Approximations of one-one measurable transformations *

by

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There have been numerous results in the theory of functions of a real variable which show that measurable functions, and even arbitrary functions, have certain continuity properties. Examples of these are the well-known theorems of Lusin [5], Saks–Sierpiński [1], Vitali–Carathéodory [7].

The form of Lusin's theorem considered here is: if f is a measurable function defined on a measurable subset S of Euclidean n-space, then for every positive ε there is a continuous function g defined on S such that $f = g$ except on a set of measure less than ε.

The related subject of measurable, or arbitrary, one-one transformations has been studied by Goffman [2].

Goffman [2] showed that if $2 \leqslant n$ and f is a one-one measurable function of the unit n-cube I_n onto I_n with a measurable inverse f^{-1}, then for every positive ε there is a homeomorphism (g, g^{-1}) between I_n and I_n such that $f = g$ and $f^{-1} = g^{-1}$ on sets of measure greater than $1-\varepsilon$. He also showed that if f is a one-one measurable function of the unit interval I_1 onto I_1 with a measurable inverse f^{-1}, then there is a one-one function g of Baire class 2 from I_1 onto I_1, whose inverse is also of Baire class 2, such that $f = g$ and $f^{-1} = g^{-1}$ almost everywhere.

Shaerf [6] showed that the result of Lusin's theorem is valid in more general spaces. If X is an arbitrary neighborhood space, Y is a topological space satisfying the second axiom of countability, and f is a function from X into Y which is measurable with respect to a measure μ on X then for every μ-measurable set E of X and every positive ε there is a closed subset F of E such that $\mu(E-F) < \varepsilon$ and such that f is continuous on F relative to F if and only if μ is a regular measure.

In this paper it is shown that if X and Y are locally compact separable metric spaces with totally finite regular Borel measures μ and ν, then any one-one function of X onto Y, which is μ-measurable and whose inverse is ν-measurable, can be approximated up to sets of meas-

* This work supported by NSF grant no GP 03515.

ure ε by a one-one function of X onto Y which is of Baire class 1 and whose inverse is of Baire class 1; and can be approximated up to sets of measure 0 by a one-one function of X onto Y which is of Baire class 2 and whose inverse is of Baire class 2. It is also shown that if X and Y are homeomorphic images of I_n for $n \geqslant 2$, and have totally finite regular Borel measures, then any one-one measurable transformation can be approximated up to sets of measure ε by a homeomorphism between X and Y.

Finally some examples are given to show that these approximations are "best possible" in the sense that the conclusions of the above statements will not be valid if "Baire 2" is replaced by "Baire 1", or "Baire 1" is replaced by "continuous". An example is also given to show that the hypothesis that the measures be totally finite cannot be eliminated.

To demonstrate these facts we will make use of the following lemmas.

LEMMA 1 (Goffman [2], p. 271). *If (f, f^{-1}) is a measurable transformation between I_n and I_m, then for every $\varepsilon > 0$ there are closed sets $S \subset I_n$ and $T \subset I_m$ of measure greater than $1-\varepsilon$ such that (f, f^{-1}) is a homeomorphism between S and T.*

LEMMA 2 (Kuratowski [3], p. 212). *If X and Y are complete separable topological spaces of the same cardinality, then there is a one-one transformation (φ, φ^{-1}) between them of Baire class $(1, 1)$.*

LEMMA 3. *If A and B are G_δ sets of complete separable topological spaces and have the same cardinality, then there is a one-one transformation (φ, φ^{-1}) between them of Baire class $(1, 1)$.*

Proof. It is known [4] that any G_δ set in a complete separable space is homeomorphic to a complete separable space. Let (f, f^{-1}) and (g, g^{-1}) be homeomorphisms between A and X and B and Y respectively. By Lemma 2 there is a one-one transformation (h, h^{-1}) between X and Y of Baire class $(1, 1)$. Then $(\varphi, \varphi^{-1}) = (g^{-1} \circ h \circ f, f^{-1} \circ h^{-1} \circ g)$ is a one-one transformation between A and B of Baire class $(1, 1)$.

THEOREM 1. *If $1 \leqslant n < m$ and (f, f^{-1}) is a one-one measurable transformation between I_n and I_m, then for every $\varepsilon > 0$ there is a one-one transformation (g, g^{-1}) between I_n and I_m such that $f = g$ and $f^{-1} = g^{-1}$ on sets of measure greater than $1-\varepsilon$ and such that (g, g^{-1}) is of Baire class $(1, 1)$.*

Proof. Let $\varepsilon > 0$. By Lemma 1 there are closed sets $S \subset I_n$ and $T \subset I_m$ of measure greater than $1-\varepsilon$ such that (f, f^{-1}) is a homeomorphism between S and T. Then $(I_n - S)$ and $(I_m - T)$ are open sets hence G_δ sets. By Lemma 3 there is a one-one transformation (φ, φ^{-1}) between $I_n - S$ and $I_m - T$ of Baire class $(1, 1)$. Then if

$$g(x) = \begin{cases} f(x), & x \in S, \\ \varphi(x), & x \in I_{n-S} \end{cases}$$

(g, g^{-1}) is a one-one transformation between I_n and I_m. Let G be an open set in I_m. Then

$$g^{-1}(G) = f^{-1}(G \cap T) \cup \varphi^{-1}\big(G \cap (I_m - T)\big)$$

where $f^{-1}(G \cap T)$ is open in S hence F_σ in I_n and $\varphi^{-1}\big(G \cap (I_m - T)\big)$ is F_σ in $I_n - S$ hence F_σ in I_n. So $g^{-1}(G)$ is an F_σ set and thus g is of Baire class 1 at most. Similarly, g^{-1} is of Baire class 1 at most.

THEOREM 2. *If (f, f^{-1}) is a one-one measurable transformation between I_n and I_m where n and m are any positive integers, then there is a one-one transformation (g, g^{-1}) between I_n and I_m such that $g = f$ a.e. in I_n and $g^{-1} = f^{-1}$ a.e. in I_m and (g, g^{-1}) is of Baire class $(2, 2)$.*

Proof. By Lemma 1, for every $\varepsilon > 0$ there are closed sets $S \subset I_n$ and $T \subset I_m$ of measure greater than $1-\varepsilon$ such that (f, f^{-1}) is a homeomorphism between S and T. For every $k = 1, 2, \ldots$, let $S_k \subset I_n$, $T_k \subset I_m$ be such sets for $\varepsilon_k = 1/k$. Then

$$S = \bigcup_{k=1}^{\infty} S_k, \qquad T = \bigcup_{k=1}^{\infty} T_k$$

are F_σ sets of measure 1 such that (f, f^{-1}) is a one-one transformation between S and T of Baire class $(1, 1)$. $I_n - S$ and $I_m - T$ are G_δ sets having the same cardinality so by Lemma 3 there is a one-one transformation (φ, φ^{-1}) between $I_n - S$ and $I_m - T$ of Baire class $(1, 1)$. Let

$$g(x) = \begin{cases} f(x), & x \in S, \\ \varphi(x), & x \in I_n - S. \end{cases}$$

Then (g, g^{-1}) is a one-one transformation between I_n and I_m. Let G be an open set in I_m. Then

$$g^{-1}(G) = f^{-1}(G \cap T) \cup \varphi^{-1}\big(G \cap (I_m - T)\big).$$

Now $f^{-1}(G \cap T)$ is an F_σ set in S, and hence an F_σ set in I_n. $\varphi^{-1}\big(G \cap (I_m - T)\big)$ is an F_σ set in $I_m - S$, a G_δ set, hence a $G_{\delta\sigma}$ set in I_n. So $g^{-1}(G)$ is a $G_{\delta\sigma}$ set and g is of Baire class 2 at most. Similarly g^{-1} is of Baire class 2 at most.

It should be noted that these approximations are "best possible" in the following sense. If n is different from m, then there does not exist a one-one transformation between I_n and I_m which is continuous or which has a continuous inverse. Hence the best possible approximation is of Baire class $(1, 1)$. C. Goffman [2] gives an example of a measurable transformation (f, f^{-1}) between $(0, 1)$ and $(0, 1)$ such that any transformation (g, g^{-1}) satisfying $f = g$ a.e. and $f^{-1} = g^{-1}$ a.e. has both g and g^{-1} everywhere discontinuous and therefore of Baire class $(2, 2)$ at least. This example readily extends to arbitrary dimensions.

These results can be extended to more general spaces using the following lemma.

LEMMA 4 (Shaerf [6], p. 35). *If X is a metric space with a totally finite regular Borel measure μ, Y is a topological space satisfying the second axiom of countability, and f is a function of X into Y which is measurable relative to μ, then for every measurable set $E \subset X$ and every positive ε there is a closed set $F \subset E$ such that $\mu(E-F) < \varepsilon$ and such that f is continuous on E relative to E.*

THEOREM 3. *If X and Y are locally compact separable metric spaces with totally finite regular Borel measures μ and ν and (f, f^{-1}) is a one-one measurable transformation between X and Y, then for every $\varepsilon > 0$ there is a one-one transformation (g, g^{-1}) such that $f \neq g$ on a set of μ measure less than ε and $f^{-1} \neq g^{-1}$ on a set of ν measure less than ε and such that (g, g^{-1}) is of Baire class $(1, 1)$.*

Proof. By Lemma 4 there is a closed set $E \subset X$ such that $\mu(X-E) < \varepsilon$ and such that f is continuous on E. Since $\mu(X) < \infty$ and X is locally compact, we can choose E to be compact. Then (f, f^{-1}) is a homeomorphism between E and $f(E)$. There is a compact set $F \subset (Y-f(E))$ such that $\nu(Y-F \cup f(E)) < \varepsilon$ and such that (f, f^{-1}) is a homeomorphism between $f^{-1}(F)$ and F. Let $S = E \cup f^{-1}(F)$ and $T = f(E) \cup F$. Then (f, f^{-1}) is a homeomorphism between S and T and by applying Lemma 3 as in the proof of Theorem 1 we get a one-one transformation (g, g^{-1}) with the desired properties.

THEOREM 4. *Under the hypothesis of Theorem 3, there is a one-one transformation (g, g^{-1}) between X and Y such that $f \neq g$ on a set of μ measure 0 and $f^{-1} \neq g^{-1}$ on a set of ν measure 0 and such that (g, g^{-1}) is of Baire class $(2, 2)$.*

Proof. Let $E_1 \subset X$ and $F_1 \subset Y$ be the sets E and F of Theorem 3 for $\varepsilon = 1$. Then for $n = 1, 2, \ldots$, let

$$E_{n+1} \subset X - \bigcup_{k=1}^{n} \left(E_k \cup f^{-1}(F_k)\right)$$

be compact and such that

$$\mu\Big(X - \bigcup_{k=1}^{n+1} E_k \ \bigcup_{k=1}^{n} f^{-1}(F_k)\Big) < 1/(n+1)$$

and such that f is continuous on E_{n+1}. Let

$$F_{n+1} \subset \Big(Y - \bigcup_{k=1}^{n+1} f(E_k) \ \bigcup_{k=1}^{n} F_k\Big)$$

be compact and such that

$$\nu\Big(Y - \bigcup_{k=1}^{n+1} f(E_k) \ \bigcup_{k=1}^{n} F_k\Big) < 1/(n+1)\,.$$

Then (f, f^{-1}) is a homeomorphism between $\bigcup_{k=1}^{n+1} \left(E_k \cup f^{-1}(F_k)\right)$ and $\bigcup_{k=1}^{n+1} \left(f(E_k) \cup F_k\right)$ for every n. Let

$$S = \bigcup_{n=1}^{\infty} \left(E_n \cup f^{-1}(F_n)\right) \quad \text{and} \quad T = \bigcup_{n=1}^{\infty} \left(f(E_n) \cup F_n\right).$$

Then $\mu(X-S) = 0$ and $\nu(Y-T) = 0$, S and T are F_σ sets and (f, f^{-1}) is a one-one transformation between S and T of Baire class $(1, 1)$. By applying Lemma 3 to $(X-S)$ and $(Y-T)$ as in Theorem 2 we get a one-one transformation (g, g^{-1}) between X and Y with the desired properties.

The question which arises from the result of Theorem 3 and the results of [2] is under what conditions on homeomorphic spaces X and Y can the approximating transformations be made homeomorphisms? This question is partially answered here.

DEFINITION 1. A subset E of I_n is called *sectionally zero dimensional* if for every hyperplane π parallel to a face of I_n and for every positive ε there is a hyperplane π^* parallel to π whose distance from π is less than ε and which contains no points of E.

LEMMA 5. *If μ and ν are totally finite regular Borel measures defined on I_n for $n \geqslant 2$, and (f, f^{-1}) is a one-one transformation of I_n onto itself such that f is μ-measurable and f^{-1} is ν-measurable, then for every positive ε there is a closed sectionally zero dimensional subset E of I_n such that $\mu(I_n - E) < \varepsilon$, $\mu(I_n - f^{-1}(E)) < \varepsilon$, $\nu(I_n - E) < \varepsilon$ and $\nu(I_n - f(E)) < \varepsilon$.*

Proof. Of all the hyperplanes π parallel to a face of I_n, there are a countable number which have positive μ measure. For if uncountably many of them had positive μ measure, parallel to a specific face of I_n there would be infinitely many hyperplanes with μ measure greater than $1/k$ for some positive integer k. This contradicts the hypothesis that $\mu(I_n)$ is finite. Similarly there are a countable number of hyperplanes π parallel to a face of I_n such that $\mu(\pi)$, $\mu(f^{-1}(\pi))$, $\nu(\pi)$ or $\nu(f(\pi))$ is positive. Consequently, parallel to each face of I_n, we can select a denumerable set of hyperplanes π, whose union is dense in I_n, and such that $\mu(\pi) = \mu(f^{-1}(\pi)) = \nu(\pi) = \nu(f(\pi)) = 0$. This collection of hyperplanes, as a finite union of denumerable sets, is denumerable and can be written as $\pi_1, \pi_2, \ldots, \pi_k, \ldots$

Let $\varepsilon > 0$ be given. For each pair of positive integers k and n, let G_{kn} be the set of points in I_n whose distance from π_k is less than $1/n$. Because $\pi_k = \bigcap_{n=1}^{\infty} G_{kn}$, the sets G_{kn} are decreasing and $\mu(\pi_k) = 0$, there is an N_{k1} such that $\mu(G_{kn}) < \varepsilon/2^k$ for all $n \geqslant N_{k1}$. Similarly there are $N_{k4} \geqslant N_{k3} \geqslant N_{k2} \geqslant N_{k1}$ such that $\nu(f(G_{kn})) < \varepsilon/2^k$ when $n \geqslant N_{k2}$, $\nu(G_{kn})$

$< \varepsilon/2^k$ when $n \geqslant N_{k3}$, and $\mu(f^{-1}(G_{kn})) < \varepsilon/2^k$ when $n \geqslant N_{k4}$. Let $n_k = N_{k4}$, $G_k = G_{kn_k}$, and $G = \bigcup_{k=1}^{\infty} G_k$. Then G is open and contains a dense set of hyperplanes parallel to each face of I_n, so $E = (I_n - G)$ is a closed sectionally zero dimensional set and has the desired properties.

LEMMA 6 (Goffman [2], p. 266). *If S and T are closed sectionally zero dimensional subsets of I_n, for $n \geqslant 2$, and (f, f^{-1}) is a homeomorphism between S and T, then there is a homeomorphism (g, g^{-1}) of I_n onto itself such that $f = g$ on S and $f^{-1} = g^{-1}$ on T.*

THEOREM 5. *If μ and ν are totally finite regular Borel measures defined on I_n for $n \geqslant 2$, and (f, f^{-1}) is a one-one transformation of I_n onto itself such that f is a μ-measurable function and f^{-1} is a ν-measurable function, then for every positive ε there is a homeomorphism (g, g^{-1}) of I_n onto itself such that $f = g$ except on a set of μ measure less than ε and $f^{-1} = g^{-1}$ except on a set of ν measure less than ε.*

Proof. Let $\varepsilon > 0$ be given. As in the proof of Theorem 3, by Lemma 4 there are compact subsets S and T of I_n such that $\mu(I_n - S) < \frac{1}{2}\varepsilon$, $\nu(I_n - T) < \frac{1}{2}\varepsilon$ and such that (f, f^{-1}) is a homeomorphism between S and T. By Lemma 5 there is a closed sectionally zero dimensional subset E of I_n such that $\mu(I_n - E) < \varepsilon/4$, $\nu(I_n - E) < \frac{1}{4}\varepsilon$, $\mu(I_n - f^{-1}(E)) < \frac{1}{4}\varepsilon$ and $\nu(I_n - f(E)) < \frac{1}{4}\varepsilon$. Then $E \cap S$ is a closed sectionally zero dimensional set and f is continuous on $E \cap S$ relative to $E \cap S$, so $f(E \cap S)$ is closed. Let $H = E \cap f(E \cap S)$. Then H is a closed sectionally zero dimensional subset of T and f^{-1} is continuous on H relative to H. Let $G = f^{-1}(H) = f^{-1}(E) \cap E \cap S$. Then G is a closed sectionally zero dimensional set and (f, f^{-1}) is a homeomorphism between G and H. Now

$$\mu(I_n - G) \leqslant \mu(I_n - f^{-1}(E)) + \mu(I_n - E) + \mu(I_n - S) < \varepsilon,$$

and

$$\nu(I_n - H) \leqslant \nu(I_n - E) + \nu(I_n - f(E)) + \nu(I_n - T) < \varepsilon.$$

By Lemma 6 there is a homeomorphism (g, g^{-1}) of I_n onto itself such that $f = g$ on G and $f^{-1} = g^{-1}$ on H.

THEOREM 6. *If two spaces X and Y, with totally finite regular Borel measures μ and ν, are both homeomorphic to I_n, for $n \geqslant 2$, and (f, f^{-1}) is a one-one transformation between X and Y such that f is μ-measurable and f^{-1} is ν-measurable, then for every positive ε there is a homeomorphism (g, g^{-1}) between X and Y such that $f = g$ except on a set of μ measure less than ε and $f^{-1} = g_1^{-1}$ except on a set of ν measure less than ε.*

Proof. Let (ζ, ζ^{-1}) be a homeomorphism between X and I_n. Define a measure μ^* on I_n by $\mu^*(E) = \mu(\zeta^{-1}(E))$. Then μ^* is a totally finite regular Borel measure defined on I_n. Similarly if (γ, γ^{-1}) is a homeomorphism between Y and I_n, we can define a totally finite regular Borel measure ν^* on I_n by $\nu^*(E) = \nu(\gamma^{-1}(E))$. Let (f^*, f^{*-1}) be the one-one transformation of I_n onto itself given by $f^* = \gamma \circ f \circ \zeta^{-1}$. Then (f^*, f^{*-1}) is a one-one measurable transformation of I_n onto I_n.

Let $\varepsilon > 0$ be given. By Theorem 5 there is a homeomorphism (g^*, g^{*-1}) of I_n onto itself such that $f^* = g^*$ except on a set of μ^* measure less than ε and $f^{*-1} = g^{*-1}$ except on a set of ν^* measure less than ε. Let (g, g^{-1}) be the homeomorphism between X and Y given by $g = \gamma^{-1} \circ g^* \circ \zeta$. If $A = \{x \in X\colon f(x) \neq g(x)\}$ then

$$\mu(A) = \mu^*(\zeta(A)) = \mu^*[x \in X\colon f^*(x) \neq g^*(x)] < \varepsilon.$$

By a similar argument, if $B = \{x \in X\colon f^{-1}(x) \neq g^{-1}(x)\}$, then $\nu(B) < \varepsilon$.

The following example is given to show that the conclusion of Theorem 6 is not valid if the measures on the spaces X and Y are σ-finite but not totally finite.

EXAMPLE 1. There exists a one-one measurable transformation (f, f^{-1}) between the plane R_2 and itself such that any homeomorphism (g, g^{-1}) between R_2 and itself is such that $f \neq g$ and $f^{-1} \neq g^{-1}$ on sets of infinite measure.

Verification. Let $S_1, S_2, \ldots$ be the open squares in the half plane $y > 0$ whose boundaries are the lines $x = \ldots, -1, 0, 1, \ldots$, $y = 0, 1, 2, \ldots$ Let $T_1, T_2, \ldots$ be the open strips in the half plane $y < 0$ given by $T_k = [(x, y)\colon -k < y < -k+1]$. For each $k = 1, 2, \ldots$, let (h_k, h_k^{-1}) be a homeomorphism between S_k and T_k.

Let

$$f(x) = \begin{cases} h_k(x), & x \in S_k, \\ h_k^{-1}(x), & x \in T_k, \\ x, & x \in \left(R_2 - \bigcup_{k=1}^{\infty} (S_k \cup T_k)\right). \end{cases}$$

Then f has an inverse function and $f^{-1} = f$. Let (g, g^{-1}) be a homeomorphism between R_2 and itself. Let F be the closure of S_1. Because F is compact and (g, g^{-1}) is a homeomorphism, both $g(F)$ and $g^{-1}(F)$ are compact hence bounded. But T_1 is a subset of both $f(F)$ and $f^{-1}(F)$, so on the set $(T_1 - g(F))$, $f^{-1} \neq g^{-1}$ and on the set $(T_1 - g^{-1}(F))$, $f \neq g$ and both of these sets have infinite measure, since T_1 has infinite measure outside of any bounded set.

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Reçu par la Rédaction le 14. 8. 1967

Une application de la méthode de Fraenkel – Mostowski

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1. Préliminaires.

1.1. Résumons d'abord certains résultats développés dans [1] et dont nous ferons usage dans la suite. A cet effet, considérons les axiomes A, B, C, D, E de Gödel [3] et plaçons-nous dans le système axiomatique (ABC). Appelons *univers* toute classe U telle que $\mathfrak{P}U = U$, où $\mathfrak{P}U$ désigne la classe des sous-ensembles de U. Par exemple, *l'*univers V (la classe de tous les ensembles) est *un* univers. On peut montrer que toute classe X est contenue dans un plus petit univers $U(X)$. Pour toute classe *transitive* A (c-à-d telle que $A \subset \mathfrak{P}A$) on peut établir les principes suivants:

(a) *Principe d'induction dans* $U(A)$:

Si $\Phi(x)$ *est une formule prédicative et si* y *ne figure pas dans* $\Phi(x)$, *alors*

$$\big[(\forall x)_A \Phi(x) \wedge (\forall x)_{U(A)-A}\big((\forall y)_x \Phi(y) \Rightarrow \Phi(x)\big)\big] \Rightarrow (\forall x)_{U(A)} \Phi(x).\ (^1)$$

(b) *Principe de récursion dans* $U(A)$:

Si $F_1\colon V \to V$ *et* $F_2\colon V \to V$, *alors il existe une et une seule fonction* $F\colon U(A) \to V$ *telle que*

$$F`x = \begin{cases} F_1`x & si \quad x \in A\,, \\ F_2`F``x & si \quad x \in U(A) - A\,.\ (^2) \end{cases}$$

1.2. Notons N l'axiome de von Neumann: "Toute classe propre est équipotente à l'univers V" et posons $K = \{x|\ x = \{x\}\}$. Nous savons que si le système (ABC) est consistant, alors le système (ABCNPr(K)) (³) l'est également (voir [2]). Mais dans ce dernier système, la classe $U(K)$

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(¹) $(\forall x)_A \Phi(x)$ est une abréviation de $(\forall x)(x \in A \Rightarrow \Phi(x))$.

(²) Remarque importante: la fonction F peut être *construite* à l'aide d'une *formule prédicative*.

(³) Pr(K) affirme que K est une classe propre.