An acyclic continuum that admits no mean

by

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If Y is a nonvoid Hausdorff space, a *mean* on Y is a continuous function $M\colon Y\times Y\to Y$ such that

(a) $M(x,y)=M(y,x)$;

and

(b) $M(x,x)=x$;

whenever $x,\ y\in Y$. In this note we verify the following conjecture of A. D. Wallace.

THEOREM. *There is no mean on the* $\sin(1/x)$*-continuum* S*, where*

$S=C\cup W$;

$C=\{(0,y)\colon\ -1\leqslant y\leqslant 1\}$;

and

$W=\{(x,\sin(1/x))\colon\ 0<x\leqslant 1\}$.

I am indebted to Kermit Sigmon for providing me a prepublication copy of [2], which contains a proof of a weaker form of the theorem above.

LEMMA. *Let I denote the number interval* $[0,1]$. *If $\{H,K\}$ is an open cover of $I\times I$ then either some component of H intersects both $I\times\{0\}$ and $I\times\{1\}$ or some component of K intersects both $\{0\}\times I$ and $\{1\}\times I$.*

Proof of the lemma. Suppose there is an open cover $\{H,K\}$ of $I\times I$ for which the conclusion of the lemma does not hold. Let H_0 be the union of all components of H that intersect $I\times\{1\}$. Since $I\times I$ is locally connected, H_0 is open, as is $H_1=H-H_0$. Evidently

$$\big(H_0\cup(I\times\{1\})\big)\cap\big(H_1\cup(I\times\{0\})\big)=\emptyset\,.$$

Similarly K is the union of disjoint open sets K_0 and K_1 such that

$$\big(K_0\cup(\{1\}\times I)\big)\cap\big(K_1\cup(\{0\}\times I)\big)=\emptyset\,.$$

Define $A_i=(I\times I)-H_i$; $B_i=(I\times I)-K_i$; for i in $\{0,1\}$. We have that $I\times\{i\}\subset A_i$, $\{i\}\times I\subset B_i$ $(i\in\{0,1\})$, $A_0\cup A_1=I\times I=B_0\cup B_1$ and no point is common to the closed sets A_0, A_1, B_0 and B_1. This is easily seen to be inconsistent with the $n = 2$ case of a combinatorial theorem due to H. W. Kuhn ([1], p. 519).

Proof of the theorem. Fitrs let us anatomize S. For each n in $\{0, 1, 2, \dots\}$ define $x_n = 2/(2n+1)\pi$. Notice that $\sin(1/x_n) = 1$ if n is even, $\sin(1/x_n) = -1$ if n is odd and the $\sin(1/x)$-function is monotonic on any interval of the form $[x_{n+1}, x_n]$. Define

$$
\begin{aligned}
A_n &= \{(x, \sin(1/x)):\ x_{n+1} \leqslant x \leqslant x_n\}, \qquad n \in \{0, 1, 2, \dots\};\\
L_n &= \{(x, y) \in S:\ x \leqslant x_n\}, \qquad n \in \{0, 1, 2, \dots\};\\
C_P &= \{(0, y):\ 0 \leqslant y \leqslant 1\};\\
C_N &= \{(0, y):\ -1 \leqslant y \leqslant 0\};\\
H_P &= \{(x, y) \in S:\ y > -(1/2)\};\\
H_N &= \{(x, y) \in S:\ y < 1/2\};\\
Y &= \{((0, y), (0, z)) \in C \times C:\ y \leqslant z\}.
\end{aligned}
$$

Let $D\colon S \to S \times S$ be the diagonal map, defined by the formula $Dp = (p, p)$, and let $T\colon S \times S \to S \times S$ be the symmetry map, defined by $T(p, q) = (q, p)$. Finally, define $w\colon S \times S \to C \times C$ by

$$w((x_1, y_1), (x_2, y_2)) = ((0, y_1), (0, y_2)).$$

Now suppose that there is a mean M on S. Since M is continuous, $\{M^{-1}H_P, M^{-1}H_N\}$ is an open cover of $S \times S$. Since MD is the identity on S, there is a component K_P of $Y \cap M^{-1}H_P$ that contains DC_P and there is a component K_N of $Y \cap M^{-1}H_N$ that contains DC_N.

Assume that K_P intersects $\{(0, -1)\} \times C$. There is a closed connected subset J of K_P that intersects $\{(0, -1)\} \times C$ and contains DC_P. Since M is symmetric, TJ is a subset of $M^{-1}H_P$ and intersects $C \times \{(0, -1)\}$. Define, for each n in $\{0, 1, 2, \dots\}$,

$$Q_n = (L_n \times L_n) \cap w^{-1}(J \cup TJ).$$

Since $\{Q_n\}_{n=0}^{\infty}$ is a tower of compact sets whose common part, $J \cup TJ$, is a subset of the open set $M^{-1}H_P$, there is a positive even integer t such that $Q_t \subset M^{-1}H_P$. For any i, j the restriction of w to $A_i \times A_j$ is a homeomorphism onto $C \times C$. So each of $Q_t \cap (A_t \times A_t)$, $Q_t \cap (A_{t+1} \times A_t)$ and $Q_t \cap (A_{t+1} \times A_{t+1})$ is connected. Since $J \cup TJ$ intersects both $\{(0, -1)\} \times C$ and $C \times \{(0, -1)\}$, $Q_t \cap (A_{t+1} \times A_t)$ intersects both $Q_t \cap (A_t \times A_t)$ and $Q_t \cap (A_{t+1} \times A_{t+1})$. The union Z of all three is a connected subset of $M^{-1}H_P$ that contains both $D(x_t, 1)$ and $D(x_{t+2}, 1)$. So MZ is a connected subset of H_P that contains both $(x_t, 1)$ and $(x_{t+2}, 1)$, which contradicts the fact that H_P is the union of two separated sets, one containing $(x_t, 1)$ and the other containing $(x_{t+2}, 1)$. Our assumption that K_P intersects $\{(0, -1)\} \times C$ is not valid. In a similar manner it may be shown that K_N does not intersect $C \times \{(0, 1)\}$. But these two statements are jointly inconsistent with the lemma. The proof is complete.

We close with the observation that S is chainable, that W is dense and open in S and that S is acyclic, in the sense of Alexander–Čech cohomology theory. This suggests some questions.

Is the arc the only chainable continuum that admits a mean?

Is the arc the only compact metric continuum containing an open dense half-line that admits a mean?

Is there an acyclic Peano continuum that admits no mean?

References

[1] H. W. Kuhn, *Some combinatorial lemmas in topology*, IBM J. Res. Develop. 4 (1960), pp. 518-524.

[2] Kermit Sigmon, *On the existence of a mean on certain continua*, Fund. Math. 63 (1968), pp. 311-319.

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