{R}^{p+1}$, so that the hypotheses of Theorem 1.1 are satisfied. Therefore, we need only find an action a of G on $\boldsymbol{R}^{p+q}/\Gamma$ whose fixed-point set is $\varphi(0 \times \boldsymbol{R}^q)$. But this is easy: first extend β over $\boldsymbol{R}^{p+q} = \boldsymbol{R}^p \times \boldsymbol{R}^q$ by the formula $\bar{\beta}_g(x, y) = (\beta_g(x), y)$, $g \in G$, $x \in \boldsymbol{R}^p$, $y \in \boldsymbol{R}^q$. Then $a_g = \varphi \bar{\beta}_g \varphi^{-1}$, $g \in G$, gives the action a.

Applying (1.4) we get

COROLLARY 3.2. *If S^{p-1} admits a fixed-point-free action of the finite group G, $p \geqslant 3$, $q \geqslant 1$, then there exist uncountably many mutually inequivalent G-actions on S^{p+q} each of which has a q-sphere for its fixed-point set.*

COROLLARY 3.3. *Let $p \geqslant 3$ and $q \geqslant 1$. Then there exist uncountably many mutually inequivalent involutions on S^{p+q} each having a wild q-sphere for its fixed-point set.*

Remark. Suspensions of Bing's examples [2] yield the cases $p = 1, 2$.

THEOREM 3.4. *Let G be a finite group. Then there exists a fixed-point-free action of G on some euclidean sphere.*

Proof. There is a faithful representation $G \to O(n)$ for some n, where $O(n)$ is the group of orthogonal $n \times n$ matrices, so we simply assume that G is a subgroup of $O(n)$. Now, $O(n)$ acts naturally on the unit sphere S of $\boldsymbol{R}^n$, hence G does; let F be the fixed-point set of this G-action. Now F is the intersection of a finite number of subspaces of $\boldsymbol{R}^n$ with S, and hence F and its orthogonal complement intersected with S are spheres. Since G restricts to an action on the orthogonal complement of F intersected with S, the proof is complete.

Remarks. 1. If S^n admits a fixed-point-free G-action, then so do all of the spheres S^{kn+k-1}, $k \geqslant 1$; new actions are constructed by taking joins.

2. Only a very restricted class of groups can act *freely* on spheres. See [13].

Corollary 3.5. *Let G be a finite group, and let $q \geqslant 1$. Then, for infinitely many integers p, there exist uncountably many mutually inequivalent G-actions on S^{p+q} each having a q-sphere for a fixed-point set.*

4. Circle actions. A "circle action" is an action of the group $SO(2)$, the group of complex numbers of modulus one under multiplication. We say that an $SO(2)$ action α on X *rotates freely about* $Y \subset X$ if (a) Y is the fixed-point set of α, and (b) for each $t \neq 1$ in $SO(2)$, $\alpha_t|(X-Y)$ is a fixed-point-free homeomorphism $(X-Y) \to (X-Y)$.

Theorem 4.1. *Let $p \geqslant 4$, $q \geqslant 1$, and $p \equiv 0 \ (\mathrm{mod}\ 2)$. Let Y be the closure of an open set in $\boldsymbol{R}^q$. Then there exist an action α of $SO(2)$ on R^{p+q} and an embedding $f\colon \boldsymbol{R}^q \to \boldsymbol{R}^{p+q}$ such that*

(1) α rotates freely about $f(\boldsymbol{R}^q)$, and

(2) The wild set of f is Y.

Corollary 4.2. *If $p \geqslant 2$ and $q \geqslant 1$, there are uncountably many mutually inequivalent $SO(2)$ actions on S^{2p+q}, each rotating freely about a wild q-sphere.*

Remark. The condition that the fixed-point set have even codimension is necessary since the associated involution α_{-1} is orientation preserving. (See [17].)

Proof of Theorem 4.1. Since p is even, we can think of $\boldsymbol{R}^p$ as the image of $\boldsymbol{C}^r$ under the "forget" functor, where $\boldsymbol{C}$ = field of complex numbers and $2r = p$. In this way we have a standard action ϱ of $SO(2)$ on $\boldsymbol{R}^p$, rotating freely about the origin, given by scalar multiplication $\boldsymbol{C} \times \boldsymbol{C}^r \to \boldsymbol{C}^r$ restricted to $SO(2) \times \boldsymbol{C}^r$. It is clear that the set

$$\Sigma^{p-1} = \{(x_1, \ldots, x_p) \in \boldsymbol{R}^p | x_j > 0 \text{ for } j = 1, \ldots, p-1, \text{ and } x_p = 0\} \cup \{0\}$$

is a "slice" of the action ϱ; that is, Σ^{p-1} intersects each orbit $\varrho\big(SO(2) \times x\big)$, $x \in \boldsymbol{R}^p$, in at most one point.

Now, as in the proof of (3.1), there is an arc A in Σ^{p-1} such that 0 is an endpoint of A and $\boldsymbol{R}^{p-1} - A$ is not simply connected. (See [3].) Since Σ^{p-1} is a slice of ϱ, the arcs $\varrho(t \times A)$ are pairwise disjoint except for their common endpoint 0, and hence $\varrho\big((SO)(2) \times A\big) = D$ is a disk in $\boldsymbol{R}^p$. Also,

if a loop in $\boldsymbol{R}^{p-1} - [A \cup (-A)]$ is contractible in $\boldsymbol{R}^p - D$, then the singular disk can be dragged back into $\boldsymbol{R}^{p-1}$ via ϱ, so that

$$\boldsymbol{R}^p - D \textit{ is not simply connected},$$

and

$$D \textit{ is invariant under the action } \varrho.$$

Finally, using the results in [7], we see that

$$D \textit{ is tame in } \boldsymbol{R}^{p+1}.$$

(In applying [7], notice that, for $k = 2$ and $n = p+1 \geqslant 5$, the approximation theorem of Homma is a triviality using general position, so that the results announced in [7] are definitely true for $k = 2$ and $n \geqslant 5$.)

The proof is now completed as in Theorem 3.1. Let $\Gamma = \Gamma(D, Y)$ be the decomposition of $\boldsymbol{R}^{p+q}$ determined by $D \subset \boldsymbol{R}^p$ and $Y \subset \boldsymbol{R}^q$. Since D is tame in $\boldsymbol{R}^{p+1}$, the hypotheses of Theorem 1.1 are satisfied by the main result of [6]. Therefore, we need only find an action α of $SO(2)$ on $\boldsymbol{R}^{p+q}/\Gamma$ which rotates freely about $\varphi(0 \times \boldsymbol{R}^q)$, where φ is the quotient map. Again, this is easy: let $\bar{\varrho}$ be given by $\bar{\varrho}(t, x, y) = \big(\varrho(t, x), y\big)$, $t \in SO(2)$, $x \in \boldsymbol{R}^p$, $y \in \boldsymbol{R}^q$, and define α by $\alpha_t = \varphi\bar{\varrho}_t\varphi^{-1}$, $t \in SO(2)$.

5. Non-euclidean fixed-point sets.

Lemma 5.1. *If $p \geqslant 3$ and $q \geqslant 0$, there is a compact space X such that X is not a (finite) polyhedron but the join $X * S^q$ is a $(p+q+1)$-sphere.*

Proof. Since join is associative, it suffices to prove the lemma assuming that $q = 0$. I.e., we need a non-polyhedron X whose suspension is a $(p+1)$-sphere, whenever $p \geqslant 3$. Let A be an arc in S^p such that the fundamental group of $S^p - A$ is infinitely generated. (See [3].) Then clearly S^p/A, S^p with A identified to a point, is not a (finite) polyhedron. Let $X = S^p/A$. As Brown observes in [5], the suspension of X is a $(p+1)$-sphere by the argument of [1].

Theorem 5.2. *If α is a fixed-point-free action of G on S^q, $q \geqslant 0$, $p \geqslant 3$, then there is an action $\bar{\alpha}$ of G on $S^{p+q+1} = S^p * S^q$ such that*

(1) $\bar{\alpha} = \alpha$ on S^q,

(2) the fixed-point set X of $\bar{\alpha}$ is not a (finite) polyhedron,

(3) $\bar{\alpha}$ restricts to a fixed-point-free action on $S^{p+q+1} - X$.

Proof. This is obvious from (5.1), by taking the "join" of the identity on X with α on S^q. Since $p \geqslant 3$, S^q is a topologically unknotted q-sphere in $X * S^q \approx S^{p+q+1}$, so that we can assume S^q to be in the standard position in $S^p * S^q$. (See [18].)

Definition. If α is an action of the (topological) group G on P, a polyhedron, call α *totally wild* if, whenever $1 \neq g \in G$, α_g is not conjugate to a piecewise linear homeomorphism; i.e., there is no (topological) homeomorphism h *of* P such that $h^{-1}\alpha_g h$ is piecewise linear.

Remark. All of the circle action constructed in Section 4 are totally wild. In Section 3, a constructed action is totally wild if the original fixed-point-free action is free.

THEOREM 5.3. *If there is a free action of G on S^q, $q \geqslant 0$, $p \geqslant 3$, then there is a totally wild action of G on S^{p+q+1}.*

Proof. Let $\bar{a}$ be action constructed in the proof of Theorem 5.2, where a is taken to be free. Then, for each $1 \neq g \in G$, X is the fixed-point set of $\bar{a}_g$. If $\bar{a}_g$ were conjugate to a piecewise linear homeomorphism, then X would be homeomorphic to the fixed-point set of a piecewise linear map, which is impossible since X is not a polyhedron.

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INSTITUTE FOR ADVANCED STUDY
Princeton, New Jersey
FLORIDA STATE UNIVERSITY
Tallahassee, Florida

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About an imbedding conjecture for k-independent sets

by

A. B. Németh (Cluj)

Following [1] we say that a subset X of the n-dimensional real Euclidean space R^n is *k-independent* ($0 \leqslant k \leqslant n-1$) if any $k+2$ distinct points of that subset are linearly independent. (¹)

In what follows the homeomorphic image of the set $\{(x^1, \ldots, x^m)\colon \sum (x^i)^2 < 1\}$ in R^m will be said to be an *open m-cell*; the homeomorphic image of the set $\{(x^1, \ldots, x^m)\colon \sum (x^i)^2 = 1\}$ will be said to be an *$m-1$-sphere*.

K. Borsuk [1] has proved the following imbedding theorem concerning k-independent sets:

If X is a compact k-independent set in R^n and if N is an open subset in X containing k distinct points, then $X \setminus N$ is homeomorphic with a subset of R^{n-k}.

In [6], p. 503 and in [4], another notion of k-independence is applied, which is useful in applications in the approximation theory and which will be called in the sequel *k-vectorial-independence.*

The subset X of R^n will be said to be *k-vectorial-independent* if for any k of its distinct points $x_1, \ldots, x_k$ the vectors $\overrightarrow{Ox_1}, \ldots, \overrightarrow{Ox_k}$, where O is the origin in R^n, are linearly independent.

OBSERVATION 1. *A k-vectorial-independent subset X in R^n is $k-2$-independent in the sense of* [1].

Indeed, if $x_1, \ldots, x_k$ are k distinct points in X, then they cannot be contained in any $k-2$-dimensional hyperplane H^{k-2}, because such a hyperplane generates a $k-1$-dimensional subspace (i.e. a $k-1$-dimensional hyperplane passing through the origin), and if $x_1, \ldots, x_k$ were in H^{k-2}, the vectors $\overrightarrow{Ox_1}, \ldots, \overrightarrow{Ox_k}$ would be linearly dependent, being in R^{k-1}.

OBSERVATION 2. *If X is a k-independent subset in R^n, then it may be considered a $k+2$-vectorial-independent subset in R^{n+1} if we consider R^n as a hyperplane H^n in R^{n+1} not passing through the origin.*

(¹) For the sake of simplicity, the affine space and the vectorial Euclidean space of dimension n are denoted by the same symbol R^n.