On countably compact reduced products I

by

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In [3] B. Jónsson and Ph. Olin have formulated the problem of determining the ideals $\mathfrak{J}$ of subsets of a set I which have the property that every $\mathfrak{J}$-reduced product of arbitrary structures is countably compact (¹) and they have shown that the Fréchet ideal, i.e. the ideal of all finite subsets of ω, has the required property. This covers an earlier result of H. J. Keisler [5] concerning products of Boolean algebras.

We present here a solution of the problem under an additional assumption that the Boolean algebra is atomless. Namely (Theorem 1) if $2^I_{\mathfrak{J}}$ is atomless, then every $\mathfrak{J}$-reduced product is countably compact if and only if

(i) $2^I_{\mathfrak{J}}$ *is countably compact*

and

(ii) I *is the union of a countable subfamily* of $\mathfrak{J}$.

Cleary for denumerable I condition (ii) is automatically satisfied. We observe that any countably compact Boolean algebra has a property called here *basic connectedness* (see Definition 2). On the other hand Theorem 2 asserts that if a Boolean algebra $2^I_{\mathfrak{J}}$ is basically connected, then it is countably compact. Theorem 3 is a reformulation of an unpublished result of F. Galvin mentioned in [3] p. 132. In [3] B. Jónsson and Ph. Olin expressed an opinion that the ideals described by F. Galvin did not exhaust all possibilities. In fact, this turns out to be true as it is shown by the Example.

The investigation of countably compact reduced products was started by H. J. Keisler [4], who described a class of ultrafilters for which ultraproducts are countably compact. This result can be easily extended to the case when $2^I_{\mathfrak{J}}$ is finite. On the other hand one can easily verify that $2^I_{\mathfrak{J}}$ is not countably compact provided an element of $2^I_{\mathfrak{J}}$ contains exactly $\aleph_0$ atoms. If $2^I_{\mathfrak{J}}$ has uncountably many atoms which of course can happen

(¹) We recall that a relational structure $\mathfrak{A}$ is countably compact if the family of all sets definable in $\mathfrak{A}$ is countably compact. In this note we rather use „countably compact structure" instead of "ω_1-saturated"; the latter being used in [3], [4], [5]. For countable languages both notions are equivalent.

for countable I, it seems to be difficult to decide whether $2^I_{\mathfrak{J}}$ is countably compact or not.

The letter $\mathfrak{A}$ (sometimes with subscripts) denotes a relational structure of a fixed similarity type and A—the universe of $\mathfrak{A}$. We will assume that every element of A has a name in $\mathfrak{A}$. Moreover "formula" means "formula of a language of $\mathfrak{A}$". The set I is always infinite. By 2 we denote the Boolean algebra $\langle \mathbf{0}, \mathbf{1}, \cup, \cap, - \rangle$. We assume that all ideals are proper. If $\mathfrak{J}$ is an ideal on I then $\mathfrak{A}^I_{\mathfrak{J}}$ denotes the $\mathfrak{J}$-reduced power of $\mathfrak{A}$. For $B-A \in \mathfrak{J}$ we write $A \subseteq B (\bmod \mathfrak{J})$ and for $a_0 \vee \ldots \vee a_{n-1}$, $a_0 \wedge \ldots \wedge a_{n-1}$ we write $\bigvee_{i<n} a_i$, $\bigwedge_{i<n} a_i$, respectively. For another information on notation and terminology used in this paper see [1].

Definition 1. We say that sequence $\mathcal{A} = \langle \mathfrak{A}_i \colon i \in I \rangle$ of similar relational structures is *rich*, when for any formula φ there exists a predicate $P(\varphi)$ such that $\mathfrak{A}_i \models \varphi \leftrightarrow P(\varphi)$ for $i \in I$.

Proposition 1. *If $\mathfrak{J}$ is an ideal on I such that $2^I_{\mathfrak{J}}$ is atomless and $\mathcal{A}$ is a rich sequence, then in the product $\mathfrak{A} = \mathfrak{P}_{\mathfrak{J}}\mathcal{A}$* (²) *every formula is equivalent to an open formula.*

Proof. By a theorem of S. Feferman and R. L. Vaught ([1], Th. 3.1) for any formula φ there exist a partitioning acceptable sequence $\zeta = \langle \varphi, \theta_0, \ldots, \theta_m \rangle$ such that for every $f \in A$

$$\mathfrak{A} \models \varphi[f] \text{ if and only if } 2^I_{\mathfrak{J}} \models \varphi[K^{\mathfrak{A}}_{\theta_0}(f), \ldots, K^{\mathfrak{A}}_{\theta_m}(f)] .$$

Since $2^I_{\mathfrak{J}}$ is atomless, φ is equivalent to some quantifierfree formula (see [6]) i.e. $2^I_{\mathfrak{J}} \models \mathrm{Part}(X_0, \ldots, X_m) \to \varphi \leftrightarrow \bigvee_{n<c} \varphi_n$, where

$$\text{(1)} \qquad \varphi_n \quad \text{is} \quad (\tau_{n,0} = \mathbf{0} \wedge \tau_{n,1} \neq \mathbf{0} \wedge \ldots \wedge \tau_{n,k_n} \neq \mathbf{0})$$

and

$\tau_{n,j} = X_{p_{n,j,0}} \cup \ldots \cup X_{p_{n,j,r_{n,j}}}$, where $X_0 \ldots, X_m$, are free variables in φ. (In (1) only one equality $\tau_{n,0} = \mathbf{0}$ appears, because every conjunction of equalities may be replaced by a single equality). Obviously,

$$\varphi_n \leftrightarrow -\tau_{n,0} = \mathbf{1} \wedge -\tau_{n,1} \neq \mathbf{1} \wedge \ldots \wedge -\tau_{n,k_n} \neq \mathbf{1} .$$

Let $\xi_j = P(\neg \theta_j)$ for $j = 0, \ldots, m$, then, by the definition of validity in a reduced product, we have

$$2^I_{\mathfrak{J}} \models (-\tau_{n,0} = \mathbf{1})[K^{\mathfrak{A}}_{\theta_0}(f), \ldots, K^{\mathfrak{A}}_{\theta_m}(f)] , \quad \text{if, and only if} \quad \mathfrak{A} \models (\xi_{p_{n,0,0}} \wedge \ldots \wedge \xi_{p_{n,0,r_{n,0}}})[f] .$$

Moreover,

$$2^I_{\mathfrak{J}} \models (-\tau_{n,j} \neq \mathbf{1})[K^{\mathfrak{A}}_{\theta_0}(f), \ldots, K^{\mathfrak{A}}_{\theta_m}(f)] \quad \text{if and only if} \quad \mathfrak{A} \models \neg (\xi_{p_{n,j,0}} \wedge \ldots \wedge \xi_{p_{n,j,r_{n,j}}})[f] .$$

(²) $\mathfrak{P}_{\mathfrak{J}}\mathcal{A}$ denotes $\mathfrak{J}$-reduced product of A.

Definition 2. A Boolean algebra is called *basically connected* if the zero element is not a meet of a countable decreasing sequence of non-zero elements.

It is known that the factor algebra of all subsets of a countable set by the Fréchet ideal is basically connected (cf. [2], p. 100).

Lemma 1. *If $2^I_{\mathfrak{J}}$ is basically connected and atomless, then for any countable sequence $\langle F_n \colon n < \omega \rangle$ such that $F_n \subseteq I$, $F_n \notin \mathfrak{J}$ there exists a sequence of mutually disjoint sets G_n for which $G_n \subseteq F_n$ and $G_n \notin \mathfrak{J}$ for $n < \omega$.*

Proof. By induction on n, we define a sequence $\langle G_n \colon n < \omega \rangle$ of subsets of I such that $G_m \cap G_n = 0$, $G_n \subseteq F_n$, $G_n \notin \mathfrak{J}$ and $F_n \setminus \bigcup_{k<m} G_k \notin \mathfrak{J}$ for $m, n < \omega$, and $m > n$. Suppose that for $n > m_n$ are already defined. Let $F_{m,0} = F_m \setminus \bigcup_{n<m} G_n$ and $D_{m,k} = F_{m,k} \cap F_{m+1}$. If $D_{m,k} \in \mathfrak{J}$, we put $F_{m,k+1} = F_{m,k}$, otherwise we select a subset $F_{m,k+1}$ of $D_{m,k}$ is such a way that $F_{m,k+1} \notin \mathfrak{J}$ and $D_{m,k} - F_{m,k+1} \notin \mathfrak{J}$. The sequence $\langle F_{m,k} \colon k < \omega \rangle$ obtained in this way is decreasing and $F_{m,k} \notin \mathfrak{J}$ for $k < \omega$. Since $2^I_{\mathfrak{J}}$ is basically connected, there exists a set $G_m \notin \mathfrak{J}$ such that $G_m \subseteq F_{m,k} (\bmod \mathfrak{J})$ for $k < \omega$; moreover, one may assume that $G_m \subseteq F_{m,0}$.

To formulate Lemma 2 the following notation is applied. Let $A = \mathfrak{P}_{i \in I} A_i$, where A_i are non-void sets. For a sequence $\mathfrak{B} = \langle B_i \colon i \in I \rangle$ such that $B_i \subseteq A_i$ we put

$$Q_{\mathfrak{B}} = \{ f \in A \colon \{ i \in I \colon f(i) \in B_i \} \notin \mathfrak{J} \} ,$$

$$R_{\mathfrak{B}} = \{ f \in A \colon \{ i \in I \colon f(i) \in B_i \} \in \mathfrak{J} \} .$$

Now, let $\mathfrak{K}_{\mathfrak{J}}$ denote the family of all sets of the form $R_{\mathfrak{B}}$ or $Q_{\mathfrak{B}}$.

Lemma 2. *If an ideal $\mathfrak{J}$ satisfies* (ii), *$2^I_{\mathfrak{J}}$ is atomless and basically connected, then $\mathfrak{K}_{\mathfrak{J}}$ is countably compact.*

Proof. Consider a countable subfamily $\mathfrak{L} \subseteq \mathfrak{K}_{\mathfrak{J}}$ with the finite intersection property. Let $\mathfrak{L} = \langle Q_{\mathfrak{B}}(n) \colon n < \omega \rangle \cup \langle R_{\mathfrak{C}}(n) \colon n < \omega \rangle$, where

$$\mathfrak{B}^{(n)} = \langle B^{(n)}_i \colon i \in I \rangle \text{ and } \mathfrak{C}^{(n)} = \langle C^{(n)}_i \colon i \in I \rangle .$$

We put

$$\text{(2)} \qquad F^{(j)}_n = \{ i \in I \colon B^{(n)}_i - (C^{(0)}_i \cup \ldots \cup C^{(j)}_i) \neq 0 \}$$

and

$$\text{(3)} \qquad I_j = \{ i \in I \colon C^{(0)}_i \cup \ldots \cup C^{(j)}_i = A_i \} .$$

It follows from the finite intersection property of $\mathfrak{L}$ that $F^{(j)}_n \notin \mathfrak{J}$ and $I_j \in \mathfrak{J}$. For any n, the sequence $\langle F^{(j)}_n \colon j < \omega \rangle$ is decreasing, whence, by basic connectedness of $2^I_{\mathfrak{J}}$, there is a set F_n such that

$$\text{(4)} \qquad F_n \subseteq F^{(j)}_n (\bmod \mathfrak{J}) \quad \text{and} \quad F_n \notin \mathfrak{J} .$$

In view of Lemma 1 one may assume that the sets F_n are mutually disjoint. Since $F_n - F_n^{(n)} \in \mathfrak{J}$, without any loss of generality we may assume that

$$F_n \subseteq F_n^{(n)}. \tag{5}$$

$\mathfrak{J}$ satisfies (ii), hence $I = \bigcup_{j<\omega} E_j$, where E_j form an increasing sequence of sets from $\mathfrak{J}$.

We are going to define an $f \in A$ such that $f \in \bigcap \mathfrak{L}$. There are two cases: $i \in F = \bigcup_{n<\omega} F_n$ or $i \notin F$.

Case 1. $i \in F$. Then for some n $i \in F_n$, we put $j_{\max} = \sup\{j: i \in F_n^{(j)}\}$ and

$$j_i = \begin{cases} j_{\max} & \text{if} \quad j_{\max} < \infty \\ j & \text{if} \quad j_{\max} = \infty, \quad i \in E_j - E_{j-1}. \end{cases}$$

Now we define $f(i)$ in such a way that

$$f(i) \in B_i^{(n)} - (C_i^{(0)} \cup \ldots \cup C_i^{(j_i)}). \tag{6}$$

Case 2. $i \notin F$. For $j < \omega$ we put $I'_j = I_j \cup E_j$ and let

$$f(i) \in A_i - \bigcup_{l \leqslant j} C_i^{(l)} \quad \text{for} \quad i \in (I'_{j+1} - I'_j) - F. \tag{7}$$

The sets occuring in (6) and (7) are non-empty by (2), (3) and (5). It remains to verify that $f \in \bigcap \mathfrak{L}$. For any $j_0 < \omega$ the following holds:

$$\begin{gathered} \{i: f(i) \in \bigcup_{l \leqslant j_0} C_i^{(l)}\} \subseteq E_{j_0} \cup \bigcup_{n \leqslant j_0} \big(\{i \notin E_{j_0}: f(i) \in \bigcup_{l \leqslant j_0} C_i^{(l)}\} \cap F_n\big) \cup \\ \cup \bigcup_{n > j_0} \big(\{i \notin E_{j_0}: f(i) \in \bigcup_{l \leqslant j_0} C_i^{(l)}\} \cap F_n\big) \cup \{i \notin E_{j_0}: f(i) \in \bigcup_{l \leqslant j_0} C_i^{(l)}\} - F. \end{gathered} \tag{8}$$

We have $E_{j_0} \in \mathfrak{J}$. For the second summand in (8) we have

$$\{i \notin E_{j_0}: f(i) \in \bigcup_{l \leqslant j_0} C_i^{(l)}\} \cap F_n \subseteq F_n - F_n^{(j_0)} \in \mathfrak{J}.$$

The third summand is empty, by (5) and (6). Finally, the last one is contained in I'_{j_0+1}, by (7); hence $\{i: f(i) \in \bigcup_{l \leqslant j_0} C_i^{(l)}\} \in \mathfrak{J}$. The fact that for any $n < \omega$ $\{i: f(i) \in B_i^{(n)}\} \supseteq F_n \notin \mathfrak{J}$ (see (4)) completes the proof that $f \in \bigcap \mathfrak{L}$.

We are now ready to prove our main result.

THEOREM 1. *If $\mathfrak{J}$ is an ideal of subsets of I such that $2^I_{\mathfrak{J}}$ is basically connected and atomless and* (ii) *holds, then for any family $\mathfrak{A}$ (indexed by I) of relational structures of the same similarity type, the reduced product $\mathfrak{A} = \mathfrak{P}_{\mathfrak{J}}\mathcal{A}$ is countably compact.*

Proof. One can assume that our family is rich. By Proposition 1, every formula is equivalent in $\mathfrak{A}$ to an open formula. It is well known that for any countably compact family of sets the closure of it with respect to finite unions and intersections is countably compact. Hence it is enough to investigate only atomic formulas and their negations. For atomic formula $P(v_0)$ and $f \in A$ we have $\mathfrak{A} \models P[f]$ if and only if $\{i: f(i) \notin P_i\} \in \mathfrak{J}$, where $P_i = \{a \in A_i: \mathfrak{A}_i \models P[a]\}$. Puting $B_i = A_i - P_i$, we obtain

$$\mathfrak{A} \models P[f] \quad \text{if and only if } \{i: f(i) \in B_i\} \in I,$$
$$\mathfrak{A} \models \neg P[f] \quad \text{if and only if } \{i: f(i) \in B_i\} \notin I.$$

By Lemma 2, this completes the proof.

COROLLARY 1. *When $2^I_{\mathfrak{J}}$ is atomless, countably compact and* (ii) *holds, then for every family $\mathcal{A}$ of relational structures, $\mathfrak{A} = \mathfrak{P}_{\mathfrak{J}}\mathcal{A}$ is countably compact.*

Let us remark that (ii) is a necessary assumption. In fact, let us consider a structure $\mathfrak{A}$ given by an infinite set A and a decreasing sequence of non-empty subsets B_n of A $(n < \omega)$ with the empty intersection. For any ideal $\mathfrak{J}$ the sets $Q_n = \{f: \{i: f(i) \notin B_n\} \in \mathfrak{J}\}$ form a decreasing sequence of non-void sets from $\mathfrak{K}_{\mathfrak{J}}$. It is an easy exercise to prove that $\bigcap_{n<\omega} Q_n \neq 0$ if and only if $\mathfrak{J}$ satisfies (ii).

LEMMA 3. *If $2^I_{\mathfrak{J}}$ is atomless and basically connected, then $\mathfrak{J}$ satisfies a condition:*

(C) *If $E \notin \mathfrak{J}$, then there is a subset E_0 of E such that $E_0 \notin \mathfrak{J}$ and E_0 is a union of countably many sets from $\mathfrak{J}$.*

Proof. Since $2^I_{\mathfrak{J}}$ is atomless, for $E \notin \mathfrak{J}$ there is a sequence $\langle E_n: 0 < n < \omega\rangle$ such that $E_n \subseteq E$, $E_n \cap E_m = 0$ for $m \neq n$ and $E_n \notin \mathfrak{J}$. Since $2^I_{\mathfrak{J}}$ is basically connected, there is a set $E_0 \subseteq E$ such that $E_0 \notin \mathfrak{J}$ and $E_0 \subseteq \bigcup_{k \geqslant n} E_k \pmod{\mathfrak{J}}$. Obviously, $E_0 = \bigcup_{k<\omega} E_0 \cap E_k$ and $E_0 \cap E_k \in \mathfrak{J}$.

One can see that (C) follows from (ii) but the example of the ideal of all finite subsets of an uncountable set shows that the converse fails.

Criterion. An atomless Boolean algebra $\mathfrak{B}$ is countably compact if and only if for any a_n, b_n, c_n, d_n from $\mathfrak{B}$ such that

$$a_n \subseteq a_{n+1} \subseteq b_{n+1} \subseteq b_n,\ b_n \cap c_m \neq 0,\ -a_n \cap d_m \neq 0 \quad \text{for } m, n < \omega, \tag{9}$$

there is an element x in $\mathfrak{B}$ such that

$$a_n \subseteq x \subseteq b_n,\ x \cap c_n \neq 0,\ -x \cap d_n \neq 0 \quad \text{for all} \quad n < \omega. \tag{10}$$

THEOREM 2. *If the Boolean algebra $2^I_{\mathfrak{J}}$ is atomless and basically connected, then $2^I_{\mathfrak{J}}$ is countably compact.*

Proof. Assume that $a_n = A_n/I$, $b_n = B_n/I$, $c_n = C_n/I$, $d_n = D_n/I$ satisfy (9). Without loss of generality we may assume that $A_n \subseteq A_{n+1} \subseteq B_{n+1} \subseteq B_n$ for $n < \omega$. In virtue of Lemma 3, one can find sets $C_{m,n} \subseteq C_m \subseteq B_n$ and $D_{m,n} \subseteq D_m - A_n$ which do not belong to $\mathfrak{J}$ but are countable unions of sets from $\mathfrak{J}$. Let us denote by $\mathfrak{J}'$ the restriction of ideal $\mathfrak{J}$ to the set $I' = \bigcup_{m,n<\omega} (C_{m,n} \cup D_{m,n})$. $\mathfrak{J}'$ satisfies (ii), hence by Theorem 1 there exists a set $X' \subseteq I'$ such that $(A_n \cap I')/\mathfrak{J}' \subseteq X'/\mathfrak{J}' \subseteq (B_n \cap I')/\mathfrak{J}'$, $X' \cap C_n \notin \mathfrak{J}'$, $-X' \cap D_n \notin \mathfrak{J}'$. The element $X/\mathfrak{J}$ of $2^I_{\mathfrak{J}}$, where $X = X' \cup (-I' \cap \bigcup_{n<\omega} A_n)$, satisfies (10).

Let us remark that Theorem 2 does not hold for every atomless Boolean algebra. In fact, the algebra of all closed and open subsets of the one-point compactification of a topological union of uncountably many disjoint copies of $\beta(\omega) - \omega$ [3] is basically connected but it is not countably compact.

THEOREM 3. *If an ideal $\mathfrak{J}$ on I has a countable basis and has the property* (ii), *then $2^I_{\mathfrak{J}}$ is atomless and basically connected.*

Proof. By the assumption, there is a countable partition of I into sets E_n from $\mathfrak{J}$ such that $E \in \mathfrak{J}$ if and only if $E \subseteq \bigcup_{n<n_0} E_n$ for some $n_0 < \omega$. Obviously $2^I_{\mathfrak{J}} = \mathfrak{P}_{n<\omega} (2^{E_n})/\mathfrak{J}_0$, where $\mathfrak{J}_0$ is the Fréchet ideal. The assertion follows from Theorem 1 and the known fact that $2^{\omega}_{\mathfrak{J}_0}$ is atomless and basically connected.

EXAMPLE. Let $\langle I_n: n < \omega\rangle$ be a partition of ω into countably many infinite subsets and let $\mathfrak{J}_1$ be a non-principal prime ideal in ω. We are going to define another ideal $\mathfrak{J}$ letting $E \in \mathfrak{J}$ if $\{n: |I_n \cap E| = \aleph_0\} \in \mathfrak{J}_1$. We will show that $2^{\omega}_{\mathfrak{J}}$ is atomless and countably compact but $\mathfrak{J}$ has no countable basis.

Let us observe that $2^{\omega}_{\mathfrak{J}}$ is isomorphic to $(2^{\omega}_{\mathfrak{J}_0})^{\omega}_{\mathfrak{J}_1}$, where $\mathfrak{J}_0$ is the Fréchet ideal, hence the algebra is atomless and by [4] countably compact. Finally $\mathfrak{J}$ has no countable basis. In fact, if $E_n \in \mathfrak{J}$, then $A_n = \{i: |I_i \cap E_n| = \aleph_0\} \in \mathfrak{J}_1$ and, since $\mathfrak{J}_1$ is maximal, there is a set A in $\mathfrak{J}_1$ which does not belong to the ideal generated by sets A_n. It is easy to see that the set $E = \bigcup_{i\in A} I_i$ is in $\mathfrak{J}$ and $E \nsubseteq E_n$ for all $n < \omega$.

References

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