Dans ce modèle, $K$ est effectivement une classe propre. Si elle y était décomposable, il existerait deux classes propres disjointes $K_1$ et $K_2$ telles que
\[ K = K_1 \cup K_2 \cap \varphi(K_1) \cap \varphi(K_2), \]
donc il existerait un ordinal $\alpha$ tel que pour tout $\beta, \gamma \geq \alpha$:
\[ J^\alpha_\beta K_1 = K_1. \]
Mais $K_1$ et $K_2$ étant propres, il devrait exister des ordinaux $\beta_0, \gamma_0 \geq \alpha$ tels que $E\beta_0 \in K_1$ et $E\gamma_0 \in K_2$, d'où
\[ J^{\beta_0}_\gamma K_1 \neq K_1, \]
ce qui est absurde.

Remarque. Toute formule purement ensembliste $\Phi$ est invariante: $\Phi^a \iff \Phi$. Il en résulte que l'axiome du choix local $E_1$ ("Tout ensemble admet une fonction-choix") est vrai dans $\mathcal{M}$. Il est cependant aisé de vérifier que l'axiome du choix universel $E$ est faux dans $\mathcal{M}$.

**Corollaire.** Si le système $(A, BC)$ est consistant, alors $(ABC)$ et $(ABC E)$ sont des systèmes non compréhensifs dans lesquels il est consistant de supposer que $K$ est une classe propre indécomposable.

**Travaux cités**


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**Pointwise limits of sequences of functions**

by

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Each of Baire [6], Lebesgue [8], and Mazurkiewicz [3] has obtained characterizations of the class $\mathcal{C}_0$ of functions which are pointwise limits of sequences from the class $\mathcal{C}_0$ of continuous functions. In this paper, we obtain a theorem which (1) gives two new characterizations of $\mathcal{C}_0$; (2) gives two characterizations of the class $\mathcal{R}_0$ of functions which are pointwise limits of sequences from the class $\mathcal{R}_0$ of continuous on the right functions; (3) gives two characterizations of the class $\mathcal{J}_0$ of functions which are pointwise limits of sequences from the class $\mathcal{J}_0$ of jump functions (A function $f$ with domain $[0, 1]$ is a jump function means that $f(0^+)$ exists, $f(1^-)$ exists, and for each $x$ in $(0, 1)$, $f(x^+)$ and $f(x^-)$ exist); (4) has the following corollary: if $\{a_x\}, \{r_x\}, \{j_x\}$ denotes the class of functions which are pointwise limits of sequences of class $\{a_x\}, \{r_x\}, \{j_x\}$, then $\mathcal{C}_0 \subset \mathcal{R}_0 \subset \mathcal{J}_0$,

$\mathcal{C}_0 \neq \mathcal{R}_0 \neq \mathcal{J}_0$.

For simplicity, all functions discussed here will be real valued and have domain $[0, 1]$. The number $p$ will be said to be a *condensation point* (limit point from the right) (limit point from the left) of the set $X$ if and only if each neighborhood of $p$ contains (uncountably many points of $X$) (a point of $X$ to the right of $p$) (a point of $X$ to the left of $p$). If $X$ is a set, then $\{\{x\} | \{r(x)\} \} \{\{x\} \} \{\{x\} \}$ will denote the union of $X$ and the set of its condensation points (limit points from the right) (limit points from the left). If $G$ is a collection of sets, $G^*$ will denote the union of the members of $G$.

**Theorem.** If $f$ is a real-valued function defined on $[0, 1]$, then

\[ I_0 = \Pi_4 = \Pi_3 \cap \Pi_2, \quad I_0 = \Pi_3 = \Pi_2, \quad \text{and} \quad I_0 = \Pi_2 = \Pi_3. \]

I. $f$ is the pointwise limit of a sequence from

\[ \{a_x\} \quad \{r_x\} \quad \{j_x\}. \]

II. If $a$ and $b$ are numbers ($a > b$), then there exist sequences $T(a,b)$, $R(a,b)$, $J(a,b)$, such that:

1. for each $n$, $T_n(a,b)$ and $R_n(a,b)$ are finite collections of $\{A\} \{B\} \{C\}$ intervals, $\{A\}$ sets closed on the left, $\{B\}$ connected number sets, such that $\{T_n(a,b)\} \cap [B_n(a,b)] = \emptyset$,
(2) \( \{ x: f(x) > a \} \subseteq \bigcup_{k=1}^{n} \bigcap_{j=k}^{m} (T_f(a, b))^j \) and
\( \{ x: f(x) < b \} \subseteq \bigcup_{k=1}^{n} \bigcap_{j=k}^{m} (B_f(a, b))^j \).

III. If \( a > b \), \( U \subseteq \{ x: f(x) > a \} \) and \( V \subseteq \{ x: f(x) < b \} \), then
\( \{(a) U \subseteq \text{cl}(V) \text{ or } V \subseteq \text{cl}(U)\} \cup \{(b) U \subseteq \text{cl}(V) \text{ or } V \subseteq \text{cl}(U)\} \).

Proof. We will first show that \( I_0 \to \Pi_0 \). Suppose \( I_0 \to \Pi_0 \). There is an \( f \) in \( f_0 \), a sequence \( f_1, f_2, \ldots \) from \( f_0 \) converging pointwise to \( f \), numbers \( a \) and \( b \) (\( a > b \)) and sets \( U \) and \( V \) such that
\( U \subseteq \{ x: f(x) > a \}, \quad V \subseteq \{ x: f(x) < b \}, \quad U \subseteq \text{cl}(V) \text{ and } V \subseteq \text{cl}(U). \)

Let \( a' \) and \( b' \) be numbers such that \( a > a' > b > b' \). For each \( j \), set
\[ u_j = \bigcap_{i=1}^{j} \{ x: f_i(x) > a' \} \] and \[ v_j = \bigcap_{i=1}^{j} \{ x: f_i(x) < b' \}. \]

Notice that \( [0, 1] \subseteq \bigcup_{k=1}^{n} (a_k \cup v_k) \). Suppose that \( I \) is an interval such that \( I \subseteq [0, 1] \), \( I \cap U \) is uncountable and \( k \) is a positive integer. There is an \( n' > k \) such that \( \{ x: f(x) > a \} \cup U \cap I \) is uncountable. Since \( f \) is in \( f_0 \), it is not discontinuous at uncountably many points and therefore there is a subinterval \( I' \) of \( I \) such that \( I' \cap U \) is uncountable and \( I' \cap \text{cl}(V) \subseteq \{ x: f(x) > a \} \). Similarly, there is a subinterval \( I'' \) of \( I' \) and an \( n'' > n' \) such that \( I'' \cap U \) is uncountable and \( I'' \cap \{ w_1 \cup v_1 \} = \emptyset \). Therefore, one can construct a monotonically decreasing sequence \( I_1, I_2, \ldots \) of subintervals of \( [0, 1] \) such that for each \( k \), \( I_k \cap (a_k \cup v_k) = \emptyset \). This constitutes a contradiction. Therefore \( I_0 \to \Pi_0 \). The fact that \( I_0 \to \Pi_0 \) and \( I_0 \to \Pi_0 \) can be established by arguments parallel to the one above. These proofs are therefore omitted from the paper.

We will now define conditions \( \Pi_0, \Pi_1, \) and \( \Pi_2 \) equivalent respectively to \( \Pi_0, \Pi_1, \) and \( \Pi_2 \).

II'. If \( a \) is a number, then there exist sequences \( T_1(a), T_2(a), \ldots \) and \( B_1(a), B_2(a), \ldots \) such that:

1. For each \( n \), \( T_n(a) \) and \( B_n(a) \) are finite collections of \( \{ (a) \text{ intervals} \} \) \( \{ (b) \text{ sets closed on the left} \} \) \( \{ (c) \text{ connected number sets} \} \) such that \( (T_n(a))^* \cap (B_n(a))^* = \emptyset \);

2. \( \{ x: f(x) > a \} \subseteq \bigcup_{k=1}^{n} \bigcap_{j=k}^{m} (T_f(a, b))^j \) and
\( \{ x: f(x) < b \} \subseteq \bigcup_{k=1}^{n} \bigcap_{j=k}^{m} (B_f(a, b))^j \).

\( \Pi_4 \) obviously implies \( \Pi_3 \). We will now show that \( \Pi_3 \to \Pi_4 \). For each pair of numbers \( a, b \) (\( a > b \)) define sequences \( T_f(a, b), T_f(b, a), \ldots \) and \( B_f(a, b), B_f(b, a), \ldots \) which satisfy \( \Pi_3 \). Suppose that \( s \) is a number. For each positive integer \( n \), let
\[ t_n = \bigcup_{i=1}^{n} \{ (T_f(s+1/i, z))^* \cap (T_f(s, s-1/i))^* \}, \]
\[ b_n = \bigcup_{i=1}^{n} \{ (B_f(s, s-1/i))^* \cap (B_f(s+1/i, z))^* \}. \]

Suppose that there is an \( n \) such that \( b_n \cap t_n \neq \emptyset \). Let \( x \) denote a member of \( b_n \cap t_n \). Since \( x \) is in \( b_n \), there is a \( k \leq n \) such that \( x \in \{ (B_f(s, s-1/k))^* \cap \bigcup_{i=1}^{n} \{ (T_f(s, z-1/i))^* \} \}. \) Since \( x \) is in \( t_n \), there is an \( L \leq n \) such that \( x \in \{ (T_f(s, z+1/L))^* \cap \bigcup_{i=1}^{n} \{ (T_f(s, z-1/i))^* \} \}. \) If \( L < h \), \( x \) is in both \( \{ (T_f(s, z+1/L))^* \} \) and \( \{ (B_f(s, s-1/k))^* \} \). If \( L > h, x \) is in both \( \{ (B_f(s, z-1/k))^* \} \) and \( \{ (T_f(s, z-1/i))^* \} \). This constitutes a contradiction. Therefore, \( b_n \cap t_n = \emptyset \).

Similarly, \( x \in \{ (T_f(s+1/k, z))^* \cap \bigcup_{i=1}^{n} \{ (T_f(s, z-1/i))^* \} \} \) and therefore \( x \in t_k \).

Similarly, \( \{ x: f(x) < c \} \subseteq \bigcup_{k=1}^{n} \bigcap_{i=1}^{n} B_f(c) \). Therefore \( T_f(c), T_f(d), \ldots \) and \( B_f(c), B_f(d), \ldots \) satisfy the conditions imposed on the sequences in \( \Pi_4 \). Notice that \( \Pi_4 \to \Pi_1 \) and \( \Pi_4 \to \Pi_2 \) by arguments parallel to the above.

It will now be demonstrated that \( \Pi_4 \) implies \( \Pi_4 \) by showing that \( \Pi_4 \) implies \( \Pi_4 \). For each number \( c \), let \( T_f(c), T_f(d), \ldots \) and \( B_f(c), B_f(d), \ldots \) denote two sequences which satisfy the conditions imposed on the sequences in \( \Pi_4 \). For each positive rational number \( c (= r/c \text{ in lowest terms}) \) and each positive integer \( n \), the sets \( E(c), U_n(c), v_n(c), U_n(c), v_n(c), \ldots \) will now be defined. \( E(c) \) is the set of non-negative rational numbers to which \( D = (D) \in \text{cl}(D) \text{ in lowest terms} \) belongs if and only if \( e \leq s \) and \( D \leq e \).

\[ U_n(c) = [0, 1] \cap \bigcup_{D \in E(c)} \{ (T_f(D))^* \}, \]
\[ v_n(c) = [0, 1] \cap \bigcup_{D \in E(c)} \{ (B_f(D))^* \}, \]
\[ V_n(c) = [0, 1] \cap \bigcup_{D \in E(c)} \{ (B_f(-D))^* \}. \]
$M_n$ is the set consisting of the centers of the components of $[0, 1] - [0, 1] \cap (T_n(0)^* \cup B_n(0)^*)$. $F_n(c)$ is the point set in the plane to which $(x, y)$ belongs if and only if either (1) $x$ is in $U_n(c)$ and $y = c$; (2) $x$ is in $V_n(c)$ and $y = -c$; or (3) $x$ is in $[0, 1] \cap M_n \cap u_n(c) \cup v_n(c)$ but not in $V_n(c) \cup U_n(c)$ and $y = 0$. Since $U_n(c), V_n(c), u_n(c), v_n(c)$, and $M_n$ are mutually exclusive closed sets with only finitely many components, $F_n(c)$ is a point set in the plane which is the union of a finite collection of degenerate sets and horizontal line intervals and no two points of $F_n(c)$ have the same abscissa. Let $f_3$ denote the polygon with vertices the boundary points of $F_n(c)$. $f_3$ is the graph of a continuous function with domain $[0, 1]$. Let $f_1, f_2, \ldots$ denote a sequence of positive rational numbers which contains each rational number only once.

We will now define a sequence $g_1, g_2, \ldots$ of continuous functions converging pointwise to $f$. For each positive integer $n$ and each $x$ in $[0, 1]$ define $g_n$ as follows: If $x$ is in $M_n$, then set $g_n(x) = 0$. If $x$ is not in $M_n$ and $[T_n(0)^* \cap \{0, 1\}]$ contains $x$, then set

$$g_n(x) = \max\{u_n(x), v_n(x), \ldots, s_n(x)\}.$$ 

If $x$ is not in $M_n$ and $[T_n(0)^* \cap \{0, 1\}]$ does not intersect the component of $[0, 1] - M_n$ containing $x$, then set

$$g_n(x) = \min\{u_n(x), v_n(x), \ldots, s_n(x)\}.$$ 

Since, for each $n$, each of $u_n, v_n, \ldots, s_n$ is continuous and vanishes at each member of $M_n$, $g_n$ is continuous.

Suppose that $x$ is a number in $[0, 1]$ such that $f(x) > 0$ and suppose that $\delta > 0$. There exist integers $j$ and $k$ such that $f(x) - \delta < f(x) < f(x) + \delta$. Write $r_k = a/b$ in lowest terms and let $Y$ denote the set of positive rational numbers to which $y$ (written in lowest terms) belongs if and only if $d < b$ and $c - 1/d < r_k < c/d$. There is a positive integer $n > c / (d - b)$ such that $n > c/d$, then

$$x \in \bigcap_{d \in Y} T_n(D)^* \cap \bigcap_{d \in Y} B_n(D)^*.$$ 

Suppose that $m > n$. Since $x \in [T_m(0)^*],

$$g_n(x) = \max\{u_n(x), v_n(x), \ldots, s_n(x)\};$$

and $x \in \bigcap_{d \in Y} T_m(D)^*.

Therefore, $x \in U_n(r_k)$. Therefore, $s_n(x) = r_k$. Therefore, $g_n(x) = r_k$. However, for each $k$ such that $r_k > r_k$, there is a member of $Y$ in $E(r_k)$. Therefore, $x \in u_n(r_k).$ Therefore, $s_n(x) = 0$. Therefore, $g_n(x) < r_k$. Therefore, $g_n(x) - f(x) < \delta$. Similarly, if $x$ is a number in $[0, 1]$ such that $f(x) < 0$, then $g_n(x) - f(x) > \delta$ for sufficiently large integers $m$. Therefore, $g_1, g_2, \ldots$ is a sequence of continuous functions converging pointwise to $f$. Therefore, $III_{1} \rightarrow I_{1}$. The fact that $II_{1} \rightarrow I_{0}$ and $II_{2} \rightarrow I_{0}$ can be established by arguments parallel to the one above. These proofs are therefore omitted from the paper.

In order to facilitate the description of certain collections of sets used in the proofs that $III_{1} \rightarrow III_{2}$, $II_{1}$ implies $II_{2}$, and $II_{2}$ implies $II_{1}$, respectively, the terms "acceptable", "determines", and "total $(M, Q)$ result" will now be introduced.

Suppose that $K$ is a collection of sets. The statement that $H$ is an acceptable subcollection of $K$ means that $H$ is $K$, or else each element of $K - H$ is a subset of every element of $H$. The word "determines" is not defined at this point but is assumed to have a meaning which satisfies the following condition: no collection of sets determines two sets, and if $H$ is a collection of sets which determines the set $x$, then $x$ is a subset of each element of $H$ and each subcollection of $H$ determines a set which has $x$ as a subset. If $M$ is a set and $Q$ is a meaning of the word determines, then $H$ is a total $(M, Q)$ result means that $H$ is a collection with the following four properties: (1) $M$ is in $H$ and each element of $H$ is a subset of $M$; (2) if $G$ is an acceptable subcollection of $H$ such that $G$ determines a set $x$ and no subcollection of $G$ determines a member of itself, then $x$ is in $H$; (3) if $y$ is a member of $H$ and $K$ is a subcollection of $H$ such that $K$ does not contain $y$ and $K$ determines $y$, then no subcollection of $K$ determines a member of itself; and (4) if $y$ is a member of $H$ distinct from $M$, then there is an acceptable subcollection $G$ of $H$ such that $G$ determines $y$ but $G$ does not contain $y$.

It can be demonstrated that if $M$ is a set and $Q$ is a meaning of the word determines, then there is only one total $(M, Q)$ result. Moreover, if $x$ precedes $y$ means that $y$ is a proper subset of $x$, then the total $(M, Q)$ result is well ordered with respect to this meaning of the word precedes. See [2].

The following notation will be used in the proof that $III_{1} \rightarrow III_{2}$. If $K$ is a collection of point sets and $i$ is an element of a set in $K$ or $i$ is a subset of a set in $K$, then $K_i$ will denote the common part of all the members of $K$ containing $i$.

We will now show that $III_{4} \rightarrow III_{4}$. Suppose that $f$ is a function defined on $[0, 1]$ and $f$ has property $III_{4}$. Let $a$ and $b$ be two numbers ($a > b$). In the non-trivial case there is an $f(x) > a$ and an $f(x) < b$. Let $U = \{x : f(x) > a\}$ and let $V = \{x : f(x) < b\}$. The statement that the collection $K$ of number sets determines the number set $A$ in the $(S \{T\})$ sense means that

$$\{A \cap \{x \in K \cap \{x \cap \{x \in A \} \} \} \subset \{A \cap \{x \in K \cap \{x \cap \{x \in A \} \} \} \} = \{A \cap \{x \in K \cap \{x \cap \{x \in A \} \} \} \}.$$ 

Let $(\theta) (H)$ denote the total $(U, S) (V, T)$ result. For each positive integer $n$, let $(\theta_n) (K_n)$ denote the collection of intervals to which the
interval $I$ belongs if and only if: (1) the length of $I$ is $1/n$; (2) the center $x$ of $I$ is in $(U \setminus E)$; and (3) 
\[(x - 4/n, x + 4/n) \cap U \cap \text{cl}(s_i) = \emptyset, (x - 2/n, x + 2/n) \cap \text{cl}(s_i) = \emptyset].\]

We will now show that 
\[\bigcup_{i=1}^{m} \bigcup_{k=1}^{m} (J_k).\]

Let $u \in U$. If $u \in \text{cl}(V \cap \text{cl}(G))$, then $G \cap \text{cl}(V \cap \text{cl}(G))$ is a member of $G$ containing $u$. Therefore, $G \cap \text{cl}(V \cap \text{cl}(G))$. Therefore, $V \cap \text{cl}(G) \text{cl}(G)$. Hence, there is an $n$ such that for all $u \in G$, 
\[(x - 4/n, x + 4/n) \cap \text{cl}(G) \cap \text{cl}(G) = \emptyset].\]

We will now show that 
\[\bigcup_{i=1}^{m} \bigcup_{k=1}^{m} (J_k).\]

Let $\theta (H)$ denote the total $(\{U, Q\}, \{V, B\})$ result. Let $G$ be the collection of sets to which $W$ belongs if and only if there is an $x$ in $(U \setminus E)$ such that $(W = G_i)(W \cap \text{cl}(G))$ for each $i \in I$. Let $(U \setminus E) \cap (V \setminus E)$ denote the set to which $x$ belongs if and only if there is a set $Y$ in $(U \setminus E)$ such that $x \in Y$ but $x$ is not a condensation point of $Y$. Notice that in these definitions, $Y$ must be $(G \cap H)$ at $b_i$. Notice also that $G = G'$, and $H = H'$. Suppose that $U$ is countable. Since each non-countable number set contains one of its condensation points, there must be uncountably many members $G_i$ of $G$ such that $x$ is in $U$. Suppose that no point $x$ of $U'$ is a limit point of a member of $G$ which is a proper subset of $G_i$. For each $p \in U'$, there is an integer $n_p$ such that no segment of length $1/n_p$ containing $p$ intersects a member of $G$ which is a proper subset of $G_i$. There is an integer $b$ and an uncountable subset $W$ of $U'$ such that, if $p \in W$, then $n_p = k$. $W$ has a condensation point $x$. Let $\beta$ denote a segment of length $1/k$ containing $x$. There must be two points $x$ and $y$ such that $G_i \neq G_j$. This constitutes a contradiction. Therefore, there is a point $x$ of $U'$ which is a limit point of the first member $S$ of $G$ which is a proper subset of $G_i$. $S = G_i \cap \text{cl}(V \cap \text{cl}(G))$. $x$ is in $S$. This constitutes a contradiction. Therefore, $U'$ is not countable. Similarly, $V'$ is not countable.

Suppose that $U'$ is not countable. As before, there is a number $x$ in $U'$ such that $x$ is a limit point of the first member $S$ of $G$ which is a proper subset of $G_i$. There is a number $y$ in $V$ such that $x = U \cap \text{cl}(G) \cap \text{cl}(G)$. $z = U \cap \text{cl}(G) \cap \text{cl}(G)$. This constitutes a contradiction. Therefore, $U'$ is not countable. Similarly, $V'$ is not countable. Let $U = U - (U' \cup V')$ and let $U = V - (V' \cup V')$. For each positive integer $n$, let $J_n$ denote the collection of intervals to which $I$ belongs if and only if: (1) the length of $I$ is $1/n$; (2) the center $x$ of $I$ is in $(U \setminus E)$; and (3) 
\[(x - 2/n, x + 2/n) \cap \text{cl}(s_i) = \emptyset].\]

We will now show that 
\[\bigcup_{i=1}^{m} \bigcup_{k=1}^{m} (J_k).\]

Let $u \in U$. If $u \in \text{cl}(V \cap \text{cl}(G))$, then $G \cap \text{cl}(V \cap \text{cl}(G))$. Therefore, $G \cap \text{cl}(V \cap \text{cl}(G))$. Hence, there is an $n$ such that for all $u \in G$, 
\[(x - 4/n, x + 4/n) \cap \text{cl}(G) \cap \text{cl}(G) = \emptyset].\]

We will now show that 
\[\bigcup_{i=1}^{m} \bigcup_{k=1}^{m} (J_k).\]

Let $\theta (H)$ denote the total $(\{U, Q\}, \{V, B\})$ result. Let $G$ be the collection of sets to which $W$ belongs if and only if there is an $x$ in $(U \setminus E)$ such that $(W = G_i)(W \cap \text{cl}(G))$ for each $i \in I$. Let $(U \setminus E) \cap (V \setminus E)$ denote the set to which $x$ belongs if and only if there is a set $Y$ in $(U \setminus E)$ such that $x \in Y$ but $x$ is not a condensation point of $Y$. Notice that in these definitions, $Y$ must be $(G \cap H)$ at $b_i$. Notice also that $G = G'$, and $H = H'$. Suppose that $U$ is countable. Since each non-countable number set contains one of its condensation points, there must be uncountably many members $G_i$ of $G$ such that $x$ is in $U$. Suppose that no point $x$ of $U'$ is a limit point of a member of $G$ which is a proper subset of $G_i$. For each $p \in U'$, there is an integer $n_p$ such that no segment of length $1/n_p$ containing $p$ intersects a member of $G$ which is a proper subset of $G_i$. There is an integer $b$ and an uncountable subset $W$ of $U'$ such that, if $p \in W$, then $n_p = k$. $W$ has a condensation point $x$. Let $\beta$ denote a segment of length $1/k$ containing $x$. There must be two points $x$ and $y$ such that $G_i \neq G_j$. This constitutes a contradiction. Therefore, there is a point $x$ of $U'$ which is a limit point of the first member $S$ of $G$ which is a proper subset of $G_i$. $S = G_i \cap \text{cl}(V \cap \text{cl}(G))$. $x$ is in $S$. This constitutes a contradiction. Therefore, $U'$ is not countable. Similarly, $V'$ is not countable.

Suppose that $U'$ is not countable. As before, there is a number $x$ in $U'$ such that $x$ is a limit point of the first member $S$ of $G$ which is a proper subset of $G_i$. There is a number $y$ in $V$ such that $x = U \cap \text{cl}(G) \cap \text{cl}(G)$. $z = U \cap \text{cl}(G) \cap \text{cl}(G)$. This constitutes a contradiction. Therefore, $U'$ is not countable. Similarly, $V'$ is not countable. Let $U = U - (U' \cup V')$ and let $U = V - (V' \cup V')$. For each positive integer $n$, let $J_n$ denote the collection of intervals to which $I$ belongs if and only if: (1) the length of $I$ is $1/n$; (2) the center $x$ of $I$ is in $(U \setminus E)$; and (3) 
\[(x - 2/n, x + 2/n) \cap \text{cl}(s_i) = \emptyset].\]

We will now show that 
\[\bigcup_{i=1}^{m} \bigcup_{k=1}^{m} (J_k).\]

Let $\theta (H)$ denote the total $(\{U, Q\}, \{V, B\})$ result. Let $G$ be the collection of sets to which $W$ belongs if and only if there is an $x$ in $(U \setminus E)$ such that $(W = G_i)(W \cap \text{cl}(G))$ for each $i \in I$. Let $(U \setminus E) \cap (V \setminus E)$ denote the set to which $x$ belongs if and only if there is a set $Y$ in $(U \setminus E)$ such that $x \in Y$ but $x$ is not a condensation point of $Y$. Notice that in these definitions, $Y$ must be $(G \cap H)$ at $b_i$. Notice also that $G = G'$, and $H = H'$. Suppose that $U$ is countable. Since each non-countable number set contains one of its condensation points, there must be uncountably many members $G_i$ of $G$ such that $x$ is in $U$. Suppose that no point $x$ of $U'$ is a limit point of a member of $G$ which is a proper subset of $G_i$. For each $p \in U'$, there is an integer $n_p$ such that no segment of length $1/n_p$ containing $p$ intersects a member of $G$ which is a proper subset of $G_i$. There is an integer $b$ and an uncountable subset $W$ of $U'$ such that, if $p \in W$, then $n_p = k$. $W$ has a condensation point $x$. Let $\beta$ denote a segment of length $1/k$ containing $x$. There must be two points $x$ and $y$ such that $G_i \neq G_j$. This constitutes a contradiction. Therefore, there is a point $x$ of $U'$ which is a limit point of the first member $S$ of $G$ which is a proper subset of $G_i$. $S = G_i \cap \text{cl}(V \cap \text{cl}(G))$. $x$ is in $S$. This constitutes a contradiction. Therefore, $U'$ is not countable. Similarly, $V'$ is not countable.
Let $v \in V$. Let $Q = \{ v : v \in U_0 \text{ and } v \in \text{cl}(G_0) \}$. If $v$ is a positive integer and $v > \infty$ and $v$ is the center of a member of $J_0$, then $|v| > 4/n$. Therefore, if $U_0 \setminus Q = \emptyset$ and $v$ is a positive integer, then $v \notin (J_0)^*$. Suppose that $U_0 \setminus Q \neq \emptyset$. There is a number $a$ in $(U_0 \setminus Q)$ such that $G_a = G_a \cap U_0 \setminus Q$. $\forall v \in \text{cl}(G_a) \cap (U_0 \setminus Q), v \notin \text{cl}(U_0 \setminus Q)$. Since $v$ is not in $\text{cl}(U_0 \setminus Q)$, there is no member of $J_0$ with center in $Q$ contains $v$, we have that $V \cap \bigcup_{k=1}^{\infty} \{ (a, b) \}$. Hence, we have that $V \cap \bigcup_{k=1}^{\infty} \{ (a, b) \}$. Therefore, $U_0 \setminus Q \neq \emptyset$. If $v \notin (J_0)^*$, then $v \notin \text{cl}(G_v)$. $\forall v \notin \text{cl}(G_v)$. The sequences $T_0, T_1, \ldots$ and $R_0, R_1, \ldots$, satisfy the conditions imposed on the sequences in $\Pi$. Therefore, $\Pi \subseteq \Pi$. We will now show that $\Pi \subseteq \Pi$. Suppose that $\pi$ has property $\Pi$, and suppose that $a$ and $b$ are numbers $(a, b)$. As before, let $U = \{ a : f(a) > a \}$ and let $V = \{ a : a < b \}$. The statement of the collection $K$ of numbers determines the set $A$ in the $(U, E)$ space means that $A = \bigcup_{k=1}^{n} \{ a \}$. Let $(0, \infty)$ denote the total $\bigcup \{ (U, E), (V, E) \}$. For each positive integer $n$, let $\epsilon_n$ denote the collection of closed, nonempty, $\epsilon_n$-neighborhood of $n$. We will now show that $U \subset \bigcup_{k=1}^{\infty} \{ (a, b) \}$. Suppose that $\epsilon \subset U$. If $\epsilon \subset \text{cl}(\bigcup_{k=1}^{\infty} \{ (a, b) \})$, then $\epsilon \subset \text{cl}(\bigcup_{k=1}^{\infty} \{ (a, b) \})$ is a member of $\Pi$ containing $\epsilon$. Therefore, $G_\epsilon \subset \text{cl}(\bigcup_{k=1}^{\infty} \{ (a, b) \})$. Therefore, $G_\epsilon \subset \text{cl}(\bigcup_{k=1}^{\infty} \{ (a, b) \})$. This contradicts $\Pi$. Therefore, there is a number $n$ such that for all $n > n$, $\epsilon$ is the left end point of a member of $\Pi$. Therefore, $U \subset \bigcup_{k=1}^{\infty} \{ (a, b) \}$. For each positive integer $n$ and each component $J$ of $\epsilon_n$, let $m_0(J)$ denote the collection of closed, nonempty, $\epsilon_n$-neighborhood of $n$. We will now show that $U \subset \bigcup_{k=1}^{\infty} \{ (a, b) \}$. Suppose that $\epsilon \subset U$. If $\epsilon \subset \text{cl}(\bigcup_{k=1}^{\infty} \{ (a, b) \})$, then $\epsilon \subset \text{cl}(\bigcup_{k=1}^{\infty} \{ (a, b) \})$ is a member of $\Pi$ containing $\epsilon$. Therefore, $G_\epsilon \subset \text{cl}(\bigcup_{k=1}^{\infty} \{ (a, b) \})$. Therefore, $G_\epsilon \subset \text{cl}(\bigcup_{k=1}^{\infty} \{ (a, b) \})$. This contradicts $\Pi$. Therefore, there is a number $n$ such that for all $n > n$, $\epsilon$ is the left end point of a member of $\Pi$. Therefore, $U \subset \bigcup_{k=1}^{\infty} \{ (a, b) \}$. For each positive integer $n$ and each component $J$ of $(a_n)^*$, let $m_0(J)$ denote the collection of closed, nonempty, $\epsilon_n$-neighborhood of $n$. We will now show that $U \subset \bigcup_{k=1}^{\infty} \{ (a, b) \}$. Suppose that $\epsilon \subset U$. If $\epsilon \subset \text{cl}(\bigcup_{k=1}^{\infty} \{ (a, b) \})$, then $\epsilon \subset \text{cl}(\bigcup_{k=1}^{\infty} \{ (a, b) \})$ is a member of $\Pi$ containing $\epsilon$. Therefore, $G_\epsilon \subset \text{cl}(\bigcup_{k=1}^{\infty} \{ (a, b) \})$. Therefore, $G_\epsilon \subset \text{cl}(\bigcup_{k=1}^{\infty} \{ (a, b) \})$. This contradicts $\Pi$. Therefore, there is a number $n$ such that for all $n > n$, $\epsilon$ is the left end point of a member of $\Pi$. Therefore, $U \subset \bigcup_{k=1}^{\infty} \{ (a, b) \}$.
By an argument parallel to the one above, one can construct a sequence $V_1, V_2, ...$ such that: (1) if $s$ is a positive integer, $V_s$ is a finite collection each member of which is a segment or a sector closed on the left; (2) $\forall V \subseteq \bigcup_{s=1}^{\infty} \left( V_s^c \right)^*$; and (3) $\exists U \subseteq \bigcup_{s=1}^{\infty} \left[ (0, \infty) \setminus (V_s)^c \right]^*$.

For each positive integer $n$, let $W_n = (U_n)^* - (V_n)^*$ and let $Z_n = (V_n)^* - (U_n)^*$. Notice that if $k$ is a positive integer, $c$ is a component of $W_k$, and $d$ is a component of $Z_k$, then there is a closed on the left sector $c'$ containing $c$ but no point of $Z_k$ and a closed on the left sector $d'$ containing $d$ but no point of $W_k$. Therefore, there exist sequences $T_k(a, b), T_k(a, b), ...$ and $B_k(a, b), B_k(a, b), ...$ which satisfy the conditions imposed on the sequences in $\Pi_1$. Therefore, $\Pi_2 \rightarrow \Pi_1$. This completes the proof of the theorem.

Observation I. It can be established by straight forward elementary arguments that each of the properties $\Pi_1, \Pi_2, \Pi_3, \Pi_4, \Pi_5, \Pi_6$, and $\Pi_7$ can be replaced by a local property. For example, $f$ has property $\Pi_5$ if and only if $f$ has the following property: if $x \in [0, 1]$, then there is a segment $s$ containing $x$ such that if $a = b$, $U \subseteq s \setminus \{x: f(x) = c \}$ and $V \subseteq s \setminus \{x: f(x) < c \}$, then $U \not\subset \text{cl}(V)$ or $V \not\subset \text{cl}(U)$.

Observation II. We have noted that the “two number” condition II can be replaced by the corresponding “one number” condition II'. There is no corresponding “one number” condition which is an equivalent modification of III. This fact is illustrated by the following function $f \in \mathbb{Q}$. If $x$ is an irrational number in $[0, 1]$, then $f(x) = 0$; if $x$ is a rational number in $[0, 1] (x = p/q)$ in lowest terms, then $f(x) = 1/q$ in case $q$ is even and $f(x) = -1/q$ in case $q$ is odd.

Observation III. $\exists \mathcal{O}_1 \subset \mathcal{O}_2 \subset \mathcal{J}_1$ and $\exists \mathcal{O}_1 \neq \mathcal{O}_2$. $\mathcal{O}_1$ is a subset of $\mathcal{O}_2$, because each function which is discontinuous on the right.

Observation IV. C. T. Tucker [5] has shown that the following two statements are equivalent: (1) $f$ is in $\mathcal{J}_1$; and (2) there is a countable subset $T$ of $[0, 1]$ and a function $g$ in $\mathcal{O}_1$ such that if $x$ is a number in $[0, 1] \setminus T$, then $f(x) = g(x)$. It follows from this result that $\mathcal{O}_1 \neq \mathcal{J}_1$. Since $\mathcal{O}_1 \subset \mathcal{O}_2 \subset \mathcal{J}_1$, $\mathcal{O}_1 = \mathcal{O}_2 = \mathcal{J}_1$.

Observation V. Using an argument similar to (but in fact simpler than) the argument that $\Pi_1 \rightarrow \Pi_2$, it can be shown that if $f \in \Pi_2$, then $f$ is the pointwise limit of a sequence of continuous functions on the right jump functions. Therefore, we have that the following two statements are equivalent: (1) $f$ is the pointwise limit of a sequence of continuous on the right jump functions; and (2) $f$ is the pointwise limit of a sequence of continuous on the right jump functions.

References


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