Dans ce modèle, K est évidemment une classe propre. Si elle y était décomposable, il existerait deux classes propres disjointes K_1 et K_2 telles que

$$K = K_1 \cup K_2 \wedge \Psi(K_1) \wedge \Psi(K_2)\,,$$

donc il existerait un ordinal α tel que pour tout $\beta, \gamma \geqslant \alpha$:

$$J_{\beta\gamma}^{``} K_1 = K_1\,.$$

Mais K_1 et K_2 étant propres, il devrait exister des ordinaux $\beta_0, \gamma_0 \geqslant \alpha$ tels que $R`\beta_0 \in K_1$ et $R`\gamma_0 \in K_2$, d'où

$$J_{\beta_0\gamma_0}^{``} K_1 \neq K_1\,,$$

ce qui est absurde.

Remarque. Toute formule *purement ensembliste* Φ est invariante: $\Phi_m \Leftrightarrow \Phi$. Il en résulte que l'axiome du choix local E_l ("Tout ensemble admet une fonction-choix") est vrai dans $\mathcal{M}$. Il est cependant aisé de vérifier que l'axiome du choix universel E est faux dans $\mathcal{M}$.

Corollaire. *Si le système* (ABC) *est consistant, alors* (ABC) *et* (ABC E_l) *sont des systèmes non compressifs dans lesquels il est consistant de supposer que K est une classe propre indécomposable.*

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Pointwise limits of sequences of functions

by

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Each of Baire [6], Lebesgue [6], and Mazurkiewicz [3] has obtained characterizations of the class c_1 of functions which are pointwise limits of sequences from the class c_0 of continuous functions. In this paper, we obtain a theorem which (1) gives two new characterizations of c_1; (2) gives two characterizations of the class r_1 of functions which are pointwise limits of sequences from the class r_0 of continuous on the right functions; (3) gives two characterizations of the class j_1 of functions which are pointwise limits of sequences from the class j_0 of jump functions (A function f with domain $[0,1]$ is a *jump function* means that $f(0+)$ exists, $f(1-)$ exists, and for each x in $(0,1)$, $f(x+)$ and $f(x-)$ exist.); (4) has the following corollary: if $\{c_2\}\{r_2\}\{j_2\}$ denotes the class of functions which are pointwise limits of sequences of class $\{c_1\}\ \{r_1\}\ \{j_1\}$, then $c_1 \subset r_1 \subset j_1$, $c_1 \neq r_1 \neq j_1$, and $c_2 = r_2 = j_2$.

For simplicity, all functions discussed here will be real valued and have domain $[0,1]$. The number p will be said to be a {*condensation point*} {*limit point from the right*} {*limit point from the left*} of the set X if and only if each neighborhood of p contains {uncountably many points of X} {a point of X to the right of p} {a point of X to the left of p}. If X is a set, $\{\mathrm{con}(X)\}$ $\{\mathrm{rt}(X)\}$ $\{\mathrm{lft}(X)\}$ will denote the union of X and the set of its {condensation points} {limit points from the right} {limit points from the left}. If G is a collection of sets, G^* will denote the union of the members of G.

Theorem. *If f is a real-valued function defined on* $[0,1]$, *then*

$$\mathrm{I}_{\mathrm{A}} \equiv \mathrm{II}_{\mathrm{A}} \equiv \mathrm{III}_{\mathrm{A}}, \quad \mathrm{I}_{\mathrm{B}} \equiv \mathrm{II}_{\mathrm{B}} \equiv \mathrm{III}_{\mathrm{B}}, \quad \text{and} \quad \mathrm{I}_{\mathrm{C}} \equiv \mathrm{II}_{\mathrm{C}} \equiv \mathrm{III}_{\mathrm{C}}\,.$$

I. *f is the pointwise limit of a sequence from*

$$\{(\mathrm{A})\ c_0\} \quad \{(\mathrm{B})\ r_0\} \quad \{(\mathrm{C})\ j_0\}\,.$$

II. *If a and b are numbers $(a > b)$, then there exist sequences $T_1(a,b)$, $T_2(a,b)$, ... and $B_1(a,b)$, $B_2(a,b)$, ... such that:*

(1) *for each n, $T_n(a,b)$ and $B_n(a,b)$ are finite collections of*

{(A) *intervals*} {(B) *sects closed on the left*} {(C) *connected number sets*}

such that $[T_n(a,b)]^* \cap [B_n(a,b)]^* = \emptyset$,

(2) $\{x\colon f(x) > a\} \subset \bigcup_{k=1}^{\infty} \bigcap_{j=k}^{\infty} [T_j(a, b)]^*$ *and*

$\{x\colon f(x) < b\} \subset \bigcup_{k=1}^{\infty} \bigcap_{j=k}^{\infty} [B_j(a, b)]^*$.

III. *If* $a > b$, $U \subset \{x\colon f(x) > a\}$ *and* $V \subset \{x\colon f(x) < b\}$, *then*

$\{(\mathrm{A})\, U \not\subset \mathrm{cl}(V)$ or $V \not\subset \mathrm{cl}(U)\}$ $\{(\mathrm{B})\, U \not\subset \mathrm{rt}(V)$ or $V \not\subset \mathrm{rt}(U)\}$

$\{(\mathrm{C})\, U \not\subset \mathrm{con}(V)$ or $V \not\subset \mathrm{con}(U)\}$.

Proof. We will first show that $\mathrm{I_C} \to \mathrm{III_C}$. Suppose $\mathrm{I_C} \not\to \mathrm{III_C}$. There is an f in j_1, a sequence $f_1, f_2, \ldots$ from j_0 converging pointwise to f, numbers a and b $(a > b)$ and sets U and V such that

$$U \subset \{x\colon f(x) > a\}, \quad V \subset \{x\colon f(x) < b\}, \quad U \subset \mathrm{con}(V) \text{ and } V \subset \mathrm{con}(U).$$

Let a' and b' be numbers such that $a > a' > b' > b$. For each j, set

$$u_j = \bigcap_{k=j}^{\infty} \{x\colon f_k(x) > b'\} \quad \text{and} \quad v_j = \bigcap_{k=j}^{\infty} \{x\colon f_k(x) < a'\}.$$

Notice that $[0, 1] \subset \bigcup_{k=1}^{\infty} (u_k \cup v_k)$. Suppose that I is an interval such that $I \subset [0, 1]$, $I \cap U$ is uncountable and k is a positive integer. There is an $n' > k$ such that $\{x\colon f_{n'}(x) > a\} \cap U \cap I$ is uncountable. Since $f_{n'}$ is in j_0, it is not discontinuous at uncountably many points and therefore there is a subinterval I' of I such that $I' \cap U$ is uncountable and $I' \subset \{x\colon f_{n'}(x) > a\}$. Similarly, there is a subinterval I'' of I' and an $n'' > n'$ such that $I'' \cap U$ is uncountable and $I'' \subset \{x\colon f_{n''}(x) > b\}$. Therefore, $I'' \subset I$, $I'' \cap U$ is uncountable and $I'' \cap (u_k \cup v_k) = \emptyset$. Therefore, one can construct a monotonically decreasing sequence $I_1, I_2, \ldots$ of subintervals of $[0, 1]$ such that for each p, $I_p \cap (u_p \cup v_p) = \emptyset$. This constitutes a contradiction. Therefore $\mathrm{I_C} \to \mathrm{III_C}$. The fact that $\mathrm{I_A} \to \mathrm{III_A}$ and $\mathrm{I_B} \to \mathrm{III_B}$ can be established by arguments parallel to the one above. These proofs are therefore omitted from the paper.

We will now define conditions $\mathrm{II'_A}$, $\mathrm{II'_B}$, and $\mathrm{II'_C}$ equivalent respectively to $\mathrm{II_A}$, $\mathrm{II_B}$, and $\mathrm{II_C}$.

II′. *If* a *is a number, then there exist sequences* $T_1(a), T_2(a), \ldots$ *and* $B_1(a), B_2(a), \ldots$ *such that:*

(1) *for each* n, $T_n(a)$ *and* $B_n(a)$ *are finite collections of* {(A) *intervals*} {(B) *sects closed on the left*} {(C) *connected number sets*} *such that* $[T_n(a)]^* \cap [B_n(a)]^* = \emptyset$;

(2) $\{x\colon f(x) > a\} \bigcup_{k=1}^{\infty} \bigcap_{j=k}^{\infty} [T_j(a)]^*$ *and*

$\{x\colon f(x) < a\} \subset \bigcup_{k=1}^{\infty} \bigcap_{j=k}^{\infty} [B_j(a)]^*$.

$\mathrm{II'_A}$ obviously implies $\mathrm{II_A}$. We will now show that $\mathrm{II_A} \to \mathrm{II'_A}$. For each pair of numbers a and b $(a > b)$ define two sequences $T_1(a, b)$, $T_2(a, b), \ldots$ and $B_1(a, b), B_2(a, b), \ldots$ which satisfy $\mathrm{II_A}$. Suppose that z is a number. For each positive integer n, let

$$t_n = \bigcup_{k=1}^{n} \Big([T_n(z+1/k, z)]^* \cap \bigcap_{j=1}^{k} [T_n(z, z-1/j)]^*\Big),$$

$$b_n = \bigcup_{k=1}^{n} \Big([B_n(z, z-1/k)]^* \cap \bigcap_{j=1}^{k} [B_n(z+1/j, z)]^*\Big).$$

Suppose that there is an n such that $b_n \cap t_n \neq \emptyset$. Let x denote a member of $b_n \cap t_n$. Since x is in b_n, there is a $k \leqslant n$ such that $x \in [B_n(z, z-1/k)]^* \cap \bigcap_{j=1}^{k} [B_n(z+1/j, z)]^*$. Since x is in t_n, there is an $L \leqslant n$ such that $x \in T_n[(z+1/L, z)]^* \cap \bigcap_{j=1}^{L} [T_n(z, z-1/j)]^*$. If $L \leqslant k$, x is in both $[T_n(z+1/L, z)]^*$ and $[B_n(z+1/L, z)]^*$. If $L > k$, x is in both $[B_n(z, z-1/k)]^*$ and $[T_n(z, z-1/k)]^*$. This constitutes a contradiction. Therefore, $b_n \cap t_n = \emptyset$. Since, for each p, b_p and t_p are closed number sets with only finitely many components, there exist two collections of intervals $T_p(z)$ and $B_p(z)$ such that $t_p \subset [T_p(z)]^*$, $b_p \subset [B_p(z)]^*$ and $[T_p(z)]^* \cap [B_p(z)]^* = \emptyset$. Suppose that x is a number such that $f(x) > z$. There is a positive integer k such that $f(x) > z+1/k$. Therefore there is an n such that for each $n' > n$,

$$x \in [T_{n'}(z+1/k, z)]^* \cap \bigcap_{j=1}^{k} [T_{n'}(z, z-1/j)]^* \quad \text{and therefore } x \in t_{n'}.$$

Similarly, $\{x\colon f(x) < z\} \subset \bigcup_{k=1}^{\infty} \bigcap_{p=k}^{\infty} b_p$. Therefore $T_1(z), T_2(z), \ldots$ and $B_1(z)$, $B_2(z), \ldots$ satisfy the conditions imposed on the sequences in $\mathrm{II'_A}$. Notice that $\mathrm{II_B} \equiv \mathrm{II'_B}$ and $\mathrm{II_C} \equiv \mathrm{II'_C}$ by arguments parallel to the above.

It will now be demonstrated that $\mathrm{II_A}$ implies $\mathrm{I_A}$ by showing that $\mathrm{II'_A}$ implies $\mathrm{I_A}$. For each number c, let $T_1(c), T_2(c), \ldots$ and $B_1(c), B_2(c), \ldots$ denote two sequences which satisfy the conditions imposed on the sequences in $\mathrm{II'_A}$. For each positive rational number c ($c = r/s$ in lowest terms) and each positive integer n, the sets $E(c)$, $U_n(c)$, $u_n(c)$, $v_n(c)$, $V_n(c)$, M_n, $P_n(c)$, and ${}_nf_c$ will now be defined. $E(c)$ is the set of non-negative rational numbers to which D ($D = u/v$ in lowest terms) belongs if and only if $v \leqslant s$ and $D \leqslant c$.

$$U_n(c) = [0, 1] \cap \bigcap_{D \in E(c)} [T_n(D)]^*.$$

$$u_n(c) = [0, 1] \cap [T_n(0)]^* \cap \bigcup_{D \in E(c)} [B_n(D)]^*.$$

$$v_n(c) = [0, 1] \cap [B_n(0)]^* \cap \bigcup_{D \in E(c)} [T_n(-D)]^*.$$

$$V_n(c) = [0, 1] \cap \bigcap_{D \in E(c)} [B_n(-D)]^*.$$

M_n is the set consisting of the centers of the components of $[0,1]-$ $-[0,1]\cap([T_n(0)]^*\cup[B_n(0)]^*)$. $P_n(c)$ is the point set in the plane to which (x,y) belongs if and only if either (1) x is in $U_n(c)$ and $y=c$; (2) x is in $V_n(c)$ and $y=-c$; or (3) x is in $\{0\}\cup\{1\}\cup M_n\cup u_n(c)\cup v_n(c)$ but not in $V_n(c)\cup U_n(c)$ and $y=0$. Since $U_n(c)$, $V_n(c)$, $u_n(c)$, $v_n(c)$ and M_n are mutually exclusive closed sets each with only finitely many components, $P_n(c)$ is a point set in the plane which is the union of a finite collection of degenerate sets and horizontal line intervals and no two points of $P_n(c)$ have the same abcissa. Let ${}_nf_c$ denote the polygon with vertices the boundary points of $P_n(c)$. ${}_nf_c$ is the graph of a continuous function with domain $[0,1]$. Let $r_1, r_2, \ldots$ denote a sequence of positive rational numbers which contains each rational number only once.

We will now define a sequence $g_1, g_2, \ldots$ of continuous functions converging pointwise to f. For each positive integer n and each number x in $[0,1]$ define g_n as follows: If x is in M_n, then set $g_n(x)=0$. If x is not in M_n and $[T_n(0)]^*$ intersects the component of $([0,1]-M_n)$ containing x, then set

$$g_n(x)=\max[{}_nf_{r_1}(x), {}_nf_{r_2}(x), \ldots, {}_nf_{r_n}(x)]\,.$$

If x is not in M_n and $[T_n(0)]^*$ does not intersect the component of $([0,1]-M_n)$ containing x, then set

$$g_n(x)=\min[{}_nf_{r_1}(x), {}_nf_{r_2}(x), \ldots, {}_nf_{r_n}(x)]\,.$$

Since, for each n, each of ${}_nf_{r_1}, {}_nf_{r_2}, \ldots, {}_nf_{r_n}$ is continuous and vanishes at each member of M_n, g_n is continuous.

Suppose that x is a number in $[0,1]$ such that $f(x)>0$ and suppose that $\delta>0$. There exist integers j and k such that $f(x)-\delta<r_j<f(x)$ $<r_k<f(x)+\delta$. Write $r_k=a/b$ in lowest terms and let Y denote the set of positive rational numbers to which c/d (written in lowest terms) belongs if and only if $d\leqslant b$ and $(c-1)/d<r_k\leqslant c/d$. There is a positive integer $n\geqslant\max(j,k)$ such that if $n'>n$, then

$$x\in\bigcap_{D\in E(r_j)}[T_{n'}(D)]^*\cap\bigcap_{D\in Y}[B_{n'}(D)]^*.$$

Suppose that $m>n$. Since $x\in[T_m(0)]^*$,

$$g_m(x)=\max[{}_mf_{r_1}(x), {}_mf_{r_2}(x), \ldots, {}_mf_{r_m}(x)]; \quad x\in\bigcap_{D\in E(r_j)}[T_m(D)]^*.$$

Therefore, $x\in U_m(r_j)$. Therefore, ${}_mf_{r_j}(x)=r_j$. Therefore, $g_m(x)\geqslant r_j$. However, for each L such that $r_L\geqslant r_k$, there is a member of Y in $E(r_L)$. Therefore, $x\in u_m(r_L)$. Therefore, ${}_mf_{r_L}(x)=0$. Therefore, $g_m(x)<r_k$. Therefore, $|g_m(x)-f(x)|<\delta$. Similarly, if x is a number in $[0,1]$ such that $f(x)\leqslant 0$, then $|g_m(x)-f(x)|<\delta$ for sufficiently large integers m. Therefore, $g_1, g_2, \ldots$ is a sequence of continuous functions converging pointwise to f. Therefore, $\mathrm{II_A}\to\mathrm{I_A}$. The fact that $\mathrm{II_B}\to\mathrm{I_B}$ and $\mathrm{II_C}\to\mathrm{I_C}$ can be established by arguments parallel to the one above. These proofs are therefore omitted from the paper.

In order to facilitate the description of certain collections of sets used in the proofs that $\mathrm{III_A}$, $\mathrm{III_B}$, and $\mathrm{III_C}$ imply $\mathrm{II_A}$, $\mathrm{II_B}$ and $\mathrm{II_C}$, respectively, the terms "acceptable", "determines", and "total (M,Q) result" will now be introduced.

Suppose that K is a collection of sets. The statement that H is an *acceptable* subcollection of K means that H is K, or else each element of $K-H$ is a subset of every element of H. The word *determines* is not defined at this point but is assumed to have a meaning which satisfies the following condition: no collection of sets determines two sets, and if H is a collection of sets which determines the set x, then x is a subset of each element of H and each subcollection of H determines a set which has x as a subset. If M is a set and Q is a meaning of the word determines, then H is a *total* (M,Q) *result* means that H is a collection with the following four properties: (1) M is in H and each element of H is a subset of M; (2) if G is an acceptable subcollection of H such that G determines a set x and no subcollection of G determines a member of itself, then x is in H; (3) if y is a member of H and K is a subcollection of H such that K does not contain y and K determines y, then no subcollection of K determines a member of itself; and (4) if y is a member of H distinct from M, then there is an acceptable subcollection G of H such that G determines y but G does not contain y.

It can be demonstrated that if M is a set and Q is a meaning of the word determines, then there is only one total (M,Q) result. Moreover, if x precedes y means that y is a proper subset of x, then the total (M,Q) result is well ordered with respect to this meaning of the word precedes [4]. See [2].

The following notation will be used in the proof that $\mathrm{III}\to\mathrm{II}$. If K is a collection of point sets and t is an element of a set in K or t is a subset of a set in K, then K_t will denote the common part of all of the members of K containing t.

We will now show that $\mathrm{III_A}\to\mathrm{II_A}$. Suppose that f is a function defined on $[0,1]$ and f has property $\mathrm{III_A}$. Let a and b be two numbers $(a>b)$. In the non-trivial case there is an $f(x)>a$ and an $f(z)<b$. Let $U=\{x\colon f(x)>a\}$ and let $V=\{x\colon f(x)<b\}$. The statement that the collection K of number sets determines the number set A in the $\{S\}\{T\}$ sense means that

$$\{A=\bigcap_{x\in K}x\cap\mathrm{cl}[V\cap\mathrm{cl}(\bigcap_{x\in K}x)]\}\quad\{A=\bigcap_{x\in K}x\cap\mathrm{cl}[U\cap\mathrm{cl}(\bigcap_{x\in K}x)]\}\,.$$

Let $\{G\}\{H\}$ denote the total $\{U,S\}\{V,T\}$ result. For each positive integer n, let $\{J_n\}\{K_n\}$ denote the collection of intervals to which the interval I belongs if and only if: (1) the length of I is $1/n$; (2) the center x of I is in $\{U\}\{V\}$; and (3)

$$\{[x-4/n,\, x+4/n] \cap V \cap \mathrm{cl}(G_x) = \emptyset\}$$

$$\{[x-4/n,\, x+4/n] \cap U \cap \mathrm{cl}(H_x) = \emptyset \text{ and } [x-2/n,\, x+2/n] \cap \mathrm{cl}[(J_n)^*] = \emptyset\}.$$

We will now show that

$$U \subset \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} (J_n)^*.$$

Let $u \in U$. If $u \in \mathrm{cl}[V \cap \mathrm{cl}(G_u)]$, then $G_u \cap \mathrm{cl}[V \cap \mathrm{cl}(G_u)]$ is a member of G containing u. Therefore, $G_u = G_u \cap [\mathrm{cl}(V) \cap \mathrm{cl}(G_u)]$. Therefore, $V \cap \mathrm{cl}(G_u) \subset \mathrm{cl}(G_u)$ and $G_u \subset \mathrm{cl}[V \cap \mathrm{cl}(G_u)]$. This contradicts $\mathrm{III_A}$. Therefore, $u \notin \mathrm{cl}[V \cap \mathrm{cl}(G_u)]$. Hence, there is an n such that for all $n' > n$, $[u-4/n',\, u+4/n'] \cap [V \cap \mathrm{cl}(G_u)] = \emptyset$. Therefore u is the center of an element of $J_{n'}$. Therefore, $U \subset \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} (J_n)^*$.

We will now show that $V \subset \bigcup_{j=1}^{\infty} \bigcap_{n=j}^{\infty} (K_n)^*$. Let $v \in V$. By an argument parallel to the one above, there is an m such that for all $m' > m$ $[v-4/m',\, v+4/m'] \cap U \cap \mathrm{cl}(H_v) = \emptyset$. Let $Q = \{x\colon x \in U$ and $v \in \mathrm{cl}(G_x)\}$. If n is a positive integer and x is a member of Q such that x is the center of a member of J_n, then $|x-v| > 4/n$. Therefore, if $U-Q = \emptyset$, then $v \in (K_{m'})^*$ for all $m' > m$. Suppose that $U-Q \neq \emptyset$. Either there is no set in G which is a proper subset of $G_{(U-Q)}$ or else there is one and therefore a first one. In either case, there is a member x of $(U-Q)$ such that $G_x = G_{(U-Q)}$. Therefore, $v \notin \mathrm{cl}[G_{(U-Q)}]$. Therefore, $v \notin \mathrm{cl}(U-Q)$ and there is an $r > m$ such that if $r' > r$, then $[v-4/r',\, v+4/r'] \cap \mathrm{cl}(U-Q) = \emptyset$. Therefore, $v \in (K_{r'})^*$. Therefore we have that $V \subset \bigcup_{j=1}^{\infty} \bigcap_{n=j}^{\infty} (K_n)^*$. Because of the second part of condition (3) in the definition of K_n, $\mathrm{cl}(K_n)^* \cap \mathrm{cl}(J_n)^* = \emptyset$ for all positive integers n.

For each positive integer n, let $\{T_n(a,b)\}\{B_n(a,b)\}$ denote the collection consisting of the components of $\{\mathrm{cl}(J_n)^*\}\{\mathrm{cl}(K_n)^*\}$. The sequences $T_1(a,b)$, $T_2(a,b)$, ... and $B_1(a,b)$, $B_2(a,b)$, ... satisfy the condition imposed on the sequences in $\mathrm{II_A}$. Therefore, $\mathrm{III_A} \to \mathrm{II_A}$.

We will now show that $\mathrm{III_C} \to \mathrm{II_C}$. Suppose that f has property $\mathrm{III_C}$ and suppose that a and b are numbers $(a > b)$. As before, let $U = \{x\colon f(x) > a\}$ and let $V = \{x\colon f(x) < b\}$. The statement that the collection K of number sets determines the number set A in the $\{Q\}\{R\}$ sense means that

$$\{A = \bigcap_{x \in K} x \cap \mathrm{con}[V \cap \mathrm{con}(\bigcap_{x \in K} x)]\}$$

$$\{A = \bigcap_{x \in K} x \cap \mathrm{con}[U \cap \mathrm{con}(\bigcap_{x \in K} x)]\}.$$

Let $\{G\}\{H\}$ denote the total $\{(U,Q)\}\{(V,R)\}$ result. Let $\{G'\}\{H'\}\{\alpha\}\{\beta\}$ denote the collection of sets to which W belongs if and only if there is an x in $\{U\}\{V\}\{V\}\{U\}$ such that $\{W = G_x\}\{W = H_x\}\{W = U \cap \mathrm{con}(H_x)\}\{W = V \cap \mathrm{con}(G_x)\}$. Let $\{U'\}\{V'\}\{U''\}\{V''\}$ denote the set to which z belongs if and only if there is a set Y in $\{G'\}\{H'\}\{\alpha\}\{\beta\}$ such that $z \in Y$ but z is not a condensation point of Y. Notice that in these definitions, Y must be $\{G'_z\}\{H'_z\}\{\alpha_z\}\{\beta_z\}$. Notice also that $G_z = G'_z$ and $H_z = H'_z$. Suppose that U' is uncountable. Since each uncountable number set contains one of its condensation points, there must be uncountably many members G_x of G' such that x is in U'. Suppose that no point x of U' is a limit point of a member of G' which is a proper subset of G_x. For each p in U', there is an integer n_p such that no segment of length $1/n_p$ containing p, intersects a member of G which is a proper subset of G_p. There is an integer k and an uncountable subset W of U' such that, if $p \in W$, then $n_p = k$. W has a condensation point x. Let s denote a segment of length $1/k$ containing x. There must be two points x and y of $s \cap W$ such that $G_x \neq G_y$. This constitutes a contradiction. Therefore, there is a point z of U' which is a limit point of the first member S of G which is a proper subset of G_z. $S = G_z \cap \mathrm{con}[V \cap \mathrm{con}(G_z)]$. z is in S. This constitutes a contradiction. Therefore, U' is not uncountable. Similarly, V' is not uncountable.

Suppose that U'' is uncountable. As before, there is a number z in U'' such that z is a limit point of the first member S of α which is a proper subset of α_z. There is a number y in V such that $\alpha_z = U \cap \mathrm{con}(H_y)$. $S = \big[U \cap \big(H_y \cap \mathrm{con}[U \cap \mathrm{con}(H_y)]\big)\big]$. z is a condensation point of $[U \cap \mathrm{con}(H_y)]$. This constitutes a contradiction. Therefore, U'' is not uncountable. Similarly, V'' is not uncountable. Let $U_c = U - (U' \cup U'')$ and let $V_c = V - (V' \cup V'')$. For each positive integer n, let J_n denote the collection of intervals to which I belongs if and only if: (1) the length of I is $1/n$; (2) the center x of I is in U_c; and (3) $[x-2/n,\, x+2/n] \cap V_c \cap \mathrm{con}(G_x) = \emptyset$.

We will now show that $U_c \subset \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} (J_n)^*$. Let $u \in U$. If $u \in \mathrm{cl}[V_c \cap \mathrm{con}(G_u)]$, then $u \in \mathrm{con}[V_c \cap \mathrm{con}(G_u)]$. Therefore, $G_u \cap \mathrm{con}[V_c \cap \mathrm{con}(G_u)]$ is a member of G containing u. Therefore, $G_u = G_u \cap \mathrm{con}[V_c \cap \mathrm{con}(G_u)]$ and we have that $G_u \subset \mathrm{con}[V_c \cap \mathrm{con}(G_u)]$ and $[V_c \cap \mathrm{con}(G_u)] \subset \mathrm{con}(G_u)$. This contradicts $\mathrm{III_C}$. Therefore, $u \notin \mathrm{cl}[V_c \cap \mathrm{con}(G_u)]$ and we have that $U_c \subset \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} (J_n)^*$.

We will now show that

$$V_c \subset \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} [(-\infty,\, \infty) - (J_n)]^*.$$

Let $v \in V_c$. Let $Q = \{x\colon x \in U_c \text{ and } v \in \operatorname{con}(G_x)\}$. If n is a positive integer and $x \in Q$ and x is the center of a member of J_n, then $|x-v| > 4/n$. Therefore, if $U_c - Q = \emptyset$ and n is a positive integer, then $v \notin (J_n)^*$. Suppose that $U_c - Q \neq \emptyset$. There is a number x in $(U_c - Q)$ such that $G_x = G_{(U_c-Q)}$. $v \notin \operatorname{con}(G_x)$. $v \notin \operatorname{con}(U_c \cap G_x)$. $v \notin \operatorname{cl}(U_c \cap G_x)$. $v \notin \operatorname{cl}(U_c - Q)$. Since v is not in $\operatorname{cl}(U_c - Q)$ and no member of J_n with center in Q contains v, we have that $V_c \subset \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} [(-\infty, \infty) - (J_n)^*]$.

If $\{U' \cup U'' \neq \emptyset\}$ $\{V' \cup V'' \neq \emptyset\}$, arrange its members into a sequence $\{u_1, u_2, \ldots\}$ $\{v_1, v_2, \ldots\}$. For each positive integer n, let $T_n(a, b)$ denote the collection consisting of the components of $[(J_n)^* \cup \bigcup_{k=1}^{n} \{u_n\}] - \bigcup_{k=1}^{n} \{v_n\}$ and let $B_n(a, b)$ denote the collection consisting of the components of $(-\infty, \infty) - T_n(a, b)^*$. The sequences $T_1(a, b)$, $T_2(a, b)$, ... and $B_1(a, b)$, $B_2(a, b)$, ... satisfy the conditions imposed on the sequences in $\mathrm{II_C}$. Therefore, $\mathrm{III_C} \to \mathrm{II_C}$.

We will now show that $\mathrm{III_B} \to \mathrm{II_B}$. Suppose that f has property $\mathrm{III_B}$ and suppose that a and b are numbers $(a > b)$. As before, let $U = \{x\colon f(x) > a\}$ and let $V = \{x\colon f(x) < b\}$. The statement that the collection K of number sets determines the set A in the $\{E\}$ $\{F\}$ sense means that

$$\{A = \bigcap_{x \in K} x \cap \operatorname{rht}[V \cap \operatorname{rht}(\bigcap_{x \in K} x)]\} \qquad \{A = \bigcap_{x \in K} x \cap \operatorname{rht}[U \cap \operatorname{rht}(\bigcap_{x \in K} x)]\}.$$

Let $\{G\}$ $\{H\}$ denote the total $\{(U, E)\}$ $\{(V, F)\}$ result.

For each positive integer n, let t_n denote the collection of sects closed on the left to which the sect s belongs if and only if: (1) the length of s is $1/n$; (2) the left end point x of s is in U; and (3) $[x, x+4/n] \cap V \cap \operatorname{rht}(G_x) = \emptyset$. For each n, let e_n denote the set consisting of the left end points of members of t_n.

We will now show that $U \subset \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} e_n$. Suppose that $u \in U$. If $u \in \operatorname{rht}[V \cap \operatorname{rht}(G_u)]$, then $G_u \cap \operatorname{rht}[V \cap \operatorname{rht}(G_u)]$ is a member of G containing u. Therefore, $G_u = G_u \cap \operatorname{rht}[V \cap \operatorname{rht}(G_u)]$. Therefore, $G_u \subset \operatorname{rht}[V \cap \operatorname{rth}(G_u)]$ and $V \cap \operatorname{rht}(G_u) \subset \operatorname{rht}(G_u)$. This contradicts $\mathrm{III_B}$. Therefore, there is an n such that for all $n' > n$, u is the left end point of a member of $t_{n'}$. Therefore, $U \subset \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} (e_n)$.

For each positive integer n and each component J of $(t_n)^*$, let $\alpha_n(J)$ denote the collection of closed on the left sects to which s belongs if and only if: (1) the length of s is $1/n$; (2) the left end point y of s is in $V \cap J$; (3) if $x \in J \cap e_n \cap [y-2/n, y]$, then $G_x \cap s = \emptyset$; and (4) if $x \in J \cap e_n \cap [y-1/n, y]$, then $y \in \operatorname{lft}(G_x)$. For each integer n, let M_n denote the set to which x belongs if and only if there is a component J of $(t_n)^*$ such that $x \in J - \operatorname{rht}([\alpha_n(J)]^*)$.

We will now show that $U \subset \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} M_n$. Suppose that $x \in U$. There is a positive integer n_1 such that $x \in \bigcap_{n=n_1}^{\infty} e_n$. For each integer $k > n_1$, let J_k denote the component of $(t_k)^*$ containing x. Let $N = \{w\colon x \notin \operatorname{cl}(G_w)\}$. If $N \neq \emptyset$, then there is a number y in N such that $G_N = G_y$. $x \notin \operatorname{cl}(G_y)$, $x \notin \operatorname{cl}(N)$. Therefore, there is a positive integer $n_2 > n_1$ such that if $w \in U$ and $x \notin \operatorname{cl}(G_w)$, then $|x-w| > 4/n_2$.

Suppose that there is an integer $k > n_2$ such that $x \in [\alpha_k(J_k)]^*$. There is a number y in $V \cap J_k$ and a sect s_y in $\alpha_k(J_k)$ such that y is the left end point of s_y and $x \in s_y$. There is a number w in J_k and a sect s_w in t_k such that w is the left end point of s_w and $y \in s_w$. Since $|w-x| < 2/k < 4/n_2$, $x \in \operatorname{cl}(G_w)$. Hence, there is a member of G_w in s_y. This contradicts part (3) of the definition of $\alpha_k(J_k)$. Therefore, if $k > n_2$, $x \notin [\alpha_k(J_k)]^*$.

Suppose that there exists a positive integer $i > n_2$ such that $x \in \operatorname{rht}[\alpha_i(J_i)]^*$. It follows from part (3) of the definition of $\alpha_i(J_i)$ that x is not a limit point of G_x from the right. Let δ denote a positive number such that $\delta < 1/i$ and $(x, x+\delta) \cap G_x = \emptyset$. There is a number y in $(x, x+\delta)$ such that y is the left end point of a sect in $\alpha_i(J_i)$. Since $G_x \cap (x, x+\delta) = \emptyset$, $y \notin \operatorname{lft}(G_x)$. This contradicts part (4) of the definition of $\alpha_k(J_k)$. Therefore, if $n > n_2$, then $x \in M_n$.

We will now show that $V \subset \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} [(-\infty, \infty) - M_n]$. Suppose that $y \in V$. Let $M = \{x\colon x \in U \text{ and } y \notin \operatorname{lft}(G_x)\}$. If $M \neq \emptyset$, then there is a number x in M such that $G_M = G_x$. $y \notin \operatorname{lft}(G_x)$. $y \notin \operatorname{lft}(M)$. Therefore, there is a positive integer n_1 such that if $x \in U \cap [y-4/n_1, y]$, then $y \in \operatorname{lft}(G_x)$. Similarly, there is a positive integer $n_2 > n_1$ such that if $x \in U \cap [y, y+4/n_2]$, then $y \in \operatorname{rht}(G_x)$. Suppose that there is a positive integer $n > n_2$ such that y is in a component J of $(t_n)^*$. Consider the sect $s = [y, y+1/n)$. We will show that s is in $\alpha_n(J)$. s has length $1/n$ and the left end point y of s is in $V \cap J$. Suppose that $x \in J \cap e_n \cap [y-2/n, y]$. Since $x \in e_n$, there is no point of $V \cap \operatorname{rht}(G_x)$ in $(x, x+4/n)$. Therefore, $y \notin \operatorname{rht}(G_x)$. Therefore, since $n > n_2$, no point of G_x is in s. Suppose that $x \in J \cap e_n \cap [y-1/n, y)$. Since $n > n_1$, $y \in \operatorname{lft}(G_x)$. Therefore, $s \in \alpha_n(J)$. Therefore, if $n > n_2$, then $y \notin M_n$.

For each positive integer n, let U_n denote the collection consisting of the components of M_n. U_1, U_2, ... is a sequence such that: (1) if n is a positive integer, U_n is a finite collection each member of which is a segment or a sect closed on the left; (2) $U \subset \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} (U_n)^*$; and (3) $V \subset \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} [(-\infty, \infty) - (U_n)^*]$.

By an argument parallel to the one above, one can construct a sequence $V_1, V_2, \ldots$ such that: (1) if n is a positive integer, V_n is a finite collection each member of which is a segment or a sect closed on the left; (2) $V \subset \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} (V_n)^*$; and (3) $U \subset \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} [(-\infty, \infty)-(V_n)^*]$.

For each positive integer n, let $W_n = (U_n)^* - (V_n)^*$ and let $Z_n = (V_n)^* - (U_n)^*$. Notice that if k is a positive integer, c is a component of W_k, and d is a component of Z_k, then there is a closed on the left sect c' containing c but no point of Z_k and a closed on the left sect d' containing d but no point of W_k. Therefore, there exist sequences $T_1(a, b)$, $T_2(a, b), \ldots$ and $B_1(a, b)$, $B_2(a, b), \ldots$ which satisfy the conditions imposed on the sequences in $\mathrm{II_B}$. Therefore, $\mathrm{III_B} \to \mathrm{II_B}$. This completes the proof of the theorem.

Observation I. It can be established by straight forward elementary arguments that each of the properties $\mathrm{II_A}$, $\mathrm{II_B}$, $\mathrm{II_C}$, $\mathrm{III_A}$, $\mathrm{III_B}$, and $\mathrm{III_C}$ can be replaced by a local property. For example, f has property $\mathrm{III_A}$ if and only if f has the following property: if $w \in [0, 1]$, then there is a segment s containing w such that if $a > b$, $U \subset s \cap \{x\colon f(x) > a\}$ and $V \subset s \cap \{x\colon f(x) < b\}$, then $U \not\subset \mathrm{cl}(V)$ or $V \not\subset \mathrm{cl}(U)$.

Observation II. We have noted that the "two number" condition II can be replaced by the corresponding "one number" condition II′. There is no corresponding "one number" condition which is an equivalent modification of III. This fact is illustrated by the following function $f \in c_1$. If x is an irrational number in $[0, 1]$, then $f(x) = 0$; if x is a rational number in $[0, 1]$ ($x = p/q$ in lowest terms), then $f(x) = 1/q$ in case q is even and $f(x) = -1/q$ in case q is odd.

Observation III. $c_1 \subset r_1 \subset j_1$ and $c_1 \neq r_1 \neq j_1$. c_1 is a subset of r_1, because each function which is continuous is continuous on the right. r_1 is a subset of q_1, because each function with property $\mathrm{II_B}$ has property $\mathrm{II_C}$. Let M denote a Cantor set in $[0, 1]$. Define $f(x)$ to be 1 if x is in $([0, 1]-M)$ and 0 otherwise. Define $g(x)$ to be 1 if x is a limit point from the right of a component of $([0, 1]-M)$ and 0 otherwise. Define $h(x)$ to be 1 if x is a limit point of a component of $([0, 1]-M)$ and 0 otherwise. Notice that f is in c_1; g is in r_1 but not in c_1; h is in j_1 but not in r_1.

Observation IV. C. T. Tucker [5] has shown that the following two statements are equivalent: (1) f is in j_1; and (2) there is a countable subset T of $[0, 1]$ and a function g in c_1 such that if x is a number in $([0, 1]-T)$, then $f(x) = g(x)$. It follows from this result that $c_2 = j_2$. Since $c_2 \subset r_2 \subset j_2$, $c_2 = r_2 = j_2$.

Observation V. Using an argument similar to (but in fact simpler than) the argument that $\mathrm{II_A} \to \mathrm{I_A}$, it can be shown that if $f \in \mathrm{II_B}$, then f is

the pointwise limit of a sequence of continuous on the right jump functions. Therefore, we have that the following two statements are equivalent: (1) f is the pointwise limit of a sequence of continuous on the right functions; and (2) f is the pointwise limit of a sequence of continuous on the right jump functions.

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